The Energy-Momentum Tensor

Consider the 'action' of the matter fields $\psi_{\mu}$ etc.: $\mathcal{L} V g = "\text{Lagrangian density}"$

$$S[\psi, g] = \int_B \mathcal{L} V g d^4x = \int_B \mathcal{L} d^4x$$

Recall:

Requiring $\frac{\delta S}{\delta \psi_i} = 0$ yields the equation of motions for the field $\psi_i$: $\frac{\partial}{\partial \psi_i} \left( \frac{\partial}{\partial \psi_i} \right) c = 0$
Recall:

Requiring \( \frac{\delta S}{\delta \psi_i} = 0 \) yields the equation of motions for the field \( \psi_i \):

\[
\frac{\partial L}{\partial \psi_i} - \left( \frac{\partial L}{\partial (\partial \psi_i / \partial \tau)} \right) \partial \psi_i / \partial \tau = 0
\]

(These Euler-Lagrange equations are same for \( L \) as for \( S \) because \( V_\gamma \) drops out)

Note: Overall prefactor of \( L, L, S \) does not affect the classical equations of motion, i.e. is arbitrary.

But in quantum theory each \( \psi \) is assigned
Recall:

Requiring \( \frac{\delta S}{\delta \psi_i} = 0 \) yields the equation of motions for the field \( \psi_i : \)

\[
\frac{\partial}{\partial t} \left( \frac{\partial}{\partial \psi_i} \right) \psi_i - \left( \frac{\partial}{\partial \psi_i} \right) \text{Lagrange equations} = 0
\]

(These Euler-Lagrange equations are same for \( k \) as for \( l \) because \( k \)damped)

Note: Overall prefactor of \( L, L, S \) does not affect the classical equations of motion, i.e. is arbitrary.

But in quantum theory each \( \psi \) is assigned a probability amplitude

\[ \psi_{\psi} \]
Consider the 'action' of the matter fields $\psi_{(0, \ldots, b)}$:

$$S_\psi = \int \mathcal{L} \, d^4x = \int \mathcal{L} \, d^4x$$

Recall:

Requiring $\frac{\delta S}{\delta \psi_i} = 0$ yields the equation of motions for the field $\psi_i$:

$$\frac{\partial F}{\partial \psi_i} - \frac{\partial}{\partial x_j} \left( \frac{\partial F}{\partial (\frac{\partial \psi_i}{\partial x_j})} \right) = 0$$

Note: Overall prefactor of $L, S, S'$ does not affect the classical equations of motion, as $i.e.$ is arbitrary.
Consider the action of the matter fields 

\[ S[\psi, \varphi] = \int_B \mathcal{L} \psi \, d^4x = \int_B \mathcal{L} \varphi \, d^4x \]

\[ \mathcal{L} \psi = \text{"Lagrangian density"} \]

\[ \mathcal{L} \varphi = \text{"Lagrangian density"} \]

**Recall:**

Requiring \( \frac{\delta S}{\delta \psi_i} = 0 \) yields the equation of motions for the field \( \psi_i \):

\[ \frac{\partial^2 \psi_i}{\partial t^2} - \left( \frac{\partial}{\partial \psi_i} \right) \mathcal{L} = 0 \]

(These Euler-Lagrange equations are same for \( \varphi \) as for \( \psi \) because \( \psi \) drops out)

**Note:** Overall prefactor of \( L, L', S' \) does not affect the classical equations of motion, i.e., is a bit of...
Consider the 'action' of the matter fields $\psi_{i,j}^{a...b}$:

$$S[\psi, g] = \int_B L \sqrt{g} \, d^4x = \int_B L \, d^4x$$

Recall:

Requiring $\frac{\delta S}{\delta \psi} = 0$ yields the equation of motions for the field $\psi_i$:

$$\frac{\partial}{\partial x^\mu} \left( \frac{\partial}{\partial \psi_i^\mu} \left( \frac{\partial S}{\partial \psi_{i}^{\mu}} \right) \right) = 0$$

Note: Overall prefactor of $L, L, S'$ does not affect the classical equations of motion, i.e., is arbitrary.
Note: Overall prefactor of $L, L, S'$ does not affect the classical equations of motion, i.e., is arbitrary. But in quantum theory each $\psi$ is assigned a probability amplitude $iS'[\psi, g]/\hbar$; i.e., quantum theory fixes the prefactor.

Now: How does $S'[\psi, g]$ change as $g$, i.e., as the shape of the manifold is varied?

Def: $g_{\nu \alpha}(x, x)$ is smooth deformation of $g_{\nu \alpha}(x)$ for $x \in B'$. 

are same for $h$ as for $\delta$ because $T_2$ drops out.
Note: Overall prefactor of $L, L, S'$ does not affect the classical equations of motion, i.e., is arbitrary.

But in quantum theory each $q$ is assigned a probability amplitude

$$iS'[q, g]/\hbar$$

i.e., quantum theory fixes the prefactor.

Now: How does $S'[q, g]$ change as $g$, i.e., as the shape of the manifold is varied?

Def: $g_{\nu\mu}(x, \lambda)$ is smooth deformation of $g_{\nu\mu}(x)$ for $x \in B$ if:

a) $g_{\nu\mu}(\lambda = 0, x) = g_{\nu\mu}(x)$

b) $g_{\nu\mu}(\lambda, x) = g_{\nu\mu}(x)$ if $x \in M - B$
i.e. quantum theory fixes the prefactor.

**Now:** How does $S'[\Psi, g]$ change as $g$, i.e., as the shape of the manifold is varied?

**Def:** $g_{\nu\mu}(\lambda, x)$ is smooth deformation of $g_{\nu\mu}(x)$ for $x \in B$ if:

a) $g_{\nu\mu}(\lambda=0, x) = g_{\nu\mu}(x)$

b) $g_{\nu\mu}(\lambda, x) = g_{\nu\mu}(x)$ if $x \in M-B$

**Def:** We say that $S$ is functionally differentiable w.r.t. to $g_{\nu\mu}$ in $B$ if

$$\delta S := \frac{dS}{d\lambda} \bigg|_{\lambda=0}$$

exists for all smooth deformations and is of the form:
**Def:** We say that $S$ is functionally differentiable w. r. s. p. to $g_{\mu\nu}$ in $B$ if

$$S_4 \equiv \frac{dS}{dx_0} \bigg|_{x_0=0}$$

exists for all smooth deformations and is of the form:

$$\frac{dS}{dx_0} \bigg|_{x_0=0} = \frac{1}{2} \int_B \mathcal{F}^{\mu\nu}(x) g_{\mu\nu}(x) \, dx$$

where $\mathcal{F}^{\mu\nu}$ is symmetric: $g_{\mu\nu} = g_{\nu\mu}$

any antisymmetric part drops out. By definition, we choose $g_{\mu\nu}$ to be symmetric.

We call $\mathcal{F}^{\mu\nu}$ the energy-momentum tensor density.

**Def:** We write

$$\frac{\delta S}{\delta g_{\mu\nu}} = \frac{1}{2} \mathcal{F}^{\mu\nu}$$

one can prove it is a tensor density.

i.e.

$$\mathcal{T}^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}}$$

this is the energy-momentum tensor.
**Def:** We say that $S$ is functionally differentiable w. r. s. p. to $g_{\mu \nu}$ in $B$ if

$$\delta S := \frac{dS}{d\lambda} |_{\lambda=0}$$

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$$\frac{dS}{d\lambda} |_{\lambda=0} = \frac{1}{2} \int_B \mathcal{F}^{\mu \nu}(x) \delta g_{\mu \nu}(x) d^4x$$

$$\text{convers.}$$

so symmetric: $g_{\mu \nu} = g_{\nu \mu}$

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i.e.: $$\mathcal{F}^{\mu \nu} = \frac{2}{\sqrt{g}} \frac{\delta}{\delta g_{\mu \nu}} S$$

This is the energy-momentum tensor.
Examine for any invariances that we can use.

\[ \frac{dS}{dx} \bigg|_{x=0} = \frac{1}{2} \int \mathcal{J}^{\mu\nu}(x) \delta g_{\mu\nu}(x) \, d^4x \]

\[ \text{symmetric: } g_{\mu\nu} = g_{\nu\mu} \]

By definition, we choose \( \mathcal{J}^{\mu\nu} \) to be symmetric.

We call \( \mathcal{J}^{\mu\nu} \) the energy-momentum tensor density.

**Def:** We write \( \frac{dS}{g_{\mu\nu}} = \frac{1}{2} \mathcal{J}^{\mu\nu} \)

\[ \text{one can prove it is a tensor density} \]

\[ \frac{\delta S}{\delta g_{\mu\nu}} = \frac{2}{\sqrt{g}} \frac{\delta}{\delta g_{\mu\nu}} \]

\[ \text{i.e. : } \mathcal{J}^{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta}{\delta g_{\mu\nu}} S' \]

This is the energy-momentum tensor.

