Abstract: In a quantum-Bayesian delineation of quantum mechanics, the Born Rule cannot be interpreted as a rule for setting measurement-outcome probabilities from an objective quantum state. (A quantum system has potentially as many quantum states as there are agents considering it.) But what then is the role of the rule? In this paper, we argue that it should be seen as an empirical addition to Bayesian reasoning itself. Particularly, we show how to view the Born Rule as a normative rule in addition to usual Dutch-book coherence. It is a rule that takes into account how one should assign probabilities to the outcomes of various intended measurements on a physical system, but explicitly in terms of prior probabilities for and conditional probabilities consequent upon the imagined outcomes of a special counterfactual reference measurement. This interpretation is seen particularly clearly by representing quantum states in terms of probabilities for the outcomes of a fixed, fiducial symmetric informationally complete (SIC) measurement. We further explore the extent to which the general form of the new normative rule implies the full state-space structure of quantum mechanics. It seems to go some way.
My Favorite
Convex Set
(My Favorite Shape)

Christopher Fuchs
êî - Perimeter Inst.

Work with:

Marcus Appleby
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"Quantum-Bayesian Coherence"
the consequence
= an experience, \( E_k \)

the catalyst
= quantum system, \( \mathcal{H}_d \)

the action
= \( \{ E : \beta \} \), a POVM
Density Operators

\[ \rho \in L(H) \]

1. \( \rho^* = \rho \)
2. \( \text{tr} \rho = 1 \)
3. \( \lambda_i(\rho) > 0 \)
the consequence
= an experience, $E_k$

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= $\{ E : 3 \}$, a POVM

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convex hull of the set \( \{ |\psi\rangle\langle\psi| : |\psi\rangle \in \mathcal{H} \} \)

linear operators
complex vector space
catalog of uncertainties
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= an experience, $E_k$

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= $\{ E: 3 \}$, a POVM

the catalyst
= quantum system, $q_{\Psi_d}$
Density Operators

\[ \rho \in \mathcal{L}(\mathcal{H}) \]

- **Linear operators**
- **Complex vector space**
- **Catalog of uncertainties**

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Convex hull of the set \( \{ |\psi\rangle\langle\psi| : |\psi\rangle \in \mathcal{H} \} \)
Calculus 1 → Character 1
Calculus 2 → Character 2
Calculus 3 → Character 3
the consequence
= an experience, $E_k$

the catalyst
= quantum system,

the action
= $\{E: \mathcal{E}, a \text{ POVM}$
A satisfactory statement about the actual (objective) characteristics of the quantum world should contain no \( |\Psi\rangle \)'s at all.

Really. None!
The Born Rule

Given $\rho$ and $\{E_i\}$,

quantum state $\rightarrow$ POVM measurement

$$p(i) = \text{tr} \rho E_i$$

"The Born Rule"
The Born Rule

Given \( \rho \) and \( \{ E_i \} \),

\[ \rho(i) = \text{tr} \rho E_i \]

"The Born Rule"

**NOT** a law of nature.

**RATHER** something we should strive for.
THE TEN COMMANDMENTS

- Thou shalt not kill.
- Thou shalt not steal.
- Thou shalt not covet thy neighbor's wife.
- The firstling of an ass thou shalt redeem with a lamb.
Defining Probability

Dutch Book

Declaring $p_E$ means one will buy or sell a lottery ticket worth $\$1$ if $E$ for $\$p_E$. 
Dutch Book

Normative Rule:

Never declare $p_E$, $p_{E'}$, $p_{EvF}$, etc. that will lead to sure loss.

Example 1:

If $p_E < 0$, bookie will sell ticket for negative money. Sure loss!

Example 2:

If $p_E > 1$, bookie will buy ticket for more than it is worth in best case. Sure loss.
Example 3:
Suppose $E$ and $F$ mutually exclusive.

- Worth $\$1$ if $E \lor F$
- Worth $\$1$ if $E$
- Worth $\$1$ if $F$

So must have $P_{E \lor F} = P_E + P_F$.

