Quantum Field Theory with a Minimal Length

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Based on

Class. Quantum Grav. 23 (2006) 1815 [arXiv:hep-th/0510245]
- Very general expectation for quantum gravity: fluctuations of spacetime itself disable resolution of small distances
- Can be found e.g. in String Theory, Loop Gravity, NCG, etc.
- Minimal length scale acts as UV cutoff
The Minimal Length Scale

- Very general expectation for quantum gravity: fluctuations of spacetime itself disable resolution of small distances
- Can be found e.g. in String Theory, Loop Gravity, NCG, etc.
- Minimal length scale acts as UV cutoff
With Large Extra Dimensions

- Lowering the Planck mass means raising the Planck length!
- Effects of a finite resolution become important at the same energy scale as other new signatures.
- Relevance for high energy and high precision experiments.

\[ m_p^2 = M_f^{d+2} R^d \]
Finite Resolution of Structures
Finite Resolution of Structures

- For large momenta, $p$, Compton-wavelength $\lambda = 1/k$ cannot get arbitrarily small $\lambda > L_{\text{min}} = 1/M_f$
A Model for the Minimal Length

- Modify wave-vector $k$ and commutation relations
  \[ k = k(p) = \hbar p + a_1 p^3 + a_2 p^5 \ldots \Rightarrow [p_i, x_j] = i \frac{\partial p_i}{\partial k_j} \]

- Results in a generalized uncertainty principle
  \[ \Delta x \Delta p \geq \frac{1}{2} \hbar \left( 1 + b_1 L_{\text{min}}^2 \langle p^2 \rangle \right) \]

- And a squeezed phase space at high energies
  \[ \langle p | p' \rangle = \frac{\partial p}{\partial k} \delta(p - p') \Rightarrow dk \rightarrow \hbar \ dp \ \frac{\partial k}{\partial p} = \hbar \ dp \ e^{-L_{\text{min}}^2 p^2} \]

- Can but need not have a varying speed of light $d \omega / dk \neq 1$. 
\[ f(\omega) = \frac{1}{\lambda^2} \]
Two Approximations

1. First order expansion

\[ k \approx p + \frac{\beta}{3} p \left( \frac{p}{M_f} \right)^2, \quad \frac{\partial k}{\partial p} \approx 1 + \beta \left( \frac{p}{M_f} \right)^2 \]

2. High energy limit

\[ k \approx M_f \operatorname{Erf} \left( -\frac{p}{M_f} \right), \quad \frac{\partial k}{\partial p} \approx \exp \left( -\frac{p^2}{M_f^2} \right) \]

First order expansion is not appropriate for integration over momentum space (higher order contributions)
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Quantisation with a Minimal Length

Parametrisation in $p_x = f_x(k)$ — input from underlying theory.

- Quantize via $k \rightarrow -i\partial$, $p \rightarrow f(-i\partial) := F(\partial)$
  
  $$p = k + \sum_{n \geq 1} a_n k^{2n+1}, \quad F(\partial) = -i\partial + \sum_{n \geq 1} a_n (-i)^{2n+1} \partial^{2n+1}$$

- The Klein-Gordon equation, alias modified dispersion relation
  
  $$E^2 - p^2 = m^2 \quad \Rightarrow \quad F^\nu(\partial) F_\nu(\partial) \psi = m^2 \psi$$

- The Dirac equation
  
  $$(F(\partial) - m) \psi = 0$$

- (Anti)-commutation relations
  
  $$\left[ a^\dagger(p), a(p') \right]_\pm = \delta(p - p') \left| \frac{\partial p}{\partial k} \right|$$
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- The Dirac equation

$$(F(\partial) - m)\psi = 0$$

- (Anti)-commutation relations

$$\left[ a^\dagger_\nu(p), a_\mu(p') \right]_{\pm} = \delta(p - p') \left| \frac{\partial p}{\partial k} \right|$$
Field Expansion

- Express everything in momentum-space

\[
\phi = \sum_f dk \left( a_k v_k + a_k^\dagger v_k^* \right)
\]

\[
\phi = \sum_f dp \left| \frac{\partial k}{\partial p} \right| \left( a_p v_p + a_p^\dagger v_p^* \right)
\]

- Momenta are observables, have usual properties, and final results are expressed in terms of momentum variables

- In intermediate steps (strong curvature regime), the relation between wave-vector and momentum is modified
Quantisation with a Minimal Length

Parametrisation in $p_V = f_V(k)$ – input from underlying theory.

