Title: Quantizing and Dequantizing Reference Frames

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Abstract: Quantum Information Workshop
Outline

The coherence as fact vs. coherence as fiction controversy

A resolution: Classical reference frames and quantum reference frames as alternative paradigms of description

The lessons I wish to draw from this:
• Quantum states describe relations
• Many, if not all, superselection rules can be circumvented in principle
Coherence: Fact or fiction?

There are many contexts in which the debate arises:

Superconductors – for superpositions of charge eigenstates
BECs – for superpositions of atom number eigenstates
Lasers – for superpositions of photon number eigenstates

We discuss the optical case, although the discussion would be similar for the others.
Optical coherence: a convenient myth?


Standard assumption:

\[ |\alpha\rangle = \sum_{n=0}^{\infty} \frac{e^{-|\alpha|^2/2} \alpha^n}{\sqrt{n!}} |n\rangle \]
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But if we quantize the atoms in the gain medium, and:
• assume the gain medium is in an energy eigenstate,
• apply energy conservation

$$|e\rangle|n\rangle \rightarrow \alpha(t)|e\rangle|n\rangle + \beta(t)|g\rangle|n+1\rangle$$

→ atoms and field evolve to an entangled state
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→ atoms and field evolve to an entangled state

$$\rho = \sum_{n=0}^{\infty} p_n |n\rangle\langle n|$$

$$p_n = \frac{e^{-|\alpha|^2} |\alpha|^{2n}}{n!}$$

Thus, coherence is a fiction!
The ensuing controversy

- S. J. van Enk and C. A. Fuchs, Quantum Information and Computation 2, 151 (2002)
- J. Smolin, quant-ph/0407009
- ...
A possible dialogue

C: The reduced density operator should be interpreted as a mixture of coherent states

\[ \rho = \sum_{n=0}^{\infty} p_n |n\rangle \langle n| = \int_{0}^{2\pi} \frac{d\phi}{2\pi} |\alpha\rangle \langle \alpha| \]
A possible dialogue

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NC: This is to commit the notorious preferred ensemble fallacy
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C: This is a proper mixture, the PEF only applies to improper mixtures
Experiments have shown that lasers have a well-defined phase
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Example: Homodyne detection

\[ |\beta\rangle |\alpha\rangle \]
\[ \langle b^\dagger a + ba^\dagger \rangle \neq 0 \]
C: Experiments have shown that lasers have a well-defined phase

NC: No they haven’t

Example: Homodyne detection

\[ |\beta\rangle|\alpha\rangle \]
\[ \langle b^\dagger a + ba^\dagger \rangle \neq 0 \]

Demonstrates coherence between states of different relative number
Can any standard optical experiment detect coherence?
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No.

N Probe fields

Optical elements

N+1 Photodetectors

signal
Can any standard optical experiment detect coherence?

No.

\[ \rho = \sum_{n,m} p_{nm} |n\rangle \langle m| \] from \[ \rho = \sum_{n} p_{nm} |n\rangle \langle n| \]

This cannot distinguish

The coherence has no operational significance!
But one can generate and detect coherence given a classical clock
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Generating coherence relative to a classical clock
Ex: classical oscillating current

\[ U(t,0) = \exp(\alpha(t)a^\dagger - \alpha(t)^*a) \]

\[ U(t,0)|\text{vac}\rangle = |\alpha(t)\rangle \]
But one can generate and detect coherence given a classical clock

Generating coherence relative to a classical clock
Ex: classical oscillating current
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\[ U(t, 0)|\text{vac}\rangle = |\alpha(t)\rangle \]

Detecting coherence relative to a classical clock
Ex: In homodyne detection, if the local oscillator is treated classically, then the interference term is
\[ \langle \beta^*a + \beta a^\dagger \rangle \]
So, both descriptions are empirically adequate!

The debate usually presumes that the quantum state of a system describes its intrinsic properties and consequently that there is a matter of fact about whether or not there is coherence.
So, both descriptions are empirically adequate!

The debate usually presumes that the quantum state of a system describes its intrinsic properties and consequently that there is a matter of fact about whether or not there is coherence.

Our suggestion: there are really only relations between systems and the quantum state describes these. In this case, the two descriptions can be consistent.
Relational view of quantum states

The quantum state describes the relation between the system and the reference frame

Coherence paradigm = classical RF paradigm

No coherence paradigm = quantum RF paradigm


<table>
<thead>
<tr>
<th>Non-eigenstate of</th>
<th>Classical RF</th>
<th>Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>linear momentum</td>
<td>spatial frame (e.g. GPS satellites)</td>
<td>HW</td>
</tr>
<tr>
<td>angular momentum</td>
<td>orientation frame (e.g. gyroscopes)</td>
<td>SU(2)</td>
</tr>
<tr>
<td>photon number</td>
<td>clock</td>
<td>U(1)</td>
</tr>
<tr>
<td>atom number</td>
<td>BEC phase</td>
<td>U(1)</td>
</tr>
<tr>
<td>charge</td>
<td>Superconducting phase</td>
<td>U(1)</td>
</tr>
</tbody>
</table>

We shall consider a general framework that works for all these cases
G = group of transformations for the relevant d.o.f.

