Title: Dealing with derivative interactions

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Abstract: Single field inflation with derivative interactions provides a class of scenarios with interesting theoretical and observational properties. I will discuss properties of correlation functions in generic single field models and the implications of those relationships for inflationary observables, as well as for eternal inflation.
DEALING WITH DERIVATIVES

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arXiv:0802.2290 (L. Leblond, S.S.)
arXiv:0812.0818 (S.S.)
work in progress...
THE PLAN

I. Generic single field inflation: the action

* What is relevant for observation?

II. Small sound speed and the Fokker-Planck equation (eternal inflation)
SINGLE FIELD APPROACHES

- Fluid (k-inflation; first derivatives) (Mukhanov and Co.)
- Effective theory of fluctuations (Cheung et al)
- Effective theory of the inflaton (Weinberg, 4-derivatives)
- Special cases: DBI inflation (Silverstein, Tong)
SINGLE FIELD APPROACHES

- Fluid (k-inflation; first derivatives) *(Mukhanov and Co.)*
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- Effective theory of the inflaton *(Weinberg, 4-derivatives)*
- Special cases: DBI inflation *(Silverstein, Tong)*

Extend these
WHY WE CARE

- Theoretical: Fundamental description of inflation
- Observational: Non-Gaussianity will distinguish between different inflationary physics
SINGLE FIELD FLUID

- Single field models with first-derivative interactions

\[ S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} M_p^2 R + P(X, \phi) \right] \]
\[ X = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \]

Armendariz-Picon, Damour, Mukhanov;
Garriga, Mukhanov;
Alishahiha, Silverstein, Tong (DBI); Chen;

Shandera: PI, May '09
SOUND SPEED

- Importance of the sum of kinetic terms = small sound speed:

\[ c_s^2 = \frac{P_{,X}}{P_{,X} + 2XP_{,XX}} \]

- Scalar fluctuations see sound horizon

\[ c_s H^{-1} \]
(FIRST) DERIVATIVE INTERACTIONS

\[ P(X, \phi) = X - V(\phi) + \frac{1}{2} \frac{X^2}{M^4} + \frac{1}{2} \frac{X^3}{M^8} + \ldots \]

\[ X = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \]

(Creminelli)

\[ M = \text{scale of new physics} \]

\[ M \ll M_p \Rightarrow \text{Terms may be important} \]

Brane Action: String theory (symmetries) sums the terms...

\[ P(X, \phi) = -M^4 \sqrt{1 - \frac{2X}{M^4}} - V(\phi) + M^4 \]
MOST GENERAL SCALAR FIELD

- Expand in number of derivatives
- Write down all (unique) terms;
- Use Eq. of Motion
- Treat as perturbations
AT 4 DERIVATIVES

- Three (unique of 7) possible terms (+ grav.):
  \[ X^2, \ X\Box \phi, \ (\Box \phi)^2 \]

- Eq. of motion eliminates \( \Box \phi \) in terms of the potential

- Only remaining term:
  \[ f_1(\phi) \frac{X^2}{M^4} \]
CONTINUE THIS PROCEDURE

- **Goals:**
  - Perturbative conditions for generic actions
  - Uncover structure of correlation functions
AT 4 DERIVATIVES

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CONTINUE THIS PROCEDURE

- Goals:
  - Perturbative conditions for generic actions
  - Uncover structure of correlation functions
AT SIX DERIVATIVES

- Two (unique of 23) terms (scalar field only)
  \[ X^3, \ g^{\mu\nu} g^{\alpha\beta} (\partial_\mu \partial_\alpha \phi)(\partial_\nu \partial_\beta \phi) X \equiv Y \]

- Gravity / mixed terms suppressed by Planck scale

- Alternative approach: Extrinsic curvature terms
  (Cheung et al)
  \[ K^\mu_\mu \propto a^{-2} \partial_i^2 (\delta \phi) \]
FOR FLUCTUATIONS...

