Abstract: I will show Abner how to construct Minkowski's space-time diagrams directly from Einstein's two postulates and some very elementary plane geometry. This geometric route into special relativity was developed while teaching the subject to nonscientists, but some of its features may be unfamiliar to physicists and philosophers.
Plane geometry in (flat) spacetime

How to construct Minkowski Diagrams (1908) *directly* from Einstein’s postulates (1905).

*Light rectangles*
Einstein’s Two Postulates (Voraussetzungen) (1905)

1. In electrodynamics, as well as in mechanics, no properties of phenomena correspond to the concept of absolute rest.

Dem Begriffe der absoluten Ruhe nicht nur in der Mechanik, sondern auch in der Elektrodynamik keine Eigenschaften der Erscheinungen entsprechen.

2. Light always propagates in empty space with a definite velocity \(c\), independent of the state of motion of the emitting body.

Sich das Licht im leeren Raume stets mit einer bestimmten, von Bewegungszustande des emittierenden Körpers unabhängigen Geschwindigkeit \(V\) fortflanze.

#1 “only apparently incompatible” with #1
(nur scheinbar unverträgliche)
Einstein's Third Postulate (1905)

3. If a clock at A runs synchronously with clocks at B and C, then the clocks at B and C also run synchronously relative to each other.

Wenn die Uhr in A sowohl mit der Uhr in B als auch mit der Uhr in C synchron läuft, so laufen auch die Uhren in B und C synchron relativ zueinander.

3’. If event A is simultaneous with event B and event C, then events B and C are also simultaneous.

3”’. If an event A happens in the same place as event B and event C, then the events B and C also happen in the same place.
An event:
Something happening at definite place and time;
A point in spacetime.

Alice’s geometric description of events:
Alice makes a plane diagram depicting events at various times and places in one spatial dimension (e.g. along a long straight railroad track).

- Lightning strikes track
- Bob turns on light
- Cow crosses tracks
- Conductor punches Alice’s ticket
- Front of train crosses highway
Alice organizes events in her diagram by time:

*Simultaneous events placed on single straight line*

- **= an event**

*Equitemps*

(lines of constant time)

*Distance between equitemps proportional to time between events*

Equitemps must be parallel.
Alice slides events along equitemps to further organize them by location:

*Events in same place lie on same straight line.*

![Equilocs](image)

- = an event

*Equilocs*

(lines of constant position)

*Distance between equilocs proportional to real space distance between events.*

Equilocs must be parallel.

Can’t be parallel to equitemps, but otherwise orientation is arbitrary.
Alice redefines the foot:

1 conventional foot* (ft) = 0.3048 m.
1 foot** (f) = 0.3048 m.
1 f/ns = 299,792,458 m/s = c, speed of light.
   (ns = nanosecond = 10^{-9} sec)

*Archaic unit still used in some backward nations.
**If you prefer, phoot (pronounced “foot”).

Alice relates spatial and temporal scales:

Equilocs representing events 1 f apart
   are same distance λ apart in diagram as
equitemps representing events 1 ns apart.
Some of Alice's equitemps and equiloccs and her scale factor $\lambda$

Conventional orientation:

Equiloccs more vertical than horizontal;
Equitemps more horizontal than vertical;
Both symmetrically disposed about $45^\circ$ lines.

Time increases with height on page
Some of Alice's equitemps and equiloces and her scale factor $\lambda$.

Conventional orientation:

Equiloces more vertical than horizontal;
Equitemps more horizontal than vertical;
Both symmetrically disposed about $45^\circ$ lines.
Time increases with height on page.
Equilocs and equitemps are characterized by two independent parameters: any two of $\lambda$, $\mu$, $\Theta$.

*Note:* Area of unit rhombus $= \lambda \mu = \mu^2 \sin \Theta$.  

Alternative scale factor $\mu$
Photon trajectory:
All events in the history of something moving at 1f/ns

Photon trajectories bisect angle \( \Theta = 2\theta \)
between equilocs and equitemps
(Equilocs and equitemps symmetrically disposed about photon trajectories)
Trajectories of oppositely moving photons are perpendicular.
Bob’s description of the same events

Bob moves uniformly with respect to Alice. He uses Alice’s diagram to depict events, but tries to impose on it his own equilocs and equitemps.

\[ v_{BA} = \frac{\mu_A g}{\mu_A h} = \frac{g}{h} \]

Pirsa: 06070037
Determining Bob's *equitemps* in Alice's diagram:

*Einstein's Train*

Bob's equilocs

- reflected photons return to middle
- photon reaches front and bounces back
- light flashes in middle of train
- photon reaches rear and bounces back

Bob's equitemp

rear of Bob's train
middle of Bob's train
front of Bob's train
Determining Bob’s *equitemp* in Alice’s diagram:

*Einstein’s Train*

Bob’s equilocs

- reflected photons return to middle
- photon reaches front and bounces back

\[ \Theta_B \]

Bob’s equitemp

- photon reaches rear and bounces back
- light flashes in middle of train

rear of Bob’s train  middle of Bob’s train  front of Bob’s train
Bob's equitemps and equilocs are straight lines that make the same angle $\theta_B = \frac{1}{2} \Theta_B$ with photon trajectories.

Alice's equitemps and equilocs are straight lines that make the same angle $\theta_A = \frac{1}{2} \Theta_A$ with photon trajectories.

Cannot tell who made the diagram first and who later added their own equitemps and equilocs.
"Relativity of Simultaneity"

\[ v_{BA} = g/h \]

Bob: \( P, R \) at same place. \( P, Q \) at same time.