**Example:**

\[ \delta' := -\frac{1}{2} \int \left( \psi_{\mu} \psi_{\nu} + g^{\mu\nu} + 2V(\psi) \right) \sqrt{g} \, d^4x \]

\[ \psi \text{ is scalar, i.e. } \psi_{\mu} = \psi_{\nu} \]

\[ e.g. \ V(\psi) = \frac{m^2}{2} \psi^2 + \frac{\lambda}{4!} \psi^4 \]
Example:

\[ S' = -\frac{1}{2} \int \left( \nabla_a \psi \nabla_b \psi g^{ab} + 2V(\psi) \right) \sqrt{g} \, d^n x \]

Then:

\[ \frac{\partial S'}{\partial \lambda_{\alpha \beta}} = -\frac{1}{2} \int \left( \nabla_a \psi \nabla_b \left( \delta g^{ab} \right) \sqrt{g} + \nabla_a \psi \nabla_b \sqrt{g} \frac{\partial V}{\partial \psi} + 2V(\psi) \frac{\partial^2 V}{\partial \psi^2} \sqrt{g} \right) \, d^n x \]

\[ \left( \delta g_{\alpha \beta} \right) = \frac{\delta^2 S}{\delta \lambda_{\alpha \beta}} \bigg|_{\lambda_{\alpha \beta}} \]

Recall: \[ \frac{\partial V}{\partial \psi} = \frac{1}{2} g^{ab} \nabla_a \nabla_b \psi \]

i.e. \[ \delta g_{\alpha \beta} = -g_{\alpha \gamma} g^{\gamma \delta} \delta g_{\delta \beta} \]

We also notice: \[ S \left( g_{\alpha \beta} g^{\alpha \beta} \right) = 0 = g_{\alpha \beta} g^{\alpha \beta} + \left( \delta g_{\alpha \beta} \right) g^{\alpha \beta} \]

Thus:

\[ \frac{\partial S'}{\partial \lambda_{\alpha \beta}} = -\frac{1}{2} \int \left( \nabla_a \psi \nabla_b \left( \delta g^{ab} \sqrt{g} \right) + \nabla_a \psi \nabla_b \sqrt{g} \frac{\partial^2 V}{\partial \psi^2} \sqrt{g} \right) \, d^n x \]
Recall: \( \frac{\partial V}{\partial g_{ij}} = \frac{1}{2} g^{ij} V \)

We also notice: \( \delta \left( g_{ij} g^{bc} \right) = 0 = g_{ij} \delta g^{bc} + (\delta g_{ij}) g^{bc} \)

Thus:
\[
\frac{\partial S}{\partial \lambda_{ij}} = -\frac{1}{2} \int \left( \nabla_{\mu} \nabla_{\nu} \sqrt{g} \left( -g^{ab} \delta g_{ij} \right) + \nabla_{\mu} \nabla_{\nu} g^{ab} \frac{1}{2} g^{ij} \sqrt{g} \right) \delta g_{ij}
\]

\[
+ 2 \sqrt{g} \left( \frac{1}{2} \nabla_{\nu} \sqrt{g} \delta g_{ij} \right) d^{i}x
\]

\[
\Rightarrow \frac{\partial S'}{\partial g_{\mu\nu}} = \frac{1}{2} \bar{J}^{\mu\nu} \quad \text{with:}
\]
\[
\bar{J}^{\mu\nu} = \left( \psi_{\mu} \psi_{\nu} - \frac{1}{2} \psi_{\mu} \gamma^{\alpha} \psi_{\alpha} g^{\mu\nu} - \sqrt{g} \bar{V} (\phi) g^{\mu\nu} \right) \sqrt{g}
\]

i.e. the energy-momentum tensor reads:

\[ \text{"Klein Gordon"} \]
\[ \Rightarrow \frac{\delta S}{\delta g_{\mu\nu}} = \frac{1}{2} \mathcal{T}^{\mu\nu} \quad \text{with:} \]

\[ \mathcal{T}^{\mu\nu} = \left( \psi_i \psi^i - \frac{1}{2} \frac{g_{\mu\nu}}{g^{\rho\sigma}} \psi^\rho \psi^\sigma - V(\psi) g^{\mu\nu} \right) \Psi^2 \]

i.e. the energy-momentum tensor reads:

\[ T^{\mu\nu} = \psi_i \psi^i - \frac{1}{2} g_{\mu\nu} \left( \psi_i \psi^i + 2 V(\psi) \right) \]

\[ \text{Note: } T^{\mu\nu} \text{ is already symmetric, i.e. need not delete any anti-symmetric part.} \]

\[ \text{Exercise: Show that for the electromagnetic field:} \]

\[ T^{\mu\nu}_{\text{EM}} = \frac{1}{4\pi} \left( F_{\mu i} F_{\nu j} g^{ij} - \frac{1}{4} g_{\mu\nu} F_{ij} F^{ij} \right) \]
Recall: \( \frac{\partial \sqrt{g}}{\partial g_{ij}} = \frac{1}{2} g^{ij} \sqrt{g} \)

We also notice: \( S (g_{\alpha \beta} g^{\beta \gamma}) = 0 = g_{\alpha \beta} \delta g^{\beta \gamma}(S g_{\alpha \beta}) \)

Thus:
\[
\frac{\partial S}{\partial g_{i j}} = -\frac{1}{2} \left[ \nabla_i \left( \Psi \nabla_j \sqrt{g} \right) \right] \left( -g^{\alpha \beta} \delta g_{\alpha j} \right) + \frac{1}{2} \nabla_i \left( \Psi \nabla_k g^{\alpha \beta} \right) \sqrt{g} \delta g_{i \beta} \]

\[
+ 2 \sqrt{g} (\Psi \frac{1}{2} g^{ij} \frac{\partial \Psi}{\partial g_{ij}}) \delta g_{ij} \sqrt{g} \]

\[
\Rightarrow \frac{\partial S}{\partial g_{\mu \nu}} = \frac{1}{2} \Gamma^{\mu \nu} \quad \text{with:}
\]
\[
\Gamma_{\mu \nu}^{\sigma} = \frac{\partial g^{\sigma \rho}}{\partial g_{\mu \nu}}
\]
\[
\Gamma^{\mu \nu} = \left( \Psi \frac{\partial \Psi}{\partial g^{\mu \nu}} - \frac{1}{2} \Psi \frac{\partial g^{\mu \nu}}{\partial g_{\mu \nu}} - \sqrt{g} \frac{\partial}{\partial g_{\mu \nu}} \right) \sqrt{g}
\]

i.e. the energy-momentum tensor reads:

\[ \left( \nabla^2 \Psi \frac{\partial \Psi}{\partial g^{\mu \nu}} - \frac{1}{2} \nabla^2 \Psi \frac{\partial g^{\mu \nu}}{\partial g_{\mu \nu}} - \sqrt{g} \frac{\partial}{\partial g_{\mu \nu}} \right) \sqrt{g} \]
\[ + 2 \nabla(4) \frac{1}{2} \gamma^{\mu \nu} \delta g_{\mu \nu} \] \[ d^4x \]

\[ \Rightarrow \quad \frac{\delta S'}{\delta g_{\mu \nu}} = \frac{1}{2} T^{\mu \nu} \quad \text{with:} \]

\[ T^{\mu \nu} = \left( \psi^{\mu} \psi^{\nu} - \frac{1}{2} \gamma^{\mu \nu} \psi^2 g_{\mu \nu} - \nabla(4) g^{\mu \nu} \right) \gamma^5 \]

i.e. the energy-momentum tensor reads:

\[ T_{\mu \nu}^{\text{Klein Gordon}} = \psi_\mu \psi_\nu - \frac{1}{2} g_{\mu \nu} \left( \psi^2 + 2 \nabla(4) \right) \]

\[ \text{\underline{Note:}} \quad T_{\mu \nu} \text{ is already symmetric, i.e. need not delete any anti-symmetric part.} \]

\[ \text{\underline{Exercise:}} \quad \text{Show that for the electromagnetic field:} \]

\[ \text{\underline{Exercise:}} \quad \text{Show that for the electromagnetic field:} \]
\[ + 2 V(4) \frac{1}{2} g^{ij} \delta g_{ij} \right) \, dx^4 \]

\[ \Rightarrow \quad \frac{\delta S'}{\delta g_{\mu \nu}} = \frac{1}{2} \, \mathcal{T}^{\mu \nu} \quad \text{with:} \]

\[ \mathcal{T}^{\mu \nu} = \left( \Psi_{\mu} \Psi_{\nu} - \frac{1}{2} \, g_{\mu \nu} (\Psi_{\mu} \Psi_{\nu} + 2 \, V(4)) \right) \eta_{\nu} \]

i.e. the energy-momentum tensor reads:

\[ \mathcal{T}^{\mu \nu} = \Psi_{\mu} \Psi_{\nu} - \frac{1}{2} \, g_{\mu \nu} (\Psi_{\mu} \Psi_{\nu} + 2 \, V(4)) \]

\[ \text{Note: } \mathcal{T}^{\mu \nu} \text{ is already symmetric, i.e. need not delete any anti-symmetric part.} \]

\[ \text{Exercise: Show that for the electromagnetic field:} \]

\[ T^{\mu \nu}_{\text{em}} = \frac{1}{4 \pi} \left( F_{\mu j} F_{\nu i} g^{ij} - \frac{1}{4} g_{\mu \nu} F_{ij} F^{ij} \right) \]
Exercise: Show that for the electromagnetic field:

\[
T^\mu_\nu = \frac{1}{4\pi} \left( F_\mu \wedge F_\nu \right) + \frac{1}{4} g_{\mu \nu} \left( F_\alpha \wedge F^\alpha \right)
\]

Perfect fluid case:

(Traditional sense: thermodynamically reversible dynamics)

A perfect (classical) fluid has at every point a unique time-like flux direction vector \( \nu^\mu \), the flux is conserved, and the fluid is completely characterized by its local energy density \( \rho \) and pressure \( p \) (i.e., e.g., no shear, no viscosity).