- Taylor expand: \[ S = S_0 + S_2 + S_3 + \ldots \]
  \[ S_0 \approx V(\phi_0) \]

- Substitute:
  \[ \zeta \rightarrow \langle \zeta^2 \rangle^{1/2} = \frac{H}{2\pi M_p \sqrt{2\epsilon c_s}} \]
  \[ \zeta = -\frac{H}{\dot{\phi}} \delta\phi \]

- Two comparisons:
  - Gradient energy small?
    \[ \frac{S_2}{S_0} < 1 \]
  - Interactions perturbative?
    \[ S_2 > S_n, \ n \geq 3 \]
    \[ \frac{S_3}{S_2} < 1 \]
6 DERIVATIVE TERMS IN THE THREE-POINT

\[
H_I^{(3)} \propto \int dt \ a^3 (\delta \phi)^3 \frac{\dot{\phi}_0}{M^4} \frac{H^3}{c_s^2} \left[ f_1 + f_2 \frac{H^2}{M^2 c_s^2} + f_3 \frac{X}{M^4} \right]
\]

\[
\begin{align*}
&f_1 \frac{X^2}{M^4} \\
&f_2 \frac{Y}{M^6} \\
&f_3 \frac{X^3}{M^8}
\end{align*}
\]
6 DERIVATIVE TERMS IN THE THREE-POINT

\[ H_I^{(3)} \propto \int dt \ a^3 (\delta \phi)^3 \ \frac{\dot{\phi}_0}{M^4} \ \frac{H^3}{c_s^2} \ \left[ f_1 + f_2 \ \frac{H^2}{M^2 c_s^2} + f_3 \ \frac{X}{M^4} \right] \]

\[ X = \frac{(\dot{\phi}_0)^2}{2} \]

\[ f_1 \ \frac{X^2}{M^4}, \quad f_2 \ \frac{Y}{M^6}, \quad f_3 \ \frac{X^3}{M^8} \]
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* Note: \[ X = \frac{(\phi_0)^2}{2} \]

Check loop corrections:

- \[ f_1 \frac{X^2}{M^4} \]
- \[ f_2 \frac{Y}{M^6} \]
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6 DERIVATIVE TERMS IN THE THREE-POINT

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\[ \begin{array}{c}
\text{Note: } X = \frac{(\phi_0)^2}{2} \\
f_1 \ \frac{X^2}{M^4} \\
f_2 \ \frac{Y}{M^6} \\
f_3 \ \frac{X^3}{M^8}
\end{array} \]

Check loop corrections:

In practice:

\[ \frac{S_3}{S_2} < 1 \]
PERTURBATIVE?

- Original terms in the action: \( \frac{X}{M^4} < 1 \)
- Slow-roll:
- First derivatives only:
- Higher derivatives:
PERTURBATIVE?

- Original terms in the action: \( \frac{X}{M^4} < 1 \)
- Slow-roll: \( \frac{H}{M_p} < 1 \)
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- Higher derivatives:
PERTURBATIVE?

- Original terms in the action: $\frac{X}{M^4} < 1$
- Slow-roll: $\frac{H}{M_p} < 1$
- First derivatives only: $P_\zeta < c_s^4$
- Higher derivatives: $\frac{H^2}{M^2 c_s^2} < 1$
PERTURBATIVE?

- Original terms in the action: \( \frac{X}{M^4} < 1 \)
- Slow-roll: \( \frac{H}{M_p} < 1 \)
- First derivatives only: \( P_\zeta < c_s^4 \)
- Higher derivatives: \( \frac{H^2}{M^2 c_s^2} < 1 \) New?

No:

\[ \frac{X}{M^4} < 1, \quad P_\zeta < c_s^4 \Rightarrow \frac{H}{M} < c_s \]
CONSEQUENCE...

- Eternal inflation is outside the perturbative regime:

\[
\frac{(\Delta \phi)_q}{(\Delta \phi)_c} = P_\zeta^{1/2} \sim 1
\]

\[
P_\zeta < c_s^4
\]

(Leblond, Shandera; Creminelli et al)
STRUCTURE OF CORRELATION FUNCTIONS

- Hierarchical structure:
  \[ \langle \zeta^n \rangle \propto \langle \zeta^2 \rangle^{(n-1)} \]

- Generated by non-linearities from gravity, local ansatz

- Multi-field, cosmic strings, etc don’t need to obey this...
STRUCTURE OF CORRELATION FUNCTIONS