Alice: \[ D_{PR} = v_{BA} T_{PR} \]
\[ (\mu_A g) \quad (\mu_A h) \]

\[ T_{PQ} = v_{BA} D_{PQ} \]
\[ (\mu_A g) \quad (\mu_A h) \]
Relations between events

Events lie on somebody's equitemp
Spacelike separated

Timelike separated

Events lie on somebody's equiloc

Events lie on photon trajectory
Lightlike separated
Two events determine a light rectangle.
Two events determine a light rectangle.

$E_1$ timelike separated $E_2$

$E_3$ spacelike separated $E_4$
Alice's unit light rectangle

Alice's clock

$\mu_A$
Area $\Omega_0$ of Alice's unit light rectangle

$$= \frac{1}{2}$$

$$\Omega_0 = \frac{1}{2} \lambda_A \mu_A$$
Relation between Alice’s and Bob’s scale factors:

Reciprocity of Doppler Effect

When Alice’s reads T
she sees Bob’s reading T

When Bob’s reads T
he sees Alice’s reading T

Both clocks read 0
\[ \frac{B}{A} = \frac{T}{t} \]
\[ \frac{b}{a} = \frac{t}{T} \]
\[ bB = aA \]

Alice’s and Bob’s light rectangles have same area.
Light rectangles have same area.

\[ T = 1 \implies \text{unit light rectangles have same area} \]

\[ \Omega_0 = \frac{1}{2} \mu \lambda \]

Product \( \mu \lambda \) of scale factors is the same for everyone:

\[ \mu_A \lambda_A = \mu_B \lambda_B = \mu_C \lambda_C = \cdots \]
Meaning of area $\Omega$ of light rectangle for any pair of events:

Timelike separated

$\Omega = \Omega_0 T^2$

Spacelike separated

$\Omega = \Omega_0 D^2$

$Timelike\ separated$: $\Omega / \Omega_0$ is square of time between events in frame in which events at same place.

$Spacelike\ separated$: $\Omega / \Omega_0$ is square of distance between events in frame in which events at same time.

$\Omega / \Omega_0$ is squared interval $I^2$
What about $I^2 = |T^2 - D^2|$?

*Light Rectangle versus Event Rhombus*

\[
2 \times \begin{array}{c}
\begin{array}{c}
\text{Q} \\
\text{P}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\text{Q} \\
\text{P}
\end{array}
\end{array}
\]

\[
I_{PQ}^2 = \frac{\text{Area}}{\Omega_0}
\]

\[
I_{PQ}^2 = \frac{\text{Area}}{2\Omega_0}
\]
Interval $I$ between events $P$ and $Q$ in terms of Alice’s time $T$ and distance $D$ between them:

Alice’s equilocs

$\mu_A T$

Alice’s equitemps

$T^2$

Areas $/ 2\Omega_0$

$D^2$

$I^2$
Interval $I$ between events $P$ and $Q$ in terms of Alice’s time $T$ and distance $D$ between them:

Alice’s equinos

$\mu_A D$

Alice’s equitemps

$\mu_A T$

Areas $/2\Omega_0$
Application (in 3+1 dimensions)

How to measure the interval between $P$ and $Q$
using only light signals and a single clock:

Alice moves uniformly with her clock;
Alice and her clock are both present at $P$.
Bob is present at $Q$.

When $P$ happens Alice’s clock reads $T_0$.
When $Q$ happens, Bob sees Alice’s clock reading $T_1$.
When Alice sees $Q$ happen, her clock reads $T_2$.

$$I_{PQ} = |(T_1 - T_0)(T_2 - T_0)|$$
\[ \Omega_{P,Q} = f \Omega_{T_2,T_0} \]
\[ \Omega_{T_1,T_0} = f \Omega_{P,Q} \]

Alice and her clock

\[ \Omega_{P,Q}^2 = \Omega_{T_2,T_0}, \Omega_{T_1,T_0} \Rightarrow \]
\[ \Gamma_{P,Q}^2 = (T_2 - T_0)(T_1 - T_0) \]
Stacking plane diagrams in orthogonal direction.

*Isotropy:* When Alice adds *second spatial dimension* perpendicular to plane, photon trajectories through a point would expand to right circular cone.

[Diagram with labels: $\mu \sin(\pi/4 - \theta)$, $\mu \cos(\pi/4 - \theta)$, Alice’s equitemp, Alice’s equiloc, $\theta$, P, X, R, Q]

Sets scale factor $\sigma$ for perpendicular dimension.
Determination of perpendicular scale factor $\sigma$

\[
\sigma^2 + \mu^2 \sin^2(\pi/4 - \theta) = \mu^2 \cos^2(\pi/4 - \theta)
\]

$\sigma$ is (invariant!) geometric measure.
Determination of perpendicular scale factor $\sigma$

\[ \sigma = \sqrt{\mu^2 \cos^2(\pi/4 - \theta) + \mu^2 \sin^2(\pi/4 - \theta)} \]
\[ = \mu^2 \cos^2(\pi/4 - \theta) \]

\[ \sigma^2 = \mu^2 \cos^2(\pi/2 - 2\theta) \]
\[ = \mu^2 \sin^2(2\theta) \]
\[ = \mu^2 \sin^2(\theta) \]
\[ = \mu \lambda \]

\[ \sigma \text{ is (invariant!) geometric mean of \(\mu\) and \(\lambda\).} \]
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