If \( p = 0 \), call it perfect “dust.”

\[
T^\mu_\nu = (\rho + p) \nu^\mu \nu_\nu + p g^\mu_\nu
\]
Perfect fluid case:

(Traditional sense: thermodynamically reversible dynamics)

\[ v^\mu v_\mu = -1 \]

A perfect (classical) fluid has at every point a unique time-like flux direction vector \( v^\mu \), the flux is conserved, and the fluid is completely characterized by its local energy density \( \rho \) and pressure \( p \) (i.e., e.g., no shear, no viscosity).

\[ \text{Then:} \quad T^{\mu\nu} = (\rho + p) v_\mu v_\nu + p g_{\mu\nu} \]

\[ \text{Note:} \quad \text{Eqn. of motion is} \quad T^{\mu\nu}_{\; j\nu} = 0 \quad \text{and dust (} p = 0 \text{) moves on geodesics} \]

Terminology: (Hawking & Ellis) Any fluid with this \( T_{\mu\nu} \) is called perfect.
Perfect fluid case:

(Traditional sense: thermodynamically reversible dynamics)

\[ v^\mu v_\mu = -1 \]

A perfect (classical) fluid has at every point a unique time-like flux direction vector \( v^\mu \), the flux is conserved, and the fluid is completely characterized by its local energy density \( \rho \) and pressure \( p \) (i.e., e.g., no shear, no viscosity).

Then, \( v^\mu = (1,0,0,0) \), i.e., \( T_{\mu\nu} = (\rho p, 0, 0, 0) \).

If \( p = 0 \), call it perfect "dust."

\[ T_{\mu\nu}^{\text{PF}} = (\rho + p) v_\mu v_\nu + p g_{\mu\nu} \]

Note: Eqn. of motion is \( T^{\mu\nu}_{\;;\nu} = 0 \) and dust \((p=0)\) travels on geodesics

Terminology: (Hawking & Ellis) Any fluid with this \( T_{\mu\nu} \) is called perfect.
A perfect (classical) fluid has at every point a unique time-like flux direction vector $\mathbf{v}$, the flux is conserved, and the fluid is completely characterized by its local energy density $\rho$ and pressure $p$ (i.e., e.g., no shear, no viscosity).

Then:

$$T_{\mu\nu} = (\rho + p) v_\mu v_\nu + p g_{\mu\nu}$$

Note: Eqn. of motion is $T^{\mu\nu} ;\nu = 0$ and dust ($p=0$) travels on geodesics.

Terminology: (Hawking & Ellis) Any fluid with this $T_{\mu\nu}$ is called perfect.
A perfect (classical) fluid has at every point a unique time-like flux direction vector \( \nu^\mu \), the flux is conserved, and the fluid is completely characterized by its local energy density \( \rho \) and pressure \( p \) (i.e., e.g. no shear, no viscosity).

Then:

\[
T_{\mu \nu}^{p.e.} = (\rho + p) \nu_\mu \nu_\nu + p g_{\mu \nu}
\]

Note: Eqn. of motion is \( T_{\mu \nu}^{p.e.} = 0 \) and dust \((p=0)\) travels on geodesics.

Terminology: (Hawking & Ellis) Any fluid with this \( T_{\mu \nu} \) is called perfect.
A perfect (isentropic) perfect fluid has an energy flux
a unique time-like flux direction vector \( \nu \), the
flux is conserved, and the fluid is completely characterized
by its local energy density \( \rho \) and pressure \( p \) (i.e., e.g.,
no shear, no viscosity).

\[ T_{\mu\nu} = (\rho + p) \nu_{\mu} \nu_{\nu} + p g_{\mu\nu} \]

Then:

if \( p = 0 \), call it perfect "dust".

Note: Eqn. of motion is \( T^{\mu}_{\nu} \delta_{\mu\nu} = 0 \) and dust \((p=0)\) travels on geodesics

Terminology: (Hawking & Ellis) Any fluid with this \( T_{\mu\nu} \) is called perfect.

Definition:
The 'equation of state' of a perfect fluid.
by its local energy density $\rho$ and pressure $p$ (i.e., e.g., no shear, no viscosity).

\[ T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu} \]

\[ \text{Then:} \]

If $p = 0$, call it perfect "dust.

\[ \text{Note: Eqn. of motion is } T^{\mu\nu}_{\;\;\nu} = 0 \text{ and } \text{dust}(p=0) \text{ travels on geodesics} \]

\[ \text{Terminology: (Hawking & Ellis) Any fluid with this } T_{\mu\nu} \text{ is called perfect.} \]

\[ \text{Definition:} \]

The "equation of state" of a perfect fluid is the relation between its energy density $\rho$ and its pressure $p$. It depends on the fluid.
**Definition:**

The "equation of state" of a perfect fluid is the relation between its energy density, \( \rho \) and its pressure, \( p \). It depends on the fluid and so one can characterize the fluids by this parameter:

\[
w := \frac{p}{\rho}
\]

**Important later for Cosmology:**

The two tensors

\[
T^\mu_\nu = \delta^\mu_\nu \sigma^i_\nu - \frac{1}{2} g_{\mu\nu} \left( \sigma^a_\alpha \sigma^a_\alpha + 2 V(\sigma) \right)
\]

\[
T^\mu_\nu = (\rho + p) g_{\mu\nu} + p g_{\mu\nu}
\]
**Definition:**

The "equation of state" of a perfect fluid is the relation between its energy density, $\rho$ and its pressure, $p$. It depends on the fluid and so one can characterize the fluids by this parameter:

$$w := \frac{p}{\rho}$$

**Important later for Cosmology:**

The two tensors

$$T^\mu_\nu = g_{\mu\nu} \Psi_i \Psi_i - \frac{1}{2} g_{\mu\nu} (\Psi_i \Psi_i + 2 V(\Psi))$$

$$T^{\mu\nu}_{\text{EM}} = (\rho + p) g_{\mu\nu} + p g_{\mu\nu}$$

are of similar form (unlike e.g. $T^\mu_\nu$).
by its local energy density $\rho$ and pressure $p$ (i.e., e.g., no shear, no viscosity).

\[ T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu} \]

Then:

if $p = 0$, call it perfect "dust."

Note: Eqn. of motion is $T^{\mu}_{\phantom{\mu};\nu} = 0$ and dust ($p = 0$) travels on geodesics

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Definition:

The "equation of state" of a perfect fluid is the relation between its energy density, $\rho$ and its pressure, $p$. It depends on the fluid.
\[ \mu_{\nu} = (p + \rho) u_{\nu} u^0 + \rho g_{\mu\nu} \]

Note: Eqn. of motion is $T^{\mu\nu} u_{\nu} = 0$ and dust ($p=0$) travels on geodesics.

Terminology: (Hawking & Ellis) Any fluid with this $T_{\mu\nu}$ is called perfect.

Definition: The "equation of state" of a perfect fluid is the relation between its energy density, $\rho$, and its pressure, $p$. It depends on the fluid and so one can characterize the fluids by this parameter:

\[ w := \frac{p}{\rho} \]

Important later for cosmology:
**Note:** Eqn. of motion is $T^u_{\;j,v} = 0$ and dust ($\rho = 0$) travels on geodesics.

**Terminology:** (Hawking & Ellis) Any fluid with this $T_{\mu\nu}$ is called perfect.

**Definition:**

The "equation of state" of a perfect fluid is the relation between its energy density, $\rho$, and its pressure, $p$. It depends on the fluid and so one can characterize the fluids by this parameter:

$$w = \frac{p}{\rho}$$

**Important later for Cosmology:**

The two tensors apply to the inflation field.
**Terminology:** (Hawking & Ellis) Any fluid with this $T_{\mu\nu}$ is called perfect.

**Definition:**

The "equation of state" of a perfect fluid is the relation between its energy density, $\rho$, and its pressure, $p$. It depends on the fluid and so one can characterize the fluids by this parameter:

$$w = \frac{p}{\rho}$$

Important later for Cosmology:

The two tensors

$$T_{\mu\nu} = \eta_{\mu\nu} \eta_{ij} - \frac{1}{2} g_{\mu\nu} (\eta_{ij} \eta_{ij} + 2 V(\phi))$$

Applies to the inflation field.
is the relation between its energy density, $\rho$ and its pressure, $p$. It depends on the fluid and so one can characterize the fluids by this parameter:

$$w := \frac{p}{\rho}$$

**Important later for cosmology:**

The two tensors

$$T_{\mu\nu}^{\text{KG}} = \psi_{,\mu} \psi_{,\nu} - \frac{1}{2} g_{\mu\nu} (\psi_{,\alpha} \psi_{,\alpha} + 2 V(\psi))$$

$$T_{\mu\nu}^{\text{TF}} = (\rho + p) \nabla_\mu \nabla_\nu + p g_{\mu\nu}$$

are of similar form (unlike e.g. $T^{\text{EM}}_{\mu\nu}$):

$\psi$ Assume $\psi$ is nearly homogeneous, i.e. $\psi_{,i} \equiv 0$. 

apply to the inflation field.
Important later for cosmology:

The two tensors

\[ T_{\mu\nu} = \psi_{,\mu} \psi_{,\nu} - \frac{1}{2} g_{\mu\nu} (\psi_{,a} \psi_{,a} + 2 V(\psi)) \]

\[ T_{\mu\nu}^{\text{eff}} = (\rho + p) \psi_{,\mu} \psi_{,\nu} + p g_{\mu\nu} \]

are of similar form (unlike e.g. \( T_{\mu\nu}^{\text{em}} \)):

- Assume \( \psi \) is nearly homogeneous, i.e. \( \psi_{,i} \approx 0 \).
- Identify:
  \[ \psi_{,i} \approx \frac{\psi_{,i}}{\sqrt{g_{ab} \psi_{,a} \psi_{,b}}} \]
  (so that \( \psi_{,\mu} \psi^{,\mu} = -1 \))
Important later for cosmology:

The two tensors

\[ T_{\mu\nu} = \Psi_{;\mu} \Psi_{;\nu} - \frac{1}{2} g_{\mu\nu} \left( \Psi_{;a} \Psi_{;a} + 2 V(\Psi) \right) \]

\[ T_{\mu\nu}^{p,T} = (p + p) \Psi_{;\mu} \Psi_{;\nu} + p g_{\mu\nu} \]

are of similar form (unlike e.g. \( T_{\mu\nu}^{\Theta \Theta} \)):

1. Assume \( \Psi \) is nearly homogeneous, i.e. \( \Psi_{;i} \approx 0 \).

2. Identify:

\[ \Psi_{;\mu} = \sqrt{g_{ab} \Psi_{;a} \Psi_{;b}} \]

(i.e.:

\[ T_{\mu\nu} = \left[ g_{ab} \Psi_{;a} \Psi_{;b} \right] \Psi_{;\mu} \Psi_{;\nu} + g_{\mu\nu} \left( -\frac{1}{2} \Psi_{;a} \Psi_{;a} - V(\Psi) \right) \]

\[ \approx + \Psi^2 \Psi_{;\mu} \Psi_{;\nu} + g_{\mu\nu} \left( \frac{1}{2} \Psi^2 - V(\Psi) \right) \]
The two tensors

\[ T^{\kappa \gamma} _{\mu \nu} = \Psi_{j \mu} \Psi_{j \nu} - \frac{1}{2} g_{\mu \nu} \left( \Psi_{i \alpha} \Psi_{i \alpha} + 2 V(\Psi) \right) \]

\[ T^{\rho \tau} _{\mu \nu} = (\rho + \phi) \nu _{\rho} \nu _{\tau} + p g_{\mu \nu} \]

are of similar form (unlike e.g. \( T^{\kappa \gamma} _{\mu \nu} \)).

Assume \( \Psi \) is nearly homogeneous, i.e. \( \Psi_{i \delta} \approx 0 \).

Identify:

\[ \nu _{\rho} = \frac{\Psi_{i \rho}}{\sqrt{g_{ab} \Psi_{i \alpha} \Psi_{i \beta}}} \]

so that \( \nu _{\rho} \nu ^{\rho} = -1 \)

i.e.:

\[ T^{\kappa \gamma} _{\mu \nu} = |g_{ab} \Psi_{i \alpha} \Psi_{i \beta}| \nu _{\rho} \nu _{\tau} + g_{\mu \nu} \left( -\frac{1}{2} \Psi_{i \alpha} \Psi_{i \alpha} - V(\Psi) \right) \]

\[ \approx + \psi ^{2} \nu _{\rho} \nu _{\tau} + g_{\mu \nu} \left( \frac{1}{2} \psi ^{2} - V(\Psi) \right) \]

Compare with \( T^{\rho \tau} _{\mu \nu} \):
Assume $\psi$ is nearly homogeneous, i.e. $\psi, \psi_i \approx 0$.

\[ \nu_{\mu} = \frac{\psi_{\mu}}{\sqrt{g_{ab} \psi_a \psi_b}} \quad \text{(so that } \nu_{\mu} \nu^\mu = -1) \]

i.e.:

\[ T_{\mu \nu} = g_{ab} \psi_a \psi_b \nu_{\mu} \nu_{\nu} + g_{\mu \nu} \left( -\frac{1}{2} \psi^2 - V(\psi) + \frac{1}{2} \psi^2 - V(\psi) \right) \]

\[ \approx \psi^2 \nu_{\mu} \nu_{\nu} + g_{\mu \nu} \left( \frac{1}{2} \psi^2 - V(\psi) \right) \]

Compare with $T^{\mu \nu}$:

\[ \frac{\nu^{\mu} \nu_{\mu}}{p} = \frac{1}{w} + 1 = \frac{\psi^2}{\frac{1}{2} \psi^2 - V(\psi)} \]

\[ \Rightarrow \quad \frac{1}{w} = \frac{\psi^2}{\frac{1}{2} \psi^2 - V(\psi)} - \frac{\psi^2 - V(\psi)}{\frac{1}{2} \psi^2 - V(\psi)} = \frac{\psi^2 + V(\psi)}{\psi^2 - V(\psi)} \]

Thus:

\[ w = \frac{\psi^{\frac{3}{2}} - V(\psi)}{\psi^{\frac{3}{2}} + V(\psi)} \in (-1, 1) \]

\[ \text{potential dominated, i.e. } V(\psi) \gg \psi^2 \text{ (see inflation later)} \]

\[ \text{no potential: } V(\psi) = 0 \]
The two tensors

\[ T_{\mu \nu}^{\text{KG}} = \Psi_{;\mu} \Psi_{;\nu} - \frac{1}{2} g_{\mu \nu} (\Psi_{;a} \Psi_{;a} + 2 V(\Psi)) \]

\[ T_{\mu \nu}^{\text{PF}} = (\rho + p) v_\mu v_\nu + p g_{\mu \nu} \]

are of similar form (unlike e.g. \( T_{\mu \nu}^{E_m} \)):

- Assume \( \Psi \) is nearly homogeneous, i.e. \( \Psi_{;i} \equiv 0 \).

- Identify:
  \[ v_\mu := \frac{\Psi_{;\mu}}{\sqrt{g^{ab} \Psi_{;a} \Psi_{;b}}} \]
  (so that \( v_\mu v^\mu = -1 \))

  \[ T_{\mu \nu}^{\text{KG}} = |g^{ab} \Psi_{;a} \Psi_{;b}| v_\mu v_\nu + g_{\mu \nu} \left( -\frac{1}{2} \Psi_{;a} \Psi_{;a} - V(\Psi) \right) \]
  \[ \approx + \Psi^2 v_\mu v_\nu + g_{\mu \nu} \left( \frac{1}{2} \Psi^2 - V(\Psi) \right) \]

- Compare with \( T_{\mu \nu}^{\text{PF}} \):
  \[ \rho + p = \frac{\Psi^2}{\Psi^2 + 1} - \frac{\Psi^2}{\Psi^2 + 1} \]
\[ T^{\mu\nu}_{\text{ref}} = (\rho + p) v_{\mu} v_{\nu} + p g_{\mu\nu} \]

are of similar form (unlike e.g. \( T^{\mu\nu}_{\text{EM}} \)):

- Assume \( \Psi \) is nearly homogeneous, i.e. \( \Psi_i, i \approx 0 \).
- Identify:
  \[ v_{\mu} = \frac{\Psi_i}{\sqrt{|g_{\mu\nu} \psi_i \psi_i|}} \]
  (so that \( v_{\mu} v^{\mu} = -1 \))

i.e.:
\[ T^{\mu\nu}_{\text{ref}} = |g_{\mu\nu} \psi_i \psi_i| v_{\mu} v_{\nu} + g_{\mu\nu} \left(-\frac{1}{2} \psi_i \psi_i - V(\Psi)\right) \]
\[ \approx \psi^2 v_{\mu} v_{\nu} + g_{\mu\nu} \left(\frac{1}{2} \psi^2 - V(\Psi)\right) \]

- Compare with \( T^{\rho\sigma}_{\text{ref}} \):
\[ \frac{\rho + p}{\rho} = \frac{1}{w} + 1 = \frac{\psi^2}{\frac{1}{2} \psi^2 - V(\Psi)} \]
\[ \Rightarrow \quad \frac{1}{w} = \frac{\psi^2}{\frac{1}{2} \psi^2 - V(\Psi)} - \frac{\psi^2 - V(\Psi)}{(\psi^2 - V(\Psi))} = \frac{\psi^2 + V(\Psi)}{\psi^2 - V(\Psi)} \]
Assume $\psi$ is nearly homogeneous, i.e. $\dot{\psi}_i \approx 0$.