- Calculating correlations

\[
\langle \delta \phi(k_1, t) \ldots \delta \phi(k_n, t) \rangle = i \int_{-\infty}^{t} dt' \langle [H_I^{(n)}(t'), \delta \phi(k_1, t) \ldots \delta \phi(k_n, t)] \rangle
\]

\[
H_I^{(n)} \sim A_n \int d^3x \, a^3 \mathcal{L}_2 \left( \frac{P_{\zeta}^{1/2}}{c_s^2} \right)^{n-2}
\]

- Estimate integral:

\[
(\Delta x)^3 \Delta t \sim (c_s/aH)^3 H^{-1}
\]

(S.S.)
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\]

- Estimate integral:

\[
(\Delta x)^3 \Delta t \sim (c_s/aH)^3 H^{-1}
\]
STRUCTURE, CONT’D

- Hierarchical, up to sound speed
  \[ \langle \zeta^n \rangle \propto \frac{\langle \zeta^2 \rangle^{n-1}}{(c_s^2)^{n-2}} \]

- Dimensionless combination (Expand Prob. Dist.)
  \[ \frac{\langle \zeta^n \rangle}{\langle \zeta^2 \rangle^{(n-1)/2}} \frac{\langle \zeta^2 \rangle^{(n-2)/2}}{2} \propto A_n \left( \frac{P_{\zeta}^{1/2}}{c_s^2} \right)^{n-2} \]

- Perturbative condition
  \[ c_s^4 > \frac{H^2}{M_p^2 \epsilon c_s} \sim P_{\zeta} \]

(S.S.)

Shandera: PI, May ’09
SUMMARIZE:

- While $\frac{X}{M^4} < 1$

1. First derivatives dominate

2. Correlation functions are hierarchical

- When $P_\zeta \sim c_s^4$

1. Higher derivatives as important as first derivatives
SUMMARIZE:

- While $\frac{H}{M_p} < 1$ and $\frac{X}{M^4} < 1$

1. First derivatives dominate
2. Correlation functions are hierarchical

- When $P_\zeta \sim c_s^4$

1. Higher derivatives as important as first derivatives
BRANE INFLATION SUGGESTS:

1. Sound speed can be very small

2. Warped geometry: $M$ decreases as inflation proceeds

\[ M^4 = (m_s^w)^4 = m_s^4 \left( \frac{r}{R} \right)^4 \]

\[
\frac{\dot{c}_s}{c_s H} \rightarrow \text{constant}
\]

\[
(c_s/c_{s,\text{max}})^{-2}
\]

\[
\begin{array}{c}
0.2 \\
0.4 \\
0.6 \\
0.8 \\
1.0
\end{array}
\begin{array}{c}
200 \\
400 \\
600 \\
800 \\
1000 \\
1200
\end{array}
\]
BRANE ACTION

- Lowest order:

\[
S = - \int d^4x \, a^3 \left[ f(\phi) \sqrt{1 - 2X f^{-1}(\phi)} - f(\phi) + V(\phi) \right]
\]

\[
f(\phi) \equiv Sh^{-1}(\phi) \left( = T_3 h^{-1}(\phi) \right) = M^4(\phi)
\]

\[
d s_{10}^2 = h^{-1/2}(y) g_{\mu\nu} \, dx^\mu \, dx^\nu + h^{1/2}(y) g_{mn} \, dy^m \, dy^n
\]
CURVATURE TERMS?

- Extrinsic Curvature of the brane
  
  \[ K^\mu_{\mu} \approx \frac{\partial_i^2 (\delta \phi)}{a^2 M^2} < M \]

  \( \checkmark \) (Under control)

- Corrections to the action? Guess...

\[ S_{DBI,guess} = - \int d^4 x \ a^3 f(\phi) \sqrt{1 - 2X/M^4} \]

\[ \times \left[ 1 + \mathcal{F}^{mnl} (X/M^4) (\partial_m \partial_n \phi)(\partial_k \partial_l \phi) + \ldots \right] \]

Guess from: Andreev, Tseytlin: field strength corrections for bosonic case (SUSY case); T-dual
Ib. IMPLICATIONS FOR OBSERVATION
HOW NON-GAUSSIAN?