- Quantize via $k \rightarrow -i\partial$, $p \rightarrow f(-i\partial) := F(\partial)$

$$p = k + \sum_{n \geq 1} a_n k^{2n+1}, \quad F(\partial) = -i\partial + \sum_{n \geq 1} a_n (-i)^{2n+1} \partial^{2n+1}$$

- The Klein-Gordon equation, alias modified dispersion relation

$$E^2 - p^2 = m^2 \quad \Rightarrow \quad F^\nu(\partial) F_v(\partial) \psi = m^2 \psi$$

- The Dirac equation

$$ (F(\partial) - m) \psi = 0 $$

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$$ \left[ \hat{a}^{\dagger}(p), \hat{a}(p') \right]_{\pm} = \delta(p-p') \left| \frac{\partial p}{\partial k} \right| $$
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\[ \phi = \sum_k dk \left( a_k \nu_k + a_k^\dagger \nu_k^* \right) \]

\[ \phi = \sum_p dp \left| \frac{\partial k}{\partial p} \right| \left( a_p \nu_p + a_p^\dagger \nu_p^* \right) \]

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The Collision Region

\[ p = k \]
\[ p = p(k) \]
\[ g = g(k) \]

\[ \psi^+ \]
\[ \psi^- \]

\[ \sqrt{s/m_p} \sim \sqrt{t/m_p} \sim 1 \]

non-negligible gravitational effects described by effective model
Alternative Description with Energy Dependent Metric*

- Even on a classical level, a particle’s energy influences the metric it propagates in

\[ R_{kV} - \frac{1}{2} g_{kV} = \frac{1}{m_p^2} T_{kV} \]

- On a quantum level, expect this dependence to be dominant
- Re-write modified dispersion relation in collision region as

\[ g^{kV}(k) k_k k_V = \mu^2 \]

- At small energies \( g^{kV} \sim \eta^{kV} \)

Quantisation with a Minimal Length

- Lagrangian for free fermions
  \[ L_f = i \bar{\Psi} (\bar{F}(k) - m) \Psi \quad L_f = i \bar{\Psi} (g^{\nu \kappa}(k) \gamma_\nu k_\kappa - m) \Psi \]

- Coupling of the gauge field via \( \partial_\nu \rightarrow D_\nu := \partial_\nu - ieA_\nu \) yields the gauge- and Lorentz-invariant higher order derivative interaction
  \[ L = \bar{\Psi} F(D) \Psi \quad L = \bar{\Psi} \gamma_\nu g^{\nu \kappa}(D) D_\kappa \Psi \]

- To first order one finds the usual \( L = L_f - e \bar{\Psi} \eta^{\nu \kappa} \gamma_\nu A_\kappa \Psi + O(eL^2_{\text{min}}) \) and the dominant modification comes from the propagators
  \[ (\bar{F}(k) - m)^{-1} \quad (g^{\nu \kappa}(k) \gamma_\nu k_\kappa - m)^{-1} \]
  \[ (\bar{F}^\nu(k) F_\nu(k) - m^2)^{-1} \quad (g^{\nu \kappa}(k) k_\nu k_\kappa - m^2)^{-1} \]
Alternative Description with Energy Dependent Metric*

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*Kimberly, Magueijo and Medeiros, Phys. Rev. D 70, 084007 (2004).*
The Collision Region

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\[ R_{\text{KV}} - \frac{1}{2} g_{\text{KV}} = \frac{1}{m_p^2} T_{\text{KV}} \]