Example 4:
Worth $\frac{m}{n}$ if $E$

$\text{Price? } \frac{m}{n} P_E \text{ of course.}$
Bayes Rule

Consider conditional lotteries:

If $E \land F$ give full price, but
if $\neg F$ return money.

Thus:

Worth $\#1$ if $E \land F$ ;
Worth $\# P_{E|F}$ if $\neg F$.

price $\# P_{E|F}$

But:

Worth $\#1$ if $E \land F$
Worth $\# P_{E|F}$ if $\neg F$

price $\# P_{E|F}$

price $\# P_{E|F} P_{F}$

recall example 4

So must have:

$P_{E|F} = P_{E|F} P + P_{E|F} P_{\neg F}$

$\Rightarrow \quad P_{E|F} = P_{F} P_{E|F}$
Example

One contemplates taking
\[ p(F) = 0.75 \]
\[ p(E \mid F) = 0.50 \]
\[ p(E \wedge F) = 0.70 . \]

One could gamble that way, but it wouldn't be too wise:
Not coherent.

Normative Rule:
Strive for coherence.
Bayes Rule

Consider conditional lotteries:
If $E \land F$ give full price, but
if $\overline{F}$ return money.

Thus:
- Worth $1$ if $E \land F$; Worth $p_{E|F}$ if $\overline{F}$.

But:
- Worth $1$ if $E \land F$; Price $p_{E|F}$
- Worth $p_{E|F}$ if $\overline{F}$

So must have:
$$p_{E|F} = p_{E|F} + p_{E|\overline{F}}$$

Recall example 4
Example

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If $E \wedge F$ give full price, but
if $\neg F$ return money.

Thus:
- Worth #1 if $E \wedge F$
- Worth $P_{E \wedge F}$ if $\neg F$

But:
- Worth #1 if $E \wedge F$
- Price $P_{E \wedge F}$
- Worth $P_{E \wedge F}$ if $\neg F$
- Price $P_{E \wedge F} \cdot P_F$

So must have:

$$P_{E \wedge F} = P_{E \wedge F} + P_{E \wedge F} \cdot P_F \Rightarrow P_{E \wedge F} = P_F P_{E \wedge F}$$
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Normative Rule:

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\[ p \leftrightarrow p(h) \]
Bureau of Standards

the "standard" quantum measurement
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$p(h)$
A Very Fundamental Mmt?

Caves, 1999
Zauner

Suppose $d^2$ projectors $\pi_i = |\psi_i\rangle\langle \psi_i|$, satisfying

$$\text{tr} \pi_i \pi_j = \frac{1}{d+i}, \quad i \neq j$$

exist.

Can prove:

1) the $\pi_i$ linearly independent

2) $\sum_i \frac{1}{d} \pi_i = I$

So good for Bureau of Standards.

Also

$$\rho(i) = \frac{1}{d} \text{tr} \rho \pi_i$$

$$\rho = \sum_i \left( (d+1)\rho(i) - \frac{1}{d} \right) \pi_i$$
Inequivalent SIC Sets

Let $d = 3$, $\omega = e^{\frac{2\pi i}{3}}$.

Set 1

\[
\begin{bmatrix}
0 & 0 & 0 \\
-1 & 1 & 1 \\
1 & -1 & 1 \\
0 & -1 & 0
\end{bmatrix}
\]

Set 2

\[
\begin{bmatrix}
-2 & -2 & -2 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{bmatrix}
\]
Evidence for Existence

Analytical Constructions
\[ d = 2 - 13, 15, 19 \]

Numerical \( (e \leq 10^{-38}) \)
\[ d = 2 - 47 \]

67
Probability Simplex

\[ \vec{p} = \begin{bmatrix} p(0) \\ p(1) \\ \vdots \\ p(n) \end{bmatrix} \]

\[ \sum_{h} p(h) = 1 \]

\[ p(h) \geq 0 \ \forall h \]
Remarkable Theorem

Jones & Linden, PRA II (2005)
Flammia, (unpub, 2004)

Let $A$ be Hermitian, $A^* = A$.