Slow roll

vs

Observation

vs

Theoretical Bound

\[ f_{NL} \propto -(n_s - 1) \approx 0.04 \]

\[ |f_{NL}^{\text{eff}}(k_{\text{CMB}})| < \mathcal{O}(100) \]

\[ |f_{NL}^{\text{eff}}| < 10^{9/2} \]
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COMPARING STATISTICS

Correlation functions

\[ \mathbf{k}_1 \quad \mathbf{k}_2 \quad \mathbf{k}_3 \]

Minkowski functionals
Cluster number counts

Anything else
III. FOKKER-PLANCK AND INTERACTIONS (ETERNAL INFLATION)
STOCHASTIC INFLATION

- Global picture
- Eternal inflation
- Useful for understanding IR-dependence in loop corrections?
- Interactions :: Non-Gaussianity :: deviations from exact dS

w/ Sash Sarangi; Tolley, Wyman
FOKKER-PLANCK

- Quadratic Fokker-Planck

\[
\frac{\partial f(\phi, t)}{\partial t} = \frac{1}{8\pi^2} \frac{\partial^2}{\partial \phi^2} [H^3 f(\phi, t)] + \frac{\partial}{\partial \phi} [2M_p^2 H' f(\phi, t)]
\]

- Probability conservation + locally peaked jump distribution (Master Equation)

\[
\frac{\partial f(\phi, t)}{\partial t} = \int_{-\infty}^{\infty} [ -f(\phi, t) W(\phi; \phi') + f(\phi', t) W(\phi'; \phi)] d\phi'
\]
FOKKER-PLANCK

- Quadratic Fokker-Planck

\[ \frac{\partial f(\phi, t)}{\partial t} = \frac{1}{8\pi^2} \frac{\partial^2}{\partial \phi^2} [H^3 f(\phi, t)] + \frac{\partial}{\partial \phi} [2M_p^2 H' f(\phi, t)] \]

**Diffusion term**

- Probability conservation + locally peaked jump distribution (Master Equation)

\[ \frac{\partial f(\phi, t)}{\partial t} = \int_{-\infty}^{\infty} [-f(\phi, t)W(\phi; \phi') + f(\phi', t)W(\phi'; \phi)] d\phi' \]
FOKKER-PLANCK

- Quadratic Fokker-Planck
  \[
  \frac{\partial f(\phi, t)}{\partial t} = \frac{1}{8\pi^2} \frac{\partial^2}{\partial \phi^2} [H^3 f(\phi, t)] + \frac{\partial}{\partial \phi} [2M_p^2 H' f(\phi, t)]
  \]

  - Diffusion term
  - Drift term

- Probability conservation + locally peaked jump distribution (Master Equation)
  \[
  \frac{\partial f(\phi, t)}{\partial t} = \int_{-\infty}^{\infty} [-f(\phi, t)W(\phi; \phi') + f(\phi', t)W(\phi'; \phi)]d\phi'
  \]
A STEP BACK...

- Taylor expand: \( \xi \equiv \phi - \phi' \)

\[
\frac{\partial f(\phi, t)}{\partial t} = -\frac{\partial}{\partial \phi} \left[ \mu_1(\phi)f(\phi, t) \right] + \frac{1}{2} \frac{\partial^2}{\partial \phi^2} \left[ \mu_2(\phi)f(\phi, t) \right] + \ldots
\]

- Jump Moments

\[
\mu_1(\phi) = \int_{-\infty}^{\infty} \xi W(\phi; \phi - \xi) \, d\xi = \lim_{\delta t \to 0} \frac{\delta \phi}{\delta t} = v_\phi
\]

\[
\mu_2(\phi) = \int_{-\infty}^{\infty} \xi^2 W(\phi; \phi - \xi) \, d\xi = \lim_{\delta t \to 0} \frac{(\delta \phi)^2}{\delta t}
\]
MASSLESS FIELD IN FIXED dS