Identify:

$$\nu_\mu = \frac{\psi_i \nu_i}{\sqrt{g^\alpha_\beta \psi_\alpha \psi_\beta}}$$

(i.e.:

$$T^{\mu\nu} = |g^\alpha_\beta \psi_\alpha \psi_\beta| \nu_\mu \nu_\nu + g_{\mu\nu} \left( -\frac{1}{2} \psi_i \psi_i^* - V(\psi) \right)$$

$$\approx + \frac{\dot{\psi}^2}{2} \nu_\mu \nu_\nu + g_{\mu\nu} \left( \frac{1}{2} \dot{\psi}^2 - V(\psi) \right)$$

Compare with $T^{\rho\sigma}$:

$$\frac{\nu^\rho + \psi^\sigma}{p^\nu} = \frac{1}{w} + 1 = \frac{\dot{\psi}^2}{\frac{1}{2} \dot{\psi}^2 - V(\psi)}$$

$$\Rightarrow \frac{1}{w} = \frac{\dot{\psi}^2}{\frac{1}{2} \dot{\psi}^2 - V(\psi)} - \frac{\psi_\mu \psi_\mu^* - V(\psi)}{\frac{1}{2} \dot{\psi}^2 - V(\psi)} = \frac{\psi_\mu \psi_\mu^* + V(\psi)}{\frac{1}{2} \dot{\psi}^2 - V(\psi)}$$

Thus:

$$w = \frac{\psi_\mu \psi_\mu^* - V(\psi)}{\psi_\mu \psi_\mu^* + V(\psi)} \in (-1, 1)$$

Potential dominated, i.e., $V(\psi) \gg \dot{\psi}^2$ (see inflation later)

Potential: $V(\psi) = 0$
\[ T^\mu_\nu = |g^{ab} \gamma_a \gamma_b| \, v_\mu v_\nu + g_{\mu \nu} \left( -\frac{1}{2} \psi_i \psi^i - V(\psi) \right) \]

\[ \approx + \psi_i^2 v_\mu v_\nu + g_{\mu \nu} \left( \frac{1}{2} \psi^2 - V(\psi) \right) \]

\[ \frac{\mu + \rho}{\rho} = \frac{1}{w} + 1 = \frac{\psi^2}{\frac{1}{2} \psi^2 - V(\psi)} \]

\[ w = \frac{\frac{\psi^2}{2} - V(\psi)}{\frac{\psi^2}{2} + V(\psi)} \]

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\[ w = \frac{\psi^2/2 - V(\psi)}{\psi^2/2 + V(\psi)} \in (-1, 1) \]

Potential dominated, i.e. \( V(\psi) \gg \psi^4 \) (see inflation)
Identify: \[ v_\mu = \frac{\psi_\mu}{\sqrt{g_{\alpha\beta} \psi_\alpha \psi_\beta}} \] (so that \( v_\mu v^\mu = -1 \))

i.e.: \[ T^{\kappa\ell}_{\mu\nu} = g^{\alpha\beta} \psi_\alpha \psi_\beta v_\mu v_\nu + g_{\mu\nu} \left( -\frac{1}{2} \psi_\alpha \psi^\alpha - V(\psi) \right) \]
\[ \approx + \dot{\psi}^2 v_\mu v_\nu + g_{\mu\nu} \left( \frac{1}{2} \dot{\psi}^2 - V(\psi) \right) \]

Compare with \( T_{\mu\nu}^{PF} \):
\[ \frac{\kappa + \rho}{\rho} = \frac{1}{w} + 1 = \frac{\dot{\psi}^2}{\frac{1}{2} \dot{\psi}^2 - V(\psi)} \]

\[ \Rightarrow \quad \frac{1}{w} = \frac{\dot{\psi}^2}{\frac{1}{2} \dot{\psi}^2 - V(\psi)} - \frac{\psi^{3/2} - V(\psi)}{\frac{1}{2} \dot{\psi}^2 - V(\psi)} = \frac{\psi^{3/2} + V(\psi)}{\psi^{3/2} - V(\psi)} \]

Thus:
\[ w = \frac{\psi^{3/2} - V(\psi)}{\psi^{3/2} + V(\psi)} \quad \epsilon (-1,1) \]

potential dominated, i.e. \( V(\psi) \gg \dot{\psi}^2 \) (see inflation later)

\[ \chi \quad \text{no potential:} \quad V(\psi) = 0 \]
\[ p = \frac{1}{2} \dot{\psi}^2 - V(\psi) \]

\[ \Rightarrow \quad \frac{1}{\dot{\psi}} = \frac{\dot{\psi}^2}{\frac{1}{2} \dot{\psi}^2 - V(\psi)} - \frac{\dot{\psi}^2 - V(\psi)}{\frac{1}{2} \dot{\psi}^2 - V(\psi)} = \frac{\dot{\psi}^2 + V(\psi)}{\dot{\psi}^2 - V(\psi)} \]

Thus:

\[ w = \frac{\dot{\psi}^2 - V(\psi)}{\dot{\psi}^2 + V(\psi)} \quad \epsilon (-1,1) \]

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Recall:

- In special relativity, energy and momentum conservation arise (through Noether's theorem) because space-time is always and everywhere the same, i.e. from time & space transl. invariance.

- In general relativity, we have:

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Recall:

1. In special relativity, energy and momentum conservation arise (through Noether's theorem) because space-time is always and everywhere the same, i.e., from time & space transl. invariance.

2. In general relativity, we have:

   1.) The theory (i.e., the action) is invariant under all diffeomorphisms (i.e., under all re-labelings of points).
   
   It gives Bianchi identities, but not conservation laws.

   2.) Only in special cases and sometimes only locally is space-time the same in some directions.
   
   It's case of Killing vectors. Then get conservation laws.
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2.) Only in special cases and sometimes only locally is space-time the same in some directions. It's case of 'Killing vectors'. Then get conserv. laws.
1) Relabeling, $\bar{x}^i = x^i(x^0, x^1, x^2, x^3)$, has 4 freely choosable functions. Thus, expect 4 equations. Indeed, proposition:

$$T^\mu_{\nu} = 0 \quad \text{for } \mu = 0, 1, 2, 3$$

(we will later see that this becomes the contracted Bianchi identity $\delta^\mu_{\nu}; \sigma = 0$)

**Proof:**

$t$ is the flow parameter, i.e.: $\phi_t = \text{id}$

Assume $\phi_t: M \rightarrow M$ is a diffeomorphism that is generated by the flow of a vector field, $\xi$, that vanishes outside the region $B \subseteq M$, i.e.

$$\phi_t(p) = p \text{ if } p \in M - B$$

(i.e. only the points in $B$ get re-labeled)
Namely:

1) Relabeling, $\bar{x}^i = \bar{x}^i(x^0, x^1, x^2, x^3)$, has 4 freely choosable functions. Thus, expect 4 equations. Indeed, proposition:

$$T^{\mu \nu} = 0 \quad \text{for} \quad \mu = 0, 1, 2, 3$$

(we will later see that this becomes the contracted Bianchi identity $R^{\mu \nu} = 0$)

Proof:

- Assume $\phi : M \rightarrow M$ is a diffeomorphism that is generated by the flow of a vector field, $\xi$, that vanishes outside the region $B \subset M$, i.e.

$$\phi(p) = p \quad \text{if} \quad p \in M - B$$
Namely:

1) Relabeling, \( \bar{X}^i = \bar{X}^i(x^0, x^1, x^2, x^3) \), has 4 freely choosable functions. Thus, expect 4 equations. Indeed, proposition:

\[
T^{\mu \nu} = 0 \quad \text{for } \mu = 0, 1, 2, 3
\]

(we will later see that this becomes the contracted Bianchi identity \( G^{\mu \nu} = 0 \))

Proof:

- Assume \( \phi_t : M \rightarrow M \) is a diffeomorphism that is generated by the flow of a vector field, \( \xi \), that vanishes outside the region \( B \subset M \), i.e.,

\[
\phi_t(p) = p \quad \text{if } p \in M - B
\]

(i.e., only the points in \( B \) get re-labeled)
Every integral, including the action integral, is invariant under the change of variable, i.e., here under the diffeomorphism \( \phi_x \), including when the diffeomorphism is infinitesimal. Thus:

\[
\int_\mathcal{B} L(\psi, \partial_\psi, g) \, d^7x = \int_\mathcal{B} L(\psi, \partial_\psi, g) \, d^7x
\]

\[
\Rightarrow 0 = \frac{1}{\varepsilon} \int_\mathcal{B} \left[ L - \phi_x^* L(\psi) \right] \, d^7x
\]

(total dependence on \( t \) and \( \partial_\psi \) vanishes because of symmetry of motion for the matter fields \( \psi \) for small \( \varepsilon \))
Every action, including the action integral, is invariant under the change of variable, i.e., here under the diffeomorphism $\phi_k$, including when the diffeomorphism is infinitesimial. Thus:

$$\int_B L(y, \partial y, g) \, d\tau^x = \int_B L(y, \partial y, g) \, d\tau^x$$

$$\Rightarrow 0 = \frac{1}{\xi} \int_B \left[ \mathcal{L} - \phi_k^* \mathcal{L} \right] \, d\tau^x$$