No classical RF for G

⇒

Operations and observables must be invariant under collective action of G (Superselection rule)
\( G = \) group of transformations for the relevant d.o.f.

No classical RF for \( G \) \hspace{1cm} \hspace{1cm} \text{Operations and observables must be invariant under collective action of } G \hspace{1cm} \text{(Superselection rule)}

Suppose \( T: G \to \text{GL}(H) \) is a collective representation of \( G \)

A \( G \)-invariant CP map \( \mathcal{O} \) satisfies

\[
\mathcal{O}[T(g)\rho T^\dagger(g)] = T(g)\mathcal{O}[\rho]T^\dagger(g) \quad \forall \ g \in G
\]

A \( G \)-invariant POVM \( \{E_k\} \) satisfies

\[
T(g)E_kT^\dagger(g) = E_k \quad \forall \ g \in G
\]
Equivalence classes of states:

\[ \rho \equiv \rho' \quad \text{if} \quad \text{Tr}[A\rho] = \text{Tr}[A\rho'] \]

for all \( G \)-invariant \( A \)

or

\[ G(\rho) = G(\rho') \]

where

\[ G[\rho] \equiv \begin{cases} 
\frac{1}{|G|} \sum_{g \in G} T(g) \rho T^\dagger(g), & \text{finite groups} \\
\int_G \text{d}v(g) T(g) \rho T^\dagger(g), & \text{Lie groups}
\end{cases} \]

Convention: represent each equivalence class by the \( G \)-invariant state

\[ \rho = G(\rho) \]
Quantizing RFs

Suppose the system state w.r.t the classical RF is:

\[ |\psi\rangle \in H_S \]

Quantize all physical objects that can serve as a RF. Introduce a Hilbert space \( H_R \)

**Naïve approach:** assign \( |\chi\rangle \otimes |\psi\rangle \in H_R \otimes H_S \)

E.g. For optical case, one could take \( |\chi\rangle \) to be a coherent state \( |\alpha\rangle \)

**Better approach:** Assign \( \rho \) on \( H_R \otimes H_S \)

\[
\rho = \frac{1}{2\pi} \int_0^{2\pi} d\phi \ |\phi\rangle \langle \phi| \otimes T(\phi)|\psi\rangle \langle \psi| T^\dagger(\phi)
\]
Problem with naïve approach to quantization:
There is no observational difference among states

\[ U(g)|\chi\rangle \otimes U(g)|\psi\rangle \]

for different \( g \in G \)

There is no real difference associated with this distinction
The only real degree of freedom is in the relative orientation
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We must find a set of \( G \)-invariant states in \( H_R \otimes H_S \)
that encode the possible relations
Can these simulate the states in \( H_S \)? Yes.

See: Kitaev, Mayers, Preskill, quant-ph/0310088
Classical RF paradigm

States \[ \rho \]
Measurements \[ \{ E_k \} \]
Transformations \[ \mathcal{O} \] \[ \text{defined on } \mathcal{H}_S \]

Quantum RF paradigm

States \[ \tilde{\rho} \] \[ \text{defined on } \mathcal{H}_R \otimes \mathcal{H}_S \]
Measurements \[ \{ \tilde{E}_k \} \] \[ \text{and } G\text{-invariant} \]
Transformations \[ \tilde{\mathcal{O}} \]

Find a mapping such that

\[ \rho \rightarrow \tilde{\rho} \]
\[ E_k \rightarrow \tilde{E}_k \]
\[ \mathcal{O} \rightarrow \tilde{\mathcal{O}} \]

\[ \text{Tr}_S[\mathcal{O}(\rho)E_k] = \text{Tr}_{RS}[\tilde{\mathcal{O}}(\tilde{\rho})\tilde{E}_k] \]
Define
\[ \tilde{\rho} = \$ (\rho) \]
\[ \tilde{E}_k = \$ (E_k) \]
\[ \tilde{A}_\mu = \$ (A_\mu) \]

where

\[ \$ : A \mapsto \int_G d\nu(g) \langle g | \otimes T(g) AT^\dagger(g) \langle g | \]

with \[ T(g') | g \rangle = | g' \circ g \rangle, \text{ for all } g, g' \in G \]
and \[ \langle g | g' \rangle = \delta(g, g') \]
Property 1: $(A)$ is $G$-invariant

Proof: $(T(g') \otimes T(g'))(A)(T^\dagger(g') \otimes T^\dagger(g'))$

\[
= \int_G d\mu(g) T(g') \langle g | T^\dagger(g') \otimes T(g') T(g) A T^\dagger(g) T^\dagger(g')
\]
\[
= \int_G d\mu(g) \langle g' \circ g | \langle g' \circ g | \otimes T(g' \circ g) A T^\dagger(g' \circ g)
\]
\[
= (A).
\]
Property 1: \( (T(g') \otimes T(g')) \mathcal{A}(A)(T^\dagger(g') \otimes T^\dagger(g')) \) 

\[ = \int_G d\mu(g) T(g') \langle g | T^\dagger(g') \otimes T(g') T(g) A T^\dagger(g) T^\dagger(g') \rangle \]

\[ = \int_G d\mu(g) \langle g' \circ g | \langle g' \circ g | \otimes T(g' \circ g) A T^\dagger(g' \circ g) \rangle \]

\[ = \mathcal{A}(A). \]