- Modes:

$$\psi_k(\eta) = \frac{\sqrt{\pi}}{2} H^{3/2} H_{3/2}^{(1)}(k\eta)$$

- Shift:

$$\lim_{\delta t \to 0} \frac{\langle \phi^2 \rangle}{\delta t} = \lim_{\delta t \to 0} \frac{1}{\delta t} \int_H e^{H \delta t} \frac{d^3 k}{(2\pi)^3} |\psi_k|^2$$

reviewed by: A. Linde, hep-th/0503203
MASSLESS FIELD CONT’D

- Diffusion term only

\[
\frac{\partial f(\phi, t)}{\partial t} = \frac{H^3}{8\pi^2} \frac{\partial^2}{\partial \phi^2} [f(\phi, t)]
\]

- Gaussian distribution, variance grows with time

\[
\sigma^2 = \frac{H^3 t}{4\pi^2}
\]
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- Gaussian distribution, variance grows with time

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\]
MASSIVE FIELD IN FIXED dS

- Conditions: $m^2 \ll H^2$
  
  $V = V_0 + \frac{1}{2} m^2 \phi^2$
  
  $V_0 \gg \frac{1}{2} m^2 \phi^2$

- Mode function shifts $H_{3/2} \rightarrow H_\nu$
  
  $\nu \approx \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}$

- Integrating:
  
  $\langle \phi^2 \rangle = \frac{3H^4}{8\pi^2m^2} \left[ 1 - \exp \left( \frac{-2m^2}{3H}\delta t \right) \right]$
MASSIVE FIELD CONT’D

- Jump moments:

\[
\lim_{\delta t \to 0} \frac{\langle \phi^2 \rangle}{\delta t} = \frac{H^3}{4\pi^2}
\]

\[\dot{\phi}_0 = -\frac{V'}{3H}\]

- Fokker-Planck

\[
\frac{\partial f(\phi, t)}{\partial t} = \frac{H^3}{8\pi^2} \frac{\partial^2}{\partial \phi^2} [f(\phi, t)] + \frac{1}{3H} \frac{\partial}{\partial \phi} [V' f(\phi, t)]
\]
MASSIVE FIELD, CONT’D

- Some checks:

\[
\lim_{\delta t \to 0} \frac{\langle \phi_{cl}^2 \rangle}{\delta t} = \lim_{\delta t \to 0} \phi_0^2 \times \delta t
\]

\[
\lim_{\delta t \to 0} \frac{\langle \phi_{cl}^n \rangle}{\delta t} = \lim_{\delta t \to 0} \phi_0^n \times (\delta t)^{n-1}
\]

\[
\lim_{\delta t \to 0} \frac{\langle \phi_q^4 \rangle}{\delta t} = \lim_{\delta t \to 0} 3 \frac{\langle \phi_q^2 \rangle^2}{\delta t} = 0
\]
Some checks:

\[
\lim_{\delta t \to 0} \frac{\langle \phi_{cl}^2 \rangle}{\delta t} = \lim_{\delta t \to 0} \phi_0^2 \times \delta t
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\]

Quadratic FP √
MASSIVE FIELD, CONT’D

- Taylor expand, integrate by parts:

\[
\frac{d\langle \phi \rangle}{dt} = -\frac{m^2}{3H}\langle \phi \rangle
\]

\[
\langle \phi \rangle(t) = \phi(0)e^{-(m^2/3H)t}
\]
MASSIVE FIELD, CONT’D

- Taylor expand, integrate by parts:

\[
\frac{d\langle \phi \rangle}{dt} = -\frac{m^2}{3H} \langle \phi \rangle
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\]

Classical EoM ✓
MASSIVE FIELD, CONT’D

- Taylor expand, integrate by parts:

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\]

\[
\langle \phi \rangle(t) = \phi(0)e^{-\left(\frac{m^2}{3H}\right)t}
\]

Classical EoM

\[
\frac{d\langle \phi^2 \rangle}{dt} = \frac{2H^3}{8\pi^2} - \frac{2m^2}{3H}\langle \phi^2 \rangle
\]
MASSIVE FIELD, CONT’D

- Taylor expand, integrate by parts:

\[
\frac{d\langle\phi\rangle}{dt} = -\frac{m^2}{3H}\langle\phi\rangle
\]

\[
\langle\phi\rangle(t) = \phi(0)e^{-\frac{m^2}{3H}t}
\]