(total dependence on $t$ and $\partial y$ vanishes because of eqn of motion for the matter fields $\psi$):

$$\sim \frac{1}{\xi} \int_B \left[ \sum_i \frac{d^2 \mathcal{L}}{d\psi_i \partial \psi_i} \right] \, d\tau^x$$

recognize:

$$\frac{1}{\xi} \partial_{\psi_i} \phi_k^*(g)_{\mu \nu}$$

becomes $\lim_{\xi \to 0} \frac{1}{\xi} (g - \phi_k^*(g))_{\mu \nu}$

$$\Rightarrow \frac{1}{\xi} \mathcal{L}(g_{\mu \nu} - \phi_k^*(g)_{\mu \nu}) + \frac{d \mathcal{L}}{d \psi_i \partial \psi_i} \, d\tau^x$$
\[ \int_{\mathcal{B}} \mathcal{L}(\mathbf{y}, \mathbf{\partial y}, g) \, d^ny = \int_{\mathcal{B}} \mathcal{L}(\mathbf{y}, \mathbf{\partial y}, g) \, d^ny \]

\[ \Rightarrow \quad 0 = \frac{1}{\mathcal{V}} \int_{\mathcal{B}} \left[ \mathcal{L} - \Phi^\ast \mathcal{L} \right] \, d^ny \]

\[ \approx \frac{1}{\mathcal{V}} \int_{\mathcal{B}} \left[ \sum T \left( \frac{d\mathbf{y}}{d\mathbf{y}} \right) \cdot \mathcal{L} \right] \, d^ny \]

\[ \text{recognize:} \quad \frac{1}{\mathcal{V}} \int_{\mathcal{B}} \left( \mathbf{g}_{\mu\nu} - \Phi^\ast \mathcal{L} \right) \, d^ny \]

\[ \text{becomes} \quad \lim_{\epsilon \to 0} \frac{1}{\mathcal{V}} \int_{\mathcal{B}} \left( \mathbf{g}_{\mu\nu} - \Phi^\ast \mathcal{L} \right) \, d^ny = \mathcal{L}_{\partial \mathbf{g}}(\mathbf{g}) \]

\[ \square \text{Take limit} \Rightarrow \text{obtain Lie derivative:} \]

\[ 0 = \int_{\mathcal{B}} \left( \frac{1}{2} T^{ab} \sqrt{g} \right) \mathcal{L}_{\partial \mathbf{g}}(\mathbf{g}_{ab}) \, d^ny \]

\[ \square \text{Need Lemma: For metric connection} \]
Take \( \lim \) \( \to 0 \) \( \Rightarrow \) obtain Lie derivative:

\[
0 = \sum_{\beta} \frac{1}{2} T^{a\beta} \sqrt{g} \, L_{g}(g_{ab}) \, d^{4}x
\]

Need Lemma: For metric connection, the Lie derivative can be written as:

\[
L_{\xi} Q^{a\cdots b}_{\cdots c\cdots d} = Q^{a\cdots b}_{\cdots c\cdots d} - \xi^{c} \cdots d \, Q^{a\cdots b}_{\cdots c\cdots k} + \xi^{a} \cdots k \, Q^{b\cdots c\cdots d}_{\cdots j\cdots k} + \cdots + Q^{a\cdots b}_{\cdots c\cdots k} \, \xi^{k} \cdots j
\]

Proof: We know its true for commutative. At origin of geodetic ccs, can write it without
Take \( \lim_{\varepsilon \to 0} \) to obtain Lie derivative:

\[
0 = \frac{1}{2} \sum_{\varepsilon} T^{\alpha \beta} \nabla^\alpha L_q(g_{\alpha \beta}) d^\varepsilon x
\]

Need lemma: For metric connection, the Lie derivative can be written as:

\[
L_q \, Q^{a \ldots b}_{c \ldots d} = Q^{a \ldots b}_{c \ldots d, \, \mu} \xi^\mu - Q^{k \ldots \mu}_{c \ldots d, \, \mu} \xi^\mu - Q^{a \ldots k}_{c \ldots d} \xi^k + Q^{a \ldots k}_{c \ldots i} \xi^i + \ldots + Q^{a \ldots k}_{c \ldots i} \xi^i
\]

Proof: We know it's true for commas. At origin of geodesic paths, can write it with j because all \( i^2 = 0 \). But with j, it is all...
\[ 0 = \frac{1}{2} \sum_{a} T^{ab} \sqrt{g} \ L_{a}(g_{ab}) \ d^{4}x \]

**Need Lemma:** For metric connection the Lie derivative can be written as:

\[ L_{\xi} Q^{a...b}_{c...d} = Q^{a...b}_{c...d;k} \xi^{k} \]

\[ -Q^{\xi...b}_{c...d;k} \xi^{k} - ... - Q^{\xi...k}_{c...d} \xi^{k} + Q^{a...b}_{\xi;k} \xi^{k} + ... + Q^{a...k}_{c...d} \xi^{k} \]

**Proof:** We know its true for commas. At origin of geodesic coords, can write it with \( j \) because all \( R^{j} = 0 \). But with \( j \) it is all convenient \( \Rightarrow \) true in all coordinate systems.
\[ \begin{align*} &- Q^{a} \epsilon_{cd} \xi_{ik} - \cdots - Q^{a} \epsilon_{cd} \xi_{jk} \\
\quad + Q^{a} \epsilon_{cd} \xi^{k} \xi_{ic} + \cdots + Q^{a} \epsilon_{cd} \xi^{k} \xi_{jc} \\
\end{align*} \]

**Proof:** We know it's true for commas. At origin of geodesic ods, can write it with \( j \) because all \( \Gamma^j = 0 \). But with \( j \), it is all covariant \( \Rightarrow \) true in all coordinate systems.

\[ \begin{align*}
\square \quad J_{m \theta} &= \left( \frac{1}{2} \right) T^{ab}_{\gamma} \xi^{\gamma} L_{\xi}(g_{ab}) d^4 x \text{ calculate now:} \\
&= \quad 0 \text{ because } \xi^{\gamma}_\theta = 0
\end{align*} \]

(We will revisit it) \( \rightarrow L_{\xi}(g_{ab}) = g_{ab} \xi^{k} \xi_{k} + g_{kb} \xi^{k} \xi_{ja} + g_{ak} \xi^{k} \xi_{jb} \)

\[ \begin{align*}
\square \quad \text{Thus:}
\end{align*} \]
\[ J_0 = \sqrt{\frac{1}{2} T^{ab} \nabla^2 L_3(g_{ab})} d^4x \text{ calculate now:} \]

\[ L_3(g_{ab}) = g_{ab} \kappa_3^k + g_{ab} \xi_{ja} + g_{ak} \xi_{ja} \]

(We will revisit this point here)

\[ 0 = \sqrt{\frac{1}{2} T^{ab} (g_{b,j} \xi_{ja} + g_{a,b} \xi_{ja}^k)} V_j^2 d^4x \]

\[ = \int_{\mathcal{B}} T^{ab} \left( \delta_{b,j} \xi_{ja} + \delta_{a,b} \xi_{ja}^k \right) V_j^2 d^4x \]

\[ = 2 \delta_{b,j a} + (\delta_{a,b} - \delta_{b,a}) \]

\[ = \int_{\mathcal{B}} 2 T^{ab} \xi_{b,j a} V_j^2 d^4x \]
Need lemma: For metric connection the Lie derivative can be written as:

\[ L_g Q^{a..b..c..d..} = Q^{a..b..c..d..j..k..} - Q^{k..j..c..d..} Q^{a..b..} - Q^{j..k..c..d..} Q^{a..b..} + Q^{a..b..c..d..j..k..} + \ldots + Q^{a..b..c..d..j..k..} \]

Proof: We know it's true for commas. At origin of geodetic paths, can write it with j because all R\(^j\) = 0. But with j it is all covariant \(\Rightarrow\) true in all coordinate systems.

\[ J_m 0 = \left( \frac{1}{2} T^{a..b..} f^a (g_{ab}) d^2x \right) \text{ calculate now:} \]

\[ 0 \text{ because } \partial_{x} g = 0 \]
In \( \theta = 0 \), calculate now:

\[
\hat{L}_g(\sigma b) = g_{ab} \chi^k \delta^k - g_{kb} \delta^k_{ja} + g_{ak} \delta^k_{ib}.
\]

(We will revisit this point here)

Thus:

\[
0 = \int_{\mathcal{B}} T^{ab} \left( g_{kb} \delta^k_{ja} + g_{ak} \delta^k_{ib} \right) V^a \, d^4x
\]

\[
= \int_{\mathcal{B}} T^{ab} \left( \delta_{bja} + \delta_{a;ib} \right) V^a \, d^4x
\]

\[
= \int_{\mathcal{B}} 2 T^{ab} \delta_{bja} V^a \, d^4x
\]

\[
= \int_{\mathcal{B}} 2T^{ab} \delta_{bja} V^a \, d^4x
\]
\( L_g(\text{gob}) = \text{g}ab, k^k g^k + \text{g}k b \xi^j a + \text{g} a c \xi^k i b \)

Thus:

\[ 0 = \int_\mathcal{B} T^{ab}(\text{g}k b \xi^j a + \text{g} a c \xi^k i b) V_g^j d^4x \]

\[ = \int_\mathcal{B} T^{ab} (\xi^j b a + \xi^a b i) V_g^j d^4x \]

\[ = 2 \xi^j b a + (\xi^a b - \xi^b a) \]

\[ = \int_\mathcal{B} 2 T^{ab} \xi^j b a V_g^j d^4x \]
because all $T^i = 0$. But with $i$, it is all covariant $\rightarrow$ true in all coordinate systems.

\[ J_m \bigg|_0 = \int \frac{1}{\sqrt{g}} T^{ab} \sqrt{g} L_g(g_{ab}) d^4x \text{ calculate now:} \]

\[ 0 \quad \text{because} \quad \sqrt{g} = 0 \]

(We will revisit this point here)

\[ L_g(g_{ab}) = g_{ab} \xi^k \xi^j + g_{kb} \xi^k \xi^j_a + g_{ac} \xi^k \xi^j_b \]

Thus:

\[ 0 = \int \frac{1}{\sqrt{g}} T^{ab} \big( g_{kb} \xi^k \xi^j_a + g_{ac} \xi^k \xi^j_b \big) V_g d^4x \]

\[ = \int \frac{1}{\sqrt{g}} T^{ab} \big( \xi^k \xi^j_{ab} + \xi^k \xi^j_{ba} \big) V_g d^4x \]

\[ = \int \frac{1}{\sqrt{g}} T^{ab} \big( \xi^k \xi^j_{ab} \big) V_g d^4x \]
covariant \Rightarrow true in all coordinate systems.