Then, $A = |\psi\rangle \langle \psi| \quad$ if and only if

$$\text{tr} \, A^* = \text{tr} \, A^3 = 1.$$
Pure States in SIC Language

Conditions

\[ \rho^+ = \rho, \quad \text{tr} \rho^2 = \text{tr} \rho^3 = 1 \]

translate to

\[ \sum_i \rho(i)^2 = \frac{2}{d(d+1)} \]

and

\[ \sum_{ijk} c_{ijk} \rho(j) \rho(k) \rho(l) = \frac{d+1}{(d+1)^3} \]

where

\[ c_{ijk} = \text{Re} \ \text{tr} \ \Pi_j \Pi_k \Pi_l \]

Could these be independently motivatable physical constants?
Laws of Probability

$H_i$ - various hypotheses one might have

$D_j$ - data values one might gather

\underline{Given}: \quad p(D_j|H_i) \quad \text{expectations for data given hypothesis}

\quad p(H_i) \quad \text{expectations for hypotheses themselves}

\underline{Question}: \quad \text{What expectations should one have for the } D_j?\quad$

\underline{Answer}: \quad P(D_j) = \sum_i p(H_i) p(D_j|H_i)$
**Pure States in SIC Language**

**Conditions**

\[ \rho^+ = \rho, \quad \text{tr} \rho^2 = \text{tr} \rho^3 = 1 \]

translate to

\[ \sum_i \rho(i)^2 = \frac{2}{d(d+1)} \]

and

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\( H_i \) - various hypotheses one might have

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**Given:**
- \( p(D_j | H_i) \) → expectations for data given hypothesis
- \( p(H_i) \) → expectations for hypotheses themselves

**Question:** What expectations should one have for the \( D_j \)?

**Answer:**
\[
P(D_j) = \sum_{i} p(H_i) p(D_j | H_i)
\]
Given $p(H_i)$

Any SIC

Given $p(D_j|H_i)$

Imaginary

But really going to do this.

What $p(D_j)$?

Any von Neumann measurement
\[ p(D_j) = (d+1) \sum_i p(H_i) p(D_j \mid H_i) - 1 \]

Quantum

(Usual) Bayesian

Magic!
Generalizations

When measurement on the ground is any other SIC:
\[ p(D_j) = (d+1) \sum_i p(H_i) p(D_j|H_i) - \frac{1}{d} \]
(Compare to unitary evolution.)

And

When measurement on the ground is a completely general POVM \( \{D_j\}_{j=1}^m \),
\[ p(D_j) = (d+1) \sum_i p(H_i) p(D_j|H_i) - \frac{1}{d} \sum_i p(D_j|H_i) \]
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Any SIC

Imaginary

But really going to do this.

Given $p(D_j|H_i)$

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When measurement on the ground is a completely general POVM \( \{D_j\} \), \( j=1, \ldots, m \),

\[ p(D_j) = (d+1) \sum_i p(H_i) p(D_j | H_i) - \frac{1}{d} \sum_i p(D_j | H_i) \]
Bayesian Perspective

No logical reason why situation with conditional lotteries should be commensurate with situation without conditional lotteries.

\[ p(D_j) \neq \sum_i p(H_i) p(D_j|H_i) \]

(Need better notation, though.)

Quantum Perspective

Nonetheless, there may be empirical reasons for adopting a relation. normative!

This is the content of the Born Rule.
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Quantum \rightarrow (Usual) Bayesian \rightarrow Magic!
Given $p(H_i)$

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But really going
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Quantum Perspective

Nonetheless, there may be empirical reasons for adopting a relation.

This is the content of the Born Rule.
But big question:

If one's probabilities do not satisfy this relation, what bad thing can happen in the single case?
Quantum Probability Theory

classical probability theory

Classical probability is "just" the commutative case.
\[ \overline{z} = f(\overline{p}, \{ \tilde{v}_i \}) \]

\[ \overline{p} \]

\[ p(i) \]

\[ i = 1, 2, \ldots, n \]

\[ r(j|i) \]

\[ j = 1, 2, \ldots, m \]

\[ \tilde{v}_j \]

\[ \text{fixed} \]

\[ \text{Ground variable} \]
\[ q(j) = (d+1) \sum_i p(i) r(j|i) - \frac{1}{d} \sum_i r(j|i) \]

\[ \bar{q} = f(\vec{p}, \{ \vec{v}_i \}) \]

\[ i = 1, 2, \ldots, m \]

\[ d^2 \]
Property of QM

Suppose initial $\rho = \frac{1}{d} I$, and actually follow path in sky arriving with click $j$ for POVM $\{G_j\}$ on ground.