Classical EoM

\[
\frac{d\langle\phi^2\rangle}{dt} = \frac{2H^3}{8\pi^2} - \frac{2m^2}{3H}\langle\phi^2\rangle
\]

Stationary solution:

\[
\langle\phi^2\rangle = \frac{3H^4}{8\pi^2m^2}
\]
MASSIVE FIELD, CONT’D

- Check higher order terms...

\[ \frac{d\langle \phi^3 \rangle}{dt} = \frac{6H^3}{8\pi^2} \langle \phi \rangle - \frac{3m^2}{H} \langle \phi^3 \rangle \]

\[ \frac{d\langle \phi^4 \rangle}{dt} = \frac{3H^3}{2\pi^2} \langle \phi^2 \rangle - \frac{4m^2}{3H} \langle \phi^4 \rangle \]
MASSIVE FIELD, CONT’D

- Check higher order terms...

\[ \frac{d\langle \phi^3 \rangle}{dt} = \frac{6H^3}{8\pi^2} \langle \phi \rangle - \frac{3m^2}{H} \langle \phi^3 \rangle \]

\[ \langle \phi^3 \rangle \to 0, \quad Ht > \frac{3H^2}{m^2} \]

\[ \frac{d\langle \phi^4 \rangle}{dt} = \frac{3H^3}{2\pi^2} \langle \phi^2 \rangle - \frac{4m^2}{3H} \langle \phi^4 \rangle \]
MASSIVE FIELD, CONT’D

- Check higher order terms...

\[
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\]

\[
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\]

\[
\frac{d\langle \phi^4 \rangle}{dt} = \frac{3H^3}{2\pi^2} \langle \phi^2 \rangle - \frac{4m^2}{3H} \langle \phi^4 \rangle
\]

\[
\langle \phi^4 \rangle \to 3 \left( \frac{3H^4}{8\pi^2m^2} \right)^2
\]

\[\checkmark\]

Shandera: PL, May ’09
MASSIVE FIELD, CONT’D

- Check higher order terms...

\[
\frac{d\langle \phi^3 \rangle}{dt} = \frac{6H^3}{8\pi^2} \langle \phi \rangle - \frac{3m^2}{H} \langle \phi^3 \rangle
\]

\[
\frac{d\langle \phi^4 \rangle}{dt} = \frac{3H^3}{2\pi^2} \langle \phi^2 \rangle - \frac{4m^2}{3H} \langle \phi^4 \rangle
\]
MASSIVE FIELD, CONT’D

- Check higher order terms...

\[
\frac{d\langle \phi^3 \rangle}{dt} = \frac{6H^3}{8\pi^2} \langle \phi \rangle - \frac{3m^2}{H} \langle \phi^3 \rangle
\]

\[
\langle \phi^3 \rangle \rightarrow 0, \ Ht > \frac{3H^2}{m^2}
\]

\[
\frac{d\langle \phi^4 \rangle}{dt} = \frac{3H^3}{2\pi^2} \langle \phi^2 \rangle - \frac{4m^2}{3H} \langle \phi^4 \rangle
\]

\[
\langle \phi^4 \rangle \rightarrow 3 \left( \frac{3H^4}{8\pi^2m^2} \right)^2
\]
TIME SCALES

- $H$ constant: $H \delta t \ll \frac{1}{\epsilon} = \frac{3H^2}{m^2} \left[ \frac{V_0}{\frac{1}{2}m^2 \phi^2} \right]$ 

- Stationary solution reached: $H \delta t > \frac{3H^2}{2m^2}$
TIME SCALES, CONT’D

- If $H \neq \text{constant}$
  - Small kick to velocity
    \[
    \left( \frac{\delta \phi}{\dot{\phi}_0} \right)_q \ll 1 \quad \Rightarrow \quad H \delta t \ll \frac{8\pi^2 M_p^2 \epsilon}{H^2 \eta^2} \equiv \frac{1}{P_\zeta \eta^2}
    \]
  - Small kick to acceleration
    \[
    \frac{\delta (\dot{\phi}_0)_q}{\delta t} = \left( \frac{\delta \phi_q}{\delta t} \right) \left( \frac{\partial \dot{\phi}_0}{\partial \phi} \right) \ll \dot{\phi}_0 < H \dot{\phi}_0 \Rightarrow P_\zeta \eta^2 < H \delta t
    \]
TIME SCALES, CONT’D

- Use an interval of order $H^{-1}$

$$P_\zeta \eta^2 \ll \delta t H \ll \frac{1}{P_\zeta \eta^2}$$

- Quantum contributions dominant?