\[ J_n \quad 0 = \left( \frac{1}{2} T^{ab} \partial_{\gamma} L_\gamma(g_{ab}) \right) d^4x \] calculate now:

\[ \because \partial_{\gamma} L_\gamma(g_{ab}) = 0 \]

(We will revisit (this point here))

\[ L_\gamma(g_{ab}) = g_{ab;}^{k} \delta^{x}_{k} + g_{k6} \delta^{x}_{ja} + g_{ka} \delta^{x}_{ji6} \]

Thus:

\[ 0 = \sum_{b} T^{ab} \left( g_{k6} \delta^{x}_{ja} + g_{ka} \delta^{x}_{ji6} \right) V_{g}^{\gamma} d^4x \]

\[ \Rightarrow \sum_{b} T^{ab} \left( \delta_{b}^{x}_{ja} + \delta_{a}^{x}_{ji6} \right) V_{g}^{\gamma} d^4x \]

\[ = 2 \delta_{b}^{x}_{ja} + (\delta_{a}^{x}_{ji6} - \delta_{a}^{x}_{ji6}) \]
\[ J_\text{m} = \int_\mathcal{B} \frac{1}{2} T^{ab} \, \nabla_s \, L_g(\gamma_{ab}) \, d^4x \]

We will revisit this point here.

Thus:

\[ \nabla_s L_g(\gamma_{ab}) = g_{ab;\ell} \nabla_\ell \gamma^\ell + g_{ab;\ell} \gamma^\ell_{;\ell \ell} + g_{\alpha \ell} \gamma^\alpha_{\ell \ell \ell \ell} \]

\[ 0 = \int_\mathcal{B} T^{ab} \left( g_{ab;\ell} \gamma^\ell_{;\ell \ell} + g_{\alpha \ell} \gamma^\alpha_{\ell \ell \ell \ell} \right) \, V_g \, d^4x \]

\[ = \int_\mathcal{B} T^{ab} \left( \gamma_{b;\ell a} + \gamma_{a;\ell b} \right) \, V_g \, d^4x \]

\[ = 2 \, \delta_{b;\ell a} + (\gamma_{a;\ell b} - \gamma_{b;\ell a}) \]

\[ = \int_\mathcal{B} 2 \, T^{ab} \, \delta_{b;\ell a} \, V_g \, d^4x \]
\[ J_m 0 = \int_0^1 \frac{1}{2} T^{a b} \mathcal{L}_g(g_{a b}) \, d^4x \] calculates now:

\[ 0 \text{ because } \mathcal{L}_g(g) = 0 \]

(We will revisit this point here)

\[ \mathcal{L}_g(g_{a b}) = g_{a b} \xi^k \xi^k + g_{a b} \xi^k \xi^j + g_{a b} \xi^j \xi^k \]

Thus:

\[ 0 = \int_0^1 T^{a b} (g_{a b} \xi^k \xi^j + g_{a b} \xi^j \xi^k) \, d^4x \]

\[ = \int_0^1 T^{a b} \xi_{(a} \xi_{b)} \, d^4x \]

\[ = 2 \xi_{b a} = (\xi_{a b} + \xi_{b a}) \text{ anti-symmetric} \]

\[ = \int_0^1 2 T^{a b} \xi_{b a} \, d^4x \]
\[
\begin{align*}
\hat{a} &= 2 \xi_{b; a} + (\xi_{a; b} - \xi_{b; a}) \\
&\quad \text{anti-symmetric} \\
&= \int_{\mathbb{B}} 2 \mathcal{T}^{ab}_{a} \xi_{b; a} \nabla \gamma \, d^4 x \\
&= 2 \int_{\mathbb{B}} \left( \mathcal{T}^{ab}_{a} \xi_{b; a} + \mathcal{T}^{ak}_{j k} \xi_{a} - \mathcal{T}^{ak}_{j k} \xi_{a} \right) \nabla \gamma \, d^4 x \\
&= 2 \int_{\mathbb{B}} \left( \mathcal{T}^{ab}_{a} \xi_{b; a} \right) \nabla \gamma - \mathcal{T}^{ak}_{j k} \xi_{a} \nabla \gamma \right) \, d^4 x \\
&= 0 \\
\text{Why? define } r^{a} := \mathcal{T}^{ab}_{a} \xi_{b}, \text{ then:} \\
\int_{\mathbb{B}} r^{a} \xi_{a} \nabla \gamma \, d^4 x = \int_{\mathbb{B}} \xi_{a} \nabla \gamma \, d^4 x = 0 \\
\text{because } r = 0 \\
&\text{on } \partial \mathbb{B} \text{ by assumption,} \\
&\text{i.e. } \text{div } \xi = \frac{\partial}{\partial t} = 0 \\
&\text{there.}
\end{align*}
\]
\[= \int_B 2 \left( T^{ab} \delta_{b;\alpha} V^\gamma \right) d^4x\]

\[= 2 \int_B \left( T^{ab} \delta_{b;\alpha} + T^{ak} \delta_{k;\alpha} - T^{ak} \delta_{k;\gamma} \right) V^\gamma d^4x\]

\[= 2 \int_B \left[ (T^{ab} \delta_{b;\alpha} V^\gamma) - T^{ak} \delta_{k;\alpha} V^\gamma \right] d^4x\]

Why? Define \( \tau^a := T^{ab} \delta_{b;\alpha} \), then:

\[\int_B \tau^a V^\gamma dx = \int_B i_\gamma \Omega = 0\]

because \( \gamma = 0 \)

on \( \partial B \) by assumption, i.e., \( \delta_{a;\alpha} = 0 \) elsewhere.

Thus:

\[\int_B \tau^a V^\gamma dx = 0\]
2) Conservation laws:

\[ \text{Assume that the manifold has a symmetry in this sense:} \]

\[ \text{Along paths that are induced by (i.e., tangent to)} \]
\[ \text{a vector field} \quad \nabla, \quad \text{the change of the manifold} \]
Thus:

\[ \int_T \sum_{i,j,k} \tilde{g}_{ij} \psi_i \psi_j \, d^4x = 0 \text{ for all } \psi \]

\( \Rightarrow \quad T_{i,j,k} = 0 \)

Consequence of diffeomorphism invariance.

2) Conservation laws:

Assume that the manifold has a symmetry in this sense:

Along paths that are induced by (i.e., tangent to) a vector field \( \xi \), the shape of the manifold,
2) Conservation laws:

Assume that the manifold has a symmetry in this sense:

Along paths that are induced by (i.e. tangent to) a vector field, $\xi$, the shape of the manifold, i.e., the metric, $g$, does not change, i.e. we have:

$$L_\xi g = 0$$

(K)

Definition: If $\xi$ obeys Eqn (K) in some region $B \subset M$ we say $\xi$ is a Killing vector field in $B$. 
2) Conservation laws:

Assume that the manifold has a symmetry in this sense:

Along paths that are induced by (i.e. tangent to) a vector field, \( \xi \), the shape of the manifold, i.e., the metric, \( g \), does not change, i.e. we have:

\[ L_\xi g = 0 \quad \text{(K)} \]

Definition: If \( g \) obeys Eqn (K) in some region \( B \subset M \) we say \( g \) is a Killing metric.
2) Conservation laws:

Assume that the manifold has a symmetry in this sense:

Along paths that are induced by (i.e. tangent to) a vector field, \( \mathbf{\xi} \), the shape of the manifold, i.e., the metric, \( g \), does not change, i.e., we have:

\[
\sum_{j,k} T_{jk} \xi_j v_j d^4x = 0 \quad \text{for all } v
\]

Thus:

\[
\Rightarrow T_{jk} = 0
\]
2) **Conservation laws:**

Assume that the manifold has a symmetry in this sense:

Along paths that are induced by (i.e., tangent to) a vector field, \( \xi \), the shape of the manifold, i.e., the metric, \( g \), does not change, i.e. we have:

\[
L_\xi g = 0 \quad \text{(K)}
\]

- **Definition:** If \( \xi \) obeys Eqn (K) in some region \( B \subset M \) we say \( \xi \) is a *Killing vector field* in \( B \).
Useful:

\[ \mathcal{L}_{\xi} g_{\mu\nu} = g_{\mu[j} \xi^{k] } + g_{k\nu} \xi^{j\mu} + g_{j\mu} \xi^{k\nu} \]

i.e. we obtain the Killing property:

\[ \varepsilon_{\mu\nu} = - \varepsilon_{\nu\mu} \]

(i.e., it is antisymmetric tensor)

(To find Killing vector fields for a given space-time, just check this differential equation for)

Def: The energy-momentum flow component, \( p^\mu \), in the \( 5 \) direction is defined as:

\[ p^\mu := T^{\mu\nu} \xi_\nu \]

Conservation law:

\[ p_{\mu j\nu} = (T^{\rho\mu\nu} \xi_\rho)_{j\nu} = T^{\rho\mu\nu} \xi_{j\rho} + T^{\rho\mu\nu} \xi_{j\rho} \]

\[ \Rightarrow 0 \]
Useful: 
\[ \log g_{\mu\nu} = \frac{1}{g_{\rho\nu} g_{\kappa\mu}} + g_{\rho\nu} g_{\kappa\mu} + g_{\rho\kappa} g_{\mu\nu} \]

i.e. we obtain the Killing property: 
\[ \mathcal{L}_\xi g_{\mu\nu} = -2\epsilon_{\mu\nu} \]

Def: The energy-momentum flow component, \( p^\nu \), in the \( \xi \) direction is defined as: 
\[ p^\nu := T^{\nu\rho} \xi_\rho \]

Conservation law: 
\[ P^\nu_{\mu \nu} = (T^{\nu\rho} \xi_\rho )_{;\mu} = T^{\mu\rho}_{\nu \mu} \xi_\rho + T^{\nu\rho}_{\mu \mu} \xi_\rho = 0 \]
In integral form:

\[ 0 = \sum_{B} P_{\nu}^{\mu} \nu^\alpha d^4x = \sum_{B} \text{div}_{\nu} \Omega \quad \text{Gauss} = \sum_{\partial B} \iota_{\nu} \Omega \]

Thus: As much of the \( \xi \) component of energy-momentum flows into a volume \( B \), as much also flows out of the space-time volume \( B \).