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- Re-write modified dispersion relation in collision region as

\[ g^{\text{KV}}(k) k_k k_v = \mu^2 \]

- At small energies \( g^{\text{KV}} \sim \eta^{\text{KV}} \)

Quantisation with a Minimal Length

- Lagrangian for free fermions
  \[ \mathcal{L}_f = \bar{\Psi}(F(k) - m)\Psi \quad \mathcal{L}_f = \bar{\Psi}(g^{\nu\kappa}(k)\gamma_{\nu}k_{\kappa} - m)\Psi \]

- Coupling of the gauge field via \( \partial_{\nu} \rightarrow D_{\nu} := \partial_{\nu} - ieA_{\nu} \) yields the gauge- and Lorentz-invariant higher order derivative interaction
  \[ \mathcal{L} = \bar{\Psi}F(D)\Psi \quad \mathcal{L} = \bar{\Psi}\gamma_{\nu}g^{\nu\kappa}(D)D_{\kappa}\Psi \]

- To first order one finds the usual \( \mathcal{L} = \mathcal{L}_f - e\bar{\Psi}\eta^{\kappa\nu}\gamma_{\kappa}A_{\nu}\Psi + O(eL_{\text{min}}^2) \) and the dominant modification comes from the propagators
  \[ (F(k) - m)^{-1} \quad (g^{\nu\kappa}(k)\gamma_{\nu}k_{\kappa} - m)^{-1} \]
  \[ (F^\nu(k)F_k(k) - m^2)^{-1} \quad (g^{\nu\kappa}(k)k_{\nu}k_{\kappa} - m^2)^{-1} \]
Quantisation with a Minimal Length

- Note: $\delta = F(\partial)$ has higher order derivatives $\rightarrow$ messy Euler-Lagrange formalism
- Make sure that $\mathcal{L} = \delta^\mu \phi \delta_{\mu} \phi$ indeed gives desired equations of motion
- Requires shifting of derivatives
- It can be shown that $\phi_{\mu} (\delta^\mu \psi) = - (\delta^\mu \phi_{\mu}) \psi +$ total divergence and everything works with $\delta \phi$ as conjugated field
- Also conserved currents can be computed this way
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Feynman rules

- Replace every
  \[ dp \quad \text{with} \quad dp \left| \frac{\partial k}{\partial p} \right| \]

- Keep vertices

- Make sure to express energy/momentum conservation in terms of momenta
  \[ \delta(p_i - p_f) \]

- Keep in mind: for asymptotic in/out-going fields applies the usual translation invariance
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The Locality Bound*

From the commutator

$$[a_p, a_{p'}^\dagger] = \delta(p - p') \left| \frac{\partial p}{\partial k} \right|$$

And the field expansion

$$\phi(x) = \int d^3 p \left| \frac{\partial k}{\partial p} \right| \left[ v_p(x) a_p + v_p^*(x) a_p^\dagger \right]$$

One finds the equal time commutator for \( x = (x, t), y = (y, t) \).

$$[\phi(x), \pi(y)] = i \int \frac{d^3 p}{(2\pi)^3} \left| \frac{\partial k}{\partial p} \right| e^{ik(x-y)} \rightarrow i \int \frac{d^3 p}{(2\pi)^3} e^{ik(x-y) - \varepsilon p^2}$$

where \( \varepsilon \sim L_{\text{min}}^2 \).

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$$[a_p, a^+_p] = \delta(p - p') \left\vert \frac{\partial p}{\partial k} \right\vert$$

And the field expansion

$$\phi(x) = \int d^3p \left\vert \frac{\partial k}{\partial p} \right\vert \left[ v_p(x) a_p + v^*_p(x) a^+_p \right]$$

One finds the equal time commutator for $x = (x, t)$, $y = (y, t)$.

$$[\phi(x), \pi(y)] = i \int \frac{d^3p}{(2\pi)^3} \left\vert \frac{\partial k}{\partial p} \right\vert e^{i k(x-y)} - i \int \frac{d^3p}{(2\pi)^3} e^{i k(x-y) - \epsilon p^2}$$

where $\epsilon \sim L^2_{\text{min}}$.