- Eternal inflation condition
WITH (CLASSICAL) GRAVITY: CHAOTIC INFLATION

- Allow $H$ to vary

\[ V = V_0 + \frac{1}{2} m^2 \phi^2 \quad \text{with} \quad V_0 \ll \frac{1}{2} m^2 \phi^2 \]

- Cannot assume the late time stationary solution:

\[ H \delta t \ll \frac{1}{\epsilon} \quad \quad \epsilon = \frac{2 M_p^2}{\phi^2} = \frac{m^2}{3 H^2} \]

- Fokker-Planck:

\[ \frac{\partial f(\phi, t)}{\partial t} = \frac{\partial^2}{\partial \phi^2} \left[ \frac{H^3}{8\pi^2} f(\phi, t) \right] + \frac{\partial}{\partial \phi} \left[ \frac{\sqrt{6} m M_p}{3} f(\phi, t) \right] \]
GENERICALLY NON-GAUSSIAN

\[ \frac{d\langle \phi \rangle}{dt} = -\frac{\sqrt{6}mM_p}{3} \]

Classical EoM

\[ \frac{d\langle \phi^2 \rangle}{dt} = \frac{m^3}{24\sqrt{6}\pi^2 M_p^3} \langle \phi^3 \rangle - 2\frac{\sqrt{6}mM_p}{3} \langle \phi \rangle \]
Figure 1. Evolution of the probability density in the chaotic inflationary scenario with a Gaussian initial probability centred at $\phi = 4M_p$ with a standard deviation of $0.4M_p$. Full curves, with quantum term; broken curves, classical evolution. 1 and A, $t = 45M_p^{-1}$; 2 and B, $t = 90M_p^{-1}$.

Bardeen, Bublik (1987)
GENERICALLY NON-GAUSSIAN

\[
\frac{d\langle \phi \rangle}{dt} = -\frac{\sqrt{6}mM_p}{3}
\]

Classical EoM \checkmark

\[
\frac{d\langle \phi^2 \rangle}{dt} = \frac{m^3}{24\sqrt{6}\pi^2 M_p^3} \langle \phi^3 \rangle - \frac{2\sqrt{6}mM_p}{3} \langle \phi \rangle
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Figure 1. Evolution of the probability density in the chaotic inflationary scenario with a Gaussian initial probability centred at $\phi = 4M_p$ with a standard deviation of $0.4M_p$. Full curves, with quantum term; broken curves, classical evolution. 1 and A, $t = 45M_p^{-1}$; 2 and B, $t = 90M_p^{-1}$.

Bardeen, Bublik (1987)
ADDING INTERACTIONS

1. Dynamical evolution of $H$

2. Higher terms in the Fokker-Planck equation

3. Quantum calculation of quantities to evolve
SLOW-ROLL

- A typical term in the (slow-roll) action

\[ \mathcal{L}_3 \supset \left( \frac{-\dot{\phi}_0}{4HM_p^2} \right) \delta \phi (\ddot{\delta} \phi)^2 \]

- Gives

\[ \frac{\langle (\delta \phi(x, t')^3 \rangle}{\delta t} \approx \frac{\sqrt{\epsilon} H^5}{M_p} \]
SLOW-ROLL, CONT’D

- Fokker-Planck is modified:

\[
\frac{\partial f(\phi, t)}{\partial t} = \frac{1}{8\pi^2} \frac{\partial^2}{\partial \phi^2} \left[ H^3 f(\phi, t) \right] + \frac{\partial}{\partial \phi} \left[ 2M_p^2 H' f(\phi, t) \right] - \frac{\partial^3}{\partial \phi^3} \left[ \frac{\sqrt{\epsilon} H^5}{16\pi^4 M_p} f(\phi, t) \right]
\]