Property 2: \( \mathcal{A}(A + B) = \mathcal{A}(A) + \mathcal{A}(B) \) and \( \mathcal{A}(AB) = \mathcal{A}(A)\mathcal{A}(B) \)

Proof: \( \int_G d\nu(g) |g\rangle \langle g| \otimes T(g) A T^\dagger(g) \int_G d\nu(g') |g'\rangle \langle g'| \otimes T(g') B T^\dagger(g') \)

\[ = \int_G d\mu(g) |g\rangle \langle g| \otimes T(g) A T^\dagger(g) T(g) B T^\dagger(g) \]

\[ = \int_G d\mu(g) |g\rangle \langle g| \otimes T(g) A B T^\dagger(g) \)
Property 3: \[ \text{Tr}_{RS}(\$A)) = \text{Tr}_{S}(A) \]

Property 4: if \( A > 0 \) then \( \$A > 0 \)

Property 5: \[ \$I_{S} = I_{RS} \]
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Property 4: if \( A > 0 \) then \( (A) > 0 \)

Property 5: \( (I_S) = I_{RS} \)

3,4 → if \( \rho \) is a density operator, so is \( \tilde{\rho} \)
2,4,5 → if \( \{E_k\} \) is a POVM, so is \( \{\tilde{E}_k\} \)
2,5 → if \( \mathcal{O} \) is a CP map, so is \( \tilde{\mathcal{O}} \)
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2,5 \rightarrow \text{if } \mathcal{O} \text{ is a CP map, so is } \tilde{\mathcal{O}}

\[
\text{Tr}_{RS}[\tilde{\mathcal{O}}(\tilde{\rho})\tilde{E}_k] = \text{Tr}_{RS}[\sum_k \$(A_\mu)\$(\rho)\$(A_\mu^\dagger)\$(E_k)] \\
= \text{Tr}_{RS}[\$(\sum_k A_\mu \rho A_\mu^\dagger E_k)] \\
= \text{Tr}_S[\mathcal{O}(\rho)E_k]
\]
Example: **Superpositions of charge eigenstates**

Consider a coherent superposition of charge eigenstates on $H_S$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

This is simulated by the U(1)-invariant state

$$\rho = \frac{1}{2\pi} \int_0^{2\pi} d\theta \ |\theta\rangle \langle \theta| \otimes T(\theta) |\psi\rangle \langle \psi| T^\dagger(\theta)$$

$$|\theta\rangle = \frac{1}{\sqrt{2\pi}} \sum_{q=-\infty}^{\infty} e^{-iq\theta} |q\rangle$$

$$T(\theta) = e^{-i\theta \hat{Q}}$$

which may be written as

$$\rho = \sum_{q=-\infty}^{\infty} |\psi_q\rangle \langle \psi_q|$$

where

$$|\psi_q\rangle = \alpha|q + 1\rangle|0\rangle + \beta|q\rangle|1\rangle$$
The relational Hilbert space

G-invariant operators have the form

\[ G(A) = \int_G d\nu(g) T(g) A T^\dagger(g). \]
The relational Hilbert space

$G(A) = \int_{G} d\nu(g)T(g)AT^{\dagger}(g)$.

Writing

$\mathcal{H} = \bigoplus_{j} \mathcal{H}_{j}^{\text{glob}} \otimes \mathcal{H}_{j}^{\text{rel}}$

Carrier space of jth irrep of $G$

Hilbert space for the multiplicity of the jth irrep of $G$
The relational Hilbert space

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\[ G(A) = \int_G d\nu(g)T(g)AT^\dagger(g). \]

Writing
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Carrier space of jth irrep of G

Hilbert space for the multiplicity of the jth irrep of G

We have, by Schur’s lemma,
\[ G(A) = \sum_j D_j^\text{glob} \otimes I_j^\text{rel} (P_j AP_j). \]

Decoherence-full subsystem

Decoherence-free subsystem
Dequantizing RFs

Wrong approach: Trace over reference frame

\[ \rho_S = \text{Tr}_R(\rho_{RS}) \]

Right approach: Project into an irrep and trace over the decoherence-full subsystem
i.e. keep only the decoherence-free subsystem

\[ \rho_S = \text{Tr}_{\text{glob}}(P_j \rho_{RS} P_j) \]
Conclusions

• Quantum states describe the relation of a system to a reference frame

• One can break superselection rules given appropriate resources
Future research

- Quantizing and dequantizing finite RFs
- Degradation of finite RFs (see poster by P. Turner)
- Possibility of condensates for novel degrees of freedom
- Connection to relationalism in quantum gravity (work with E. Livine and F. Girelli)