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The Minimal Length as UV-Regulator

- Minimal length regularizes loop-integrals

\[
\int d^d p \frac{1}{p^2(p-q)^2} \sim \ln \frac{\Lambda}{\mu_0} \text{ for } d = 0
\]
\[
\int d^{d+2} p \frac{1}{p^2(p-q)^2} \sim \left( \frac{\Lambda}{\mu_0} \right)^d \text{ for } d > 1?
\]
\[
\rightarrow \int d^{d+2} p \left| \frac{\partial k}{\partial p} \right| \frac{1}{p^2(p-q)^2} < \infty
\]

- Regulararizes also higher-dimensional QFT's.
- Natural interpretation of regulator.

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The Minimal Length as UV-Regulator

Example: Running Coupling

- Looking close, the propagator exhibits complex structures
  \[ D^{\mu\nu}(q) = -g^{\mu\nu}/q^2 \]

- Expansion in series of one-particle irreducible contributions
  \[ i \tilde{D}^{\mu\nu} = i D^{\mu\nu} + i \tilde{D}^{\mu\lambda} [i e^2 \Pi_{\lambda\sigma}] \tilde{D}^{\sigma\nu} + i \tilde{D}^{\mu\lambda} [i e^2 \Pi_{\lambda\sigma}] i \tilde{D}^{\sigma\kappa} [i e^2 \Pi_{\kappa\rho}] i \tilde{D}^{\rho\nu} + \ldots \]

- Can be summarized in \[ [q^2 g_{\mu\nu} - e^2 \Pi_{\mu\nu}] \tilde{D}^\nu_\lambda = -g_{\mu\lambda} \]

- Loop-integral is dampened at high energies from squeezing of the momentum space measure → becomes finite
Example: Running Coupling

- Feynman rules yield

\[
\Pi_{\mu\nu}(q,d) = \epsilon^2 \frac{(2\pi)^d}{\Omega_d} \epsilon^{d/2} \int_0^\infty \frac{d^4 p}{(2\pi)^4} \Tr\left[\frac{\gamma_\mu}{p^\mu} \gamma_\nu \frac{1}{p^\nu - q^\nu}\right] e^{-\epsilon p^2},
\]

- After Wick-rotation

\[
\pi(q,d) = \frac{3 b_i}{2\pi} \frac{(2\pi)^d}{\Omega_d} (\pi\epsilon)^{d/2} \left[ \int_0^1 dx x (1-x)^{1+d/2} \int_0^\infty dz e^{-z x q^2} z^{1-d/2} \right] + \frac{1}{q^2} \frac{(d+4)}{2(d+3)} \epsilon^{1-d/2} \int_0^1 dx (1-x)^{1+d/2} \left( e^{-\epsilon x q^2} - 1 \right)
\]

with \(\Pi_{\mu\nu}(q,d) = \pi(q,d)(q_\mu q_\nu - g_{\mu\nu} q^2)\) and \(\epsilon \sim L_{\text{min}}^2\).

- In 4-dim \((d=0)\) Schwinger parametrization, results in std. log-running.
Example: Running Coupling

- Interpretation: finite resolution of vacuum structure.
- In extra dimensions, power-law running accelerates unification
Example: Running Coupling

- Feynman rules yield

\[ \Pi_{\mu \nu}(q, d) = e^2 \frac{(2\pi)^d}{\Omega_d} \epsilon^{d/2} \int \frac{d^{4+d} p}{(2\pi)^4} \text{Tr} \left[ \frac{\gamma_\mu}{p} \right] \left[ \frac{\gamma_\nu}{p-q} \right] e^{-\epsilon p^2} \]

- After Wick-rotation

\[ \pi(q, d) = 3b_i \alpha_i \frac{(2\pi)^d}{\Omega_d} \left( \pi \epsilon \right)^{d/2} \left[ \int_0^1 dx x (1-x)^{1+d/2} \int_{\epsilon}^\infty dz e^{-zxq^2} z^{-1-d/2} \right. \]
\[ + \left. \frac{1}{q^2} \frac{(d+4)}{2(d+3)} \epsilon^{-1-d/2} \int_0^1 dx (1-x)^{1+d/2} \left( e^{-\epsilon x q^2} - 1 \right) \right] \]

with \( \Pi_{\mu \nu}(q, d) = \pi(q, d)(q_\mu q_\nu - g_{\mu \nu} q^2) \) and \( \epsilon \sim L_{\text{min}}^2 \).