- Require modifications are small:

\[
\frac{H}{M_p} < 1
\]
SLOW-ROLL, CONT’D

- Fokker-Planck is modified:

\[
\frac{\partial f(\phi, t)}{\partial t} = \frac{1}{8\pi^2} \frac{\partial^2}{\partial \phi^2} \left[ H^3 f(\phi, t) \right] + \frac{\partial}{\partial \phi} \left[ 2M_p^2 H' f(\phi, t) \right] \\
- \frac{\partial^3}{\partial \phi^3} \left[ \frac{\sqrt{\epsilon} H^5}{16\pi^4 M_p} f(\phi, t) \right]
\]

- Require modifications are small:

\[
\frac{H}{M_p} < 1 \quad \frac{S_2}{S_0} < 1
\]

(S.S.)
SMALL SOUND SPEED

- When $\epsilon < 1 - c_s^2$

- Dominant terms are modified by powers of sound speed:
  \[ \mathcal{L}_3 \supset \frac{\delta \phi (\dot{\delta \phi})^2}{2M_p \sqrt{2\epsilon c_s c_s^3}} \]

- Which gives:
  \[ \frac{\langle (\delta \phi)^3 \rangle}{\delta t} \propto \frac{H^5}{M_p \sqrt{2\epsilon c_s c_s^3}} \]
SMALL SOUND SPEED

- Keep corrections to quadratic Fokker-Planck small:

\[
\frac{H^2}{M_p^2} < c_s^3
\]
SMALL SOUND SPEED

- Keep corrections to quadratic Fokker-Planck small:

\[
\frac{H^2}{M_p^2} < c_s^3
\]

**Gradient energy bound** \( \checkmark \)

\[
\frac{S_2}{S_0} < 1
\]
SMALL SOUND SPEED

- Keep corrections to quadratic Fokker-Planck small:

\[
\frac{H^2}{M^2_P} < c_s^3
\]

Gradient energy bound \( S_2 < S_0 \) < 1

Fourth order gives same condition

(S.S.)
SMALL SOUND SPEED

- Keep corrections to quadratic Fokker-Planck small:

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\frac{H^2}{M_p^2} < c_s^3
\]
SMALL SOUND SPEED

- Keep corrections to quadratic Fokker-Planck small:

\[ \frac{H^2}{M_p^2} < c_s^3 \]

Gradient energy bound \( \checkmark \)

\[ \frac{S_2}{S_0} < 1 \]

Fourth order gives same condition

(S.S.)

Shandera: PL, May ’09
One-loop correction to the two point, small sound speed:

\[
(\delta \phi(x))^2 = \frac{H^2 (H \delta t)}{4\pi^2} \int_H^{H e^{H \delta t}} \frac{dp}{p} \left[ 1 + \frac{bH^2}{16\pi^2 M_p^2 \epsilon c_s^5} \log \left( 1 + \frac{\Lambda_{UV}}{p} \right) + \ldots \right]
\]
QUANTUM CORRECTIONS

- One-loop correction to the two point, small sound speed:

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\]
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\]

\[\Rightarrow P_\zeta < c_s^4\]

Perturbative bound

(S.S.)

Shandera: PL, May ’09
CAN WE GET AROUND THIS?

- Fokker-Planck can’t handle eternal inflation with small sound speed?
- Still useful in other contexts?
- What implications for tunneling/landscape?
QUANTUM CORRECTIONS

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QUANTUM CORRECTIONS

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\]

\[\Rightarrow P_\zeta < c_s^4 \quad \text{Perturbative bound}\]
CAN WE GET AROUND THIS?

- Fokker-Planck can’t handle eternal inflation with small sound speed?
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CONCLUSIONS I

- Non-Gaussianity: fundamental inflationary physics
- Need to understand the predictions of fundamental models
- For a *generic single-field*:
  - Sound speed perturbatively bounded
  - Structure of correlation functions is hierarchical
- Add modification to initial state? (Jan Pieter’s talk)
CONCLUSIONS II

- Stochastic inflation is sensitive to the gradient and perturbative bounds
- Corrections due to small sound speed cannot be calculated in the eternally inflating regime
- How to see the global picture for the small sound speed case?