Bayes' Rule gives us a posterior for which click $i$ occurred in sky:

$$\text{Prob}(i|j) = \frac{p(i)r_{ji}}{\sum_k p(k)r_{jk}} = \frac{\text{tr} G_j \Pi_i}{d \text{tr} G_j} = \frac{1}{d} \text{tr} \rho_j \Pi_i$$

But that is just the SIC representation of $\rho_j = \frac{G_j}{\text{tr} G_j}$.

ANY $\rho_j$ can be gotten this way by suitable choice of $\{G_j\}$. 
Reciprocity Axiom

- Starting from a state of maximal uncertainty for the sky, one can use the posterior supplied by Bayes' rule
  \[ \text{Prob}(i \mid j) = \frac{r(j \mid i)}{\sum_k r(j \mid k)} \]
as a valid prior. Moreover all valid priors can be generated in this way.

**Consequence:** Rewriting

\[ q(j) = \left( \sum_k r(j \mid k) \right) \left[ (d+1) \sum_i p(i) \text{Prob}(i \mid j) - \frac{1}{d} \right] \]

for any two valid priors \( p(i) \) and \( s(i) \),

\[ \sum_i s(i) = \sum_i p(i) s(i) \geq \frac{1}{d(d+1)} \, . \]
Basis States

Consider case where

\[ \text{ground} = \text{sky} \]

Consistency requires for any valid \( \bar{\rho} \),

\[ p(j) = (d+1) \sum_i p(i) r(j|i) - \frac{1}{d} \]

Consequently,

\[ r(j|i) = \frac{1}{d+1} \left( S_{ij} + \frac{1}{d} \right) \]

and by Reciprocity Axiom, all \( \bar{\rho} \) of the form

\[ \bar{e}_k = \left[ \frac{1}{d(d+1)}, \ldots, \frac{1}{d}, \ldots, \frac{1}{d(d+1)} \right] \]

must be valid priors.

Note: \( \bar{e}_k \cdot \bar{e}_k = \frac{2}{d(d+1)} \).
Homework containing the $\mathbf{e}_k$

Call a set $\mathcal{S} \subseteq \Delta_d$ within the probability simplex

a) **consistent** if for any $\vec{p}, \vec{q} \in \mathcal{S}$

$$\frac{1}{d(d+1)} \leq \vec{p} \cdot \vec{q} \leq \frac{2}{d(d+1)}$$

b) **maximal** if adding any further $\vec{p} \in \Delta_d$ makes it inconsistent

**Example**: If $\mathcal{S}$ is set of quantum states, it is consistent $\neq$ maximal.

**Problem**: Characterize all such $\mathcal{S}$; compare to quantum.
Examples

1) Take \( \vec{q} = \vec{p} \). Consequently must have
\[
\vec{p} \cdot \vec{p} \leq \frac{2}{d(d+1)}
\]
Same as quantum.

2) Consider a subset \( \{ \vec{p}_k \} \subseteq \mathbb{S} \) with \( k = 1, \ldots, m \) such that
\[
\vec{p}_k \cdot \vec{p}_k = \frac{2}{d(d+1)}
\]
\[
\vec{p}_k \cdot \vec{p}_l = \frac{1}{d(d+1)} \quad k \neq l.
\]
How large can \( m \) be?

Answer: \( d \), same as quantum
Property of QM

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$$\text{Prob}(i|j) = \frac{\rho(i) r(i|j)}{\Sigma_k \rho(k) r(j|k)} = \frac{\text{tr} G_i \Pi_i}{d \text{tr} G_j}$$

$$= \frac{1}{d} \text{tr} \rho_j \Pi_i$$

But that is just the SIC representation of

$$\rho_j = \frac{G_j}{\text{tr} G_j}$$

ANY $\rho_j$ can be gotten this way by suitable choice of $\{G_i\}$. 
Basis States

Consider case where

ground = sky.