Geodesic observer:

Assume \( \xi \) is Killing vector field and \( \gamma \) is a geodesic with tangent vector \( u \). Then, \( S^\nu u_\nu \) is conserved along \( \gamma \):

\[ \nabla_x (S^\nu u_\nu) = u^k (S^\nu u_\nu)_{;k} = u^k S^j_{\ k} u_\nu + u^k \Gamma^\nu_{\ jk} u_\nu u_j = 0 \quad \text{because} \]

because
In integral form:

\[ 0 = \int_\mathcal{B} \rho \nu^\mu \, d^4x = \int_\mathcal{B} \text{div}_\mathcal{B} \Omega = \int_{\partial \mathcal{B}} i_\nu \Omega \]

Thus: As much of the \( \nu \) component of energy-momentum flows into a volume \( \mathcal{B} \), as much also flows out of the space-time volume \( \mathcal{B} \).

Geodesic observer:
Assume \( \nu \) is Killing vector field and \( \gamma^\nu \) is a geodesic with tangent vector \( u^\mu \). Then, \( \gamma^\nu u_\mu \) is conserved along \( \gamma^\nu \):

\[ \nabla_\mu (\gamma^\nu u_\mu) = u^k (\gamma^\nu u_\mu)_{,k} = u^k \gamma^\nu_{,k} u_\mu + u^k \gamma^\nu u_{\mu;k} = 0 \]

rate of change of \( \gamma^\nu u_\mu \)

because \( u^k \gamma^\nu_{,k} = \nabla_\mu u^\nu = 0 \) because \( u^\nu \) is Killing vector.
Thus: As much of the $\delta$ component of energy-momentum flows into a volume $B$, as much also flows out of the space-time volume $B$.

Geodesic observer:
Assume $\delta$ is Killing vector field and $\gamma$ is a geodesic with tangent vector $u$. Then, $\delta^\mu u_\mu$ is conserved along $\gamma$:

$$\frac{d}{dt}(\delta^\mu u_\mu) = u^k (\delta^\mu u_\mu)_{,k} = u^k g^{\mu\nu} u_\nu + u^k g^{\mu\nu} u_{,\nu} u_j = 0$$

because $u^k g_{\mu\nu} u_{\nu} = 0$ because geodesic.
Thus: As much of the $\mathbf{g}$ component of energy-momentum flows into a volume $\mathbf{B}$, as much also flows out of the space-time volume $\mathbf{B}$.

**Geodesic observer:**

Assume $\mathbf{g}$ is Killing vector field and $\gamma$ is a geodesic with tangent vector $\mathbf{u}$. Then, $\mathbf{g}^\mu u_\mu$ is conserved along $\gamma$:

$$\nabla_\nu (g^\mu u_\mu) = u^k (g^\mu u_\mu) ; _\nu = u^k g^\mu j_\nu u_r + u^k g^\mu u_r j_\nu = 0 \quad \checkmark$$

Rate of change of $\mathbf{g}^\mu u_\mu$ along the geodesic $\gamma$. 

$\nabla u = 0$ because $\mathbf{g}$ is symmetric.

$\nabla u = 0$ because $\mathbf{u}$ is geodesic.
\[
0 = \sum_B P^r_i \nu_g d^4 x = \sum_B \text{div}_a p = \sum_{\partial B} i_r n^2
\]

Thus: As much of the \( \xi \) component of energy-momentum flows into a volume \( B \), as much also flows out of the space-time volume \( B \).

Geodesic observer:
Assume \( \xi \) is Killing vector field and \( \gamma \) is a geodesic with tangent vector \( u \). Then, \( S^\mu u_\mu \) is conserved along \( \gamma \):

\[
\nabla_\nu (S^\mu u_\mu) = u^k (S^\nu u_\nu)_,\mu = u^k (S^\nu g_{jk} u_\mu + u^\nu g^{jk} u_\mu) = 0
\]

because

\[
\nabla_\nu u = 0 \quad \text{because} \quad \text{geodesic}
\]

Rate of change of \( S^\mu u_\mu \) along the geodesic \( \gamma \)
In region \( B \) of \( M \) we say \( z \) is a Killing vector field in \( B \).

**Useful:**

\[
\xi_{\mu;\nu} = g_{\nu;\xi}^{\kappa} + g_{\kappa;\xi}^{\nu;\mu} + g_{\mu;\xi}^{\nu;\kappa} + g_{\nu;\xi}^{\mu;\kappa}.
\]

i.e. we obtain the Killing property:

\[
\xi_{\mu;\nu} = -\xi_{\nu;\mu}.
\]

(i.e. it is antisymmetric tensor)

**Def:** The energy-momentum flow component, \( p^\mu \), in the \( \xi \) direction is defined as:

\[
p^\mu := \Gamma^{\rho\sigma\nu} g_{\nu}^\mu.
\]

**Conservation law:**

\[
p^\mu : = (T^\mu{}_{\xi})_{|\xi} + T^\mu{}_{\rho} - T^{\rho}{}_{\mu} + T^{\sigma\rho} \xi_{\sigma}.
\]
\( p^\nu := 1 \), \( \xi^\nu \)

**Conservation Law:**

\[
\begin{aligned}
P^\nu_{\mu \nu} &= (T^\nu_{\mu \nu} \xi^\nu)_{\mu \nu} = T^\nu_{\mu \nu} \xi^\nu + T^\nu_{\mu \nu} \xi^\nu \\
&= 0
\end{aligned}
\]

**In integral form:**

\[
0 = \sum_B P^\nu_{\mu \nu} v^\mu d^4x = \sum_B \text{div}_\nu p \stackrel{\text{Complement}}{=} \sum_{\partial B} v^\mu \Sigma \nu
\]

**Thus:** As much of the \( \xi^\nu \) component of energy-momentum flows into a volume \( B \) as much also flows out.
Thus: As much of the $\mathbf{g}$ component of energy-momentum flows into a volume $B$, as much also flows out of the space-time volume $B$.

Geodesic observer:
Assume $\mathbf{g}$ is Killing vector field and $\gamma$ is a geodesic with tangent vector $u$. Then, $\mathbf{g}^\nu u_\nu$ is conserved along $\gamma$:

$$\nabla_\mu (\mathbf{g}^\nu u_\nu) = u^k (\mathbf{g}^\nu u_\nu)_{;\nu} = u^k \mathbf{g}^\nu_{;\nu} u_\nu + u^k \mathbf{g}^\nu u_{\nu;\nu} = 0$$

rate of change of $\mathbf{g}^\nu u_\nu$ along the geodesic $\gamma$
Thus: As much of the $\mathcal{E}$ component of energy-momentum flows into a volume $B$, as much also flows out of the space-time volume $B$.

Geodesic observer:

Assume $\mathcal{E}$ is Killing vector field and $\gamma$ is a geodesic with tangent vector $u$. Then, $\xi^\mu u_\mu$ is conserved along $\gamma$:

$$\nabla_\nu (\xi^\mu u_\mu) = u^k (\xi^\mu u_\mu)_{;k} = u^k \xi^\mu_{;k} u_\mu + u^k \xi_{;ik} u_\mu = 0$$

Rate of change of $\xi^\mu u_\mu$ along the geodesic $\gamma$
\[ \mathfrak{g}_{\mu} = \mathfrak{g}_{\nu} \mathfrak{g} + \mathfrak{g}_{\kappa} \mathfrak{g}_{\iota} \mathfrak{g} + \mathfrak{g}_{\eta} \mathfrak{g}_{\zeta} \mathfrak{g} \]

i.e. we obtain the Killing property:

\[ \mathfrak{g}_{\mu} ;_{\nu} = - \mathfrak{g}_{\nu} ;_{\mu} \]

(i.e., it is antisymmetric)

**Def:** The energy-momentum flow component, \( p^\nu \), in the \( 5 \) direction is defined as:

\[ p^\nu := T^{\mu \nu} \mathfrak{g}_\mu \]

**Conservation Law:**

\[ p^\nu_{,\nu} = (T^{\mu \nu} \mathfrak{g}_\nu)_{,\nu} = T^{\mu \nu} \mathfrak{g}_{\nu ;\nu} + T^{\mu \nu} \mathfrak{g}_{\nu} + T^{\mu \nu} \mathfrak{g}_{\nu} + \Rightarrow = 0 \]

= 0