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![Graphs showing running coupling](image)

References:

Tree-Level Amplitudes

- **S-Matrix elements**
  \[
  \tilde{S}_{li} = (2\pi)^4 \tilde{M}_{li} \delta(p_i - p_f) \left. \frac{\partial p}{\partial k} \right|_{p_i = p_f}
  \]
  where the amplitude is unmodified.

- This yields (2 → 2 process in lab frame) the diff. cross-sections
  \[
  d\tilde{\sigma}(i \rightarrow f) = \hbar^2 (2\pi)^4 |\tilde{M}_{li}|^2 \frac{E_{i1} E_{i2}^3}{m E_{i2}} \left. \frac{\partial k}{\partial p} \right| d\Omega.
  \]

- Ratio of total cross-section relative to std. result
  \[
  \frac{d\tilde{\sigma}}{d\sigma} = \prod_n \frac{E_n}{\omega_n} \left. \frac{\partial k}{\partial p} \right|_{p_i = p_f}
  \]
Deformed Special Relativity

- Minimal length $L_{\text{min}}$ requires new Lorentz-transformations
- New transformations have 2 invariants: $c$ and $L_{\text{min}}$
- Generalized Uncertainty $\iff$ Deformed Special Relativity
  * When relation $k(p)$ is known and $p$'s (usual) transformation, then also the transformation of $k$ is known.
  * When the new transformation on $k$ is known, then one gets $k(p)$ by boosting in and out of the restframe where $k = p$.

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SH, Class. Quantum Grav. 23 (2006) 1815.
Deformed, Non-linear Action on Momentum Space

- Lorentz-algebra remains unmodified

\[ [J^i, K^j] = \epsilon^{ijk} K_k, \quad [K^i, K^j] = \epsilon^{ijk} K_k, \quad [J^i, J^j] = \epsilon^{ijk} J_k \]

- But it acts non-linearly on momentum space, e.g.:

\[ e^{-iL_{ab}\omega_{ab}} \rightarrow U^{-1}(p_0)e^{-iL_{ab}\omega_{ab}}U(p_0) \quad \text{with} \quad U(p_0) = e^{L_{\min}p_0p_a\partial_a} \]

- Leads to Lorentz-boost (z-direction)

\[
\begin{align*}
    p'_0 &= \frac{\gamma(p_0 - \nu p_z)}{1 + L_{\min}(\gamma - 1)p_0 - L_{\min}\gamma\nu p_z} \\
    p'_z &= \frac{\gamma(p_z - \nu p_0)}{1 + L_{\min}(\gamma - 1)p_0 - L_{\min}\gamma\nu p_z}
\end{align*}
\]

which transforms \((1/L_{\min}, 1/L_{\min}) \rightarrow (1/L_{\min}, 1/L_{\min})\)

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Interpretation of an Invariant Minimal Length

Besides $c$ there is a second invariant $L_{\text{min}}$ for all observers

- Standard DSR approach:
  * DSR applies for each observer to agree on minimal-ness ... ?
  * Therefore deformed transformation applies to free particles
  * If caused by quantum gravity effects what sets the scale?

- Here:
  * Two observers can not compare lengths without interaction
  * The strength of gravitational effects sets the scale for the importance of quantum gravity
  * Free particles do not experience any quantum gravity or DSR
  * Effects apply for particles in the interaction region only

→ Propagator of exchange particles is modified
Problems of DSR

The Soccer-Ball-Problem

* **Usual DSR:** A particle has an upper bound on its energy. There better were no such bound for composite objects. How to get the correct multi-particle limit for DSR-trafos?