Consistency requires for any valid $\overline{p}$,

$$p(j) = (d+1) \sum_i p(i) r(j|i) - \frac{1}{d}.$$ 

Consequently,

$$r(j|i) = \frac{1}{d+1} (s_{ij} + \frac{1}{d})$$

and by Reciprocity Axiom, all $\overline{p}$ of the form

$$\overline{e}_k = [\frac{1}{d(d+1)}, \ldots, \frac{1}{d}, \ldots, \frac{1}{d(d+1)}]$$

must be valid priors.

Note: $\overline{e}_k \cdot \overline{e}_k = \frac{2}{d(d+1)}.$
Homework

Call a set $x \subseteq \Delta_d$ within the probability simplex

a) **consistent** if for any $\bar{p}, \bar{q} \in x$

\[
\frac{1}{d(d+1)} \leq \bar{p} \cdot \bar{q} \leq \frac{2}{d(d+1)},
\]

b) **maximal** if adding any further $\bar{p} \in \Delta_d$ makes it inconsistent

**Example**: If $x$ is set of quantum states, it is consistent $\neq$ maximal.

**Problem**: Characterize all such $x$; compare to quantum.
Examples

1) Take $\vec{q} = \vec{p}$. Consequently must have
   $$\vec{p} \cdot \vec{p} \leq \frac{2}{d(d+1)}$$
   Same as quantum.

2) Consider a subset $\{\vec{p}_k\} \subseteq 8$ with $k = 1, \ldots, m$ such that
   $$\vec{p}_k \cdot \vec{p}_k = \frac{2}{d(d+1)}$$
   $$\vec{p}_k \cdot \vec{p}_l = \frac{1}{d(d+1)} \quad k \neq l.$$
   How large can $m$ be?

   Answer: $d$, same as quantum
Challenge

What further postulates must be made to recover precisely quantum state space?

I.e., the convex hull of

1) \( \sum_i p(i)^2 = \frac{2}{d(d+1)} \)

2) \( \sum_{ijk} C_{ijk} p(i)p(j)p(k) = \frac{d+7}{(d+1)^3} \)

with \( C_{ijk} \) possessing correct properties
Examples

1) Take \( \mathbf{\mathcal{Z}} = \mathbf{\mathcal{P}} \). Consequently, must have
\[
\mathbf{\mathcal{P}} \cdot \mathbf{\mathcal{P}} \leq \frac{2}{d(d+1)}
\]
Same as quantum.

2) Consider a subset \( \{ \mathbf{\mathcal{P}}_k \} \subseteq \mathcal{S} \)
with \( k = 1, \ldots, m \) such that
\[
\mathbf{\mathcal{P}}_k \cdot \mathbf{\mathcal{P}}_k = \frac{2}{d(d+1)}
\]
\[
\mathbf{\mathcal{P}}_k \cdot \mathbf{\mathcal{P}}_\ell = \frac{1}{d(d+1)} \quad k \neq \ell.
\]
How large can \( m \) be?

Answer: \( d \), same as quantum.
**Homework**

Call a set \( S \subseteq \Delta_d \) within the probability simplex

a) **consistent** if for any \( \vec{p}, \vec{q} \in S \)

\[
\frac{1}{d(d+1)} \leq \vec{p} \cdot \vec{q} \leq \frac{2}{d(d+1)}
\]

b) **maximal** if adding any further \( \vec{p} \in \Delta_d \) makes it inconsistent

**Example:** If \( S \) is set of quantum states, it is consistent \( \neq \) maximal.

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with \( C_{ijk} \) possessing correct properties
Think SIC thoughts!

... and maybe by way of it we'll come to understand quantum mechanics a little better.
the consequence
= an experience, \( E_k \)

the catalyst
= quantum system, \( \Psi_A \)

the action
= \( \{ E : \beta \} \), a POVM