* **Here:** Composite objects don’t experience anything funny as long as the gravitational interaction among the components is weak.

The Conservation-Law Problem

* **Usual DSR:** the physical momentum \( p \) is the one that experiences DSR. But then \( p \) is not additive and \( f(p_1 + p_2) \neq f(p_1) + f(p_2) \). Which quantity is to be conserved in multi-particle interactions?*

* **Here:** the physical momenta \( p_1, p_2 \) are the asymptotic momenta, they transform, are added, and are conserved in the usual way‡.


Observables of a Minimal Length - High Precision

- Hydrogen atom: shift of energy levels

- $g$-2
  U. Harbach, SH, M. Bleicher and H. Stöcker

- Varying speed of light – energy dependent time of flight
  Amelino-Camelia, Phys. Rev. D 64 (2001) 036005
Advertisement Break
Large Extra Dimensions

- More volume: the volume factor makes gravity appear weak at large distances $\gg$ radius
- True strength: at small distances ($\sim 1$TeV) gravity becomes stronger
- 2 in 1: graviton- and black hole production at the LHC becomes possible*

* Restrictions apply
Evaporation proceeds in 3 stages:

1. Balding phase: hair loss – the black holes radiates off angular momentum and multipole moments
2. Hawking phase: thermal radiation into all particles of the standard model as well as gravitons
3. Final decay (remaining black hole relic?)

Black hole thermodynamics: \( T = \frac{\kappa}{2\pi} \sim 200 \text{ GeV} \)

Numerical investigation: black hole event generator CHARYBDIS
Big Bang Machine: Will it destroy Earth?

The London Times July 18, 1999

Creation of a black hole on Long Island?

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John Nelson, professor of nuclear physics at Birmingham University who is leading the British scientific team at RHIC, said the chances of an accident were infinitesimally small - but Brookhaven had a duty to assess them. "The big question is whether the planet will disappear in the twinkling of an eye. It is astonishingly unlikely that there is any risk - but I could not prove it," he said.
Angst vor dem großen Knall
Physiker wollen bei New York den Anfang des Universums erforschen und lösen Endzeitstimmung aus


So schick ein Schicksal steht vielleicht der Erde bevor, fürchten jetzt viele Amerikaner, wenn ein neuer Teilchenbeschleuniger bei New York ab Herbst.


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back to the main feature
Observables of a Minimal Length - High Energy

- Suppression of cross-section from $\sqrt{s} \sim M_f$ on.
  

- Black hole production becomes more difficult
  

- Cosmic ray puzzle? (modified threshold)
Outlook

- Thermodynamics with minimal length
  * Finite minimal density?
  * Modified equation of state?
- MDR from mean field approximation
  * Integrating out the graviton soup
  * Non-local effects compatible with stringy limits?
Detour: Thermodynamics*

- Quantum Field Theory at finite temperature:
  Compactify timelike coordinate to radius $R = \frac{1}{2\pi} T$.
- Stringy T-duality:
  $R$ has a dual radius $R \rightarrow \frac{R^2}{R}$
- Together: $T$ has a dual temperature

$$T \rightarrow \frac{T_d^2}{T}$$

Detour: Thermodynamics

BUT
Detour: Thermoduality*

Standard Thermodynamics is not invariant under this duality

- Even if some quantities (free energy) exhibit symmetries, others (entropy, specific heat) don’t.
- Reason is that $d/dT$ does not respect the symmetry.
- Thermoduality requires a "covariant" derivative $D_T$ which leads to modifications.
- Low $T \ll T_d$ limit reproduces the familiar thermodynamics

→ Early universe? Equation of state?
Summary

- Effective theory that allows to examine phenomenology
- Related to DSR but different interpretation of free particles
- Input $p(k)$, rspt. $\sqrt{g(p)}$, should eventually be provided by the underlying theory
- With more investigation, it should be possible to further classify and constrain the model