The tetrad formulation of GR

Why?

1. Need a formalism that is suited for study of local versus global entities:

   E.g.: For suitably isolated local systems there should exist a notion of global energy and momentum conservation, even in the absence of Killing vector fields.

   \[ \text{Recall: since antisymmetric, they transform like the \textbf{field \textbf{of the}} integral \textbf{measure} \textbf{d}\textbf{A}}. \]

\[ \Rightarrow \text{Strategy}: \text{Because differential forms are integrable, try to view our tensors as tensor-valued forms as far as possible.} \]
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   E.g.: For suitably isolated local systems there should exist a notion of global energy and momentum conservation, even in the absence of Killing vector fields.

   \[ \text{Recall:} \text{If a system is antisymmetric, then its transform is the product of the integral measure of the } \]

   \[ \text{two fields.} \]

   \[ \rightarrow \text{Strategy:} \text{Because differential forms are integrable, try to view our tensors as tensor-valued forms as far as possible.} \]
energy and momentum conservation, even in the absence of Killing vector fields.

\[ \text{\underline{Strategy:} Because differential forms are integrable, try to view our tensors as tensor-valued forms as far as possible.} \]

2. Need a formalism that is suited for the study of fermions:

\[ \text{E.g.: In special relativity, the Dirac equation} \]
\[ (\frac{\partial}{\partial x^v} - m \gamma^v) \psi = 0 \text{ with } \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \]

where the "Dirac matrices" \( \gamma^v \), are
to view our tensors as tensor-valued forms as far as possible.

2. Need a formalism that is suited for the study of fermions:

E.g.: In special relativity, the Dirac equation (for electrons and quarks etc.) has the form:

$\left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m^1\right) \psi = 0$ with $\psi = \left(\begin{array}{c} \psi_1 \\ \psi_2 \end{array}\right)$

where the "Dirac matrices", $\gamma^\mu$, are $4 \times 4$ matrices which must obey:

$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^\mu \nu = 2 \left(\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array}\right)$
2. Need a formalism that is suited for the study of fermions:

E.g.: In special relativity, the Dirac equation (for electrons and quarks etc) has the form:

\[
(i \gamma^\mu \frac{\partial}{\partial x^\mu} - m) \psi = 0 \quad \text{with} \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}
\]

where the "Dirac matrices", \( \gamma^\mu \), are 4\times4 matrices which must obey:

\[
\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu} = 2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

\[\Rightarrow \text{Strategy: At each } p \in M \text{ choose an orthonormal basis } \{ \Theta_i \} \text{ of } T_p(M), \text{ and dual basis } \{ e_i \} \text{ so that } g_{\mu\nu} = g(e_\mu, e_\nu) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]
3. **Important benefit:**

General relativity takes the form of a so-called "gauge theory", which is analogous to the gauge theories of electromagnetism, the weak and the strong force. Useful for Quantum Gravity?

Recall the math:

- **Frames** $\{e^i, e_\nu\}$:

Often, one uses as the bases of $T_p(M)$, and $T_p(M)$, the canonical bases $\{dx^\mu\}$ and $\{\frac{2}{\partial x^\nu}\}$ respectively, which suggest themselves when one chooses coordinates, say $(x^0, \ldots, x^3)$. 
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Often, one uses as the bases of $T_p(M)$ and $T_p(M)^*$, the canonical bases $\{dx^i\}$ and $\{\frac{\partial}{\partial x^i}\}$ respectively, which suggest themselves when one chooses coordinates, say $(x^0, ..., x^3)$. Thus, when changing coordinate system, $x \to \tilde{x}$, one also usually automatically changes basis in $T_p(M)$, $T_p(M)^*$.
Recall the math:

1. Frames $\{\Theta^3, e_\nu\}$:

Often, one uses as the bases of $T_p(M)$ and $T_p(M)'$, the canonical bases $\{dx^\nu\}$ and $\{\frac{\partial}{\partial x^\nu}\}$ respectively, which suggest themselves when one chooses coordinates, say $(x^0, ..., x^3)$. Thus, when changing coordinate system, $x \rightarrow \tilde{x}$, one also usually automatically changes basis in $T_p(M)$, $T_p(M)'$.

**Important:** The only reason why the components of a tensor can change when we change coordinates is that we can change basis in the (co-)tangent spaces, namely...
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**Important:** The only reason why the components of a tensor can change when we change coordinates is that we can change basis in the (co-) tangent spaces, namely from one canonical basis to another canonical basis, when we change coord. system.
Often, one uses as the bases of $T_p(M)$, and $T'_p(M)$, the canonical bases $\{dx^i\}$ and $\{\frac{\partial}{\partial x^i}\}$ respectively, which suggest themselves when one chooses coordinates, say $(x^0, \ldots, x^3)$. Thus, when changing coordinate system, $x \rightarrow \tilde{x}$, one also usually automatically changes basis in $T_p(M)$, $T'_p(M)$.

Important: The only reason why the components of a tensor can change when we change coordinates in that we can change basis in the (co-)tangent spaces, namely from one canonical basis to another canonical basis, when we change coord. system.

Recall:

\[
\begin{pmatrix}
\frac{\partial}{\partial x^0} \\
\frac{\partial}{\partial x^1} \\
\frac{\partial}{\partial x^2} \\
\frac{\partial}{\partial x^3}
\end{pmatrix} =
\begin{pmatrix}
\frac{\partial}{\partial \tilde{x}^0} \\
\frac{\partial}{\partial \tilde{x}^1} \\
\frac{\partial}{\partial \tilde{x}^2} \\
\frac{\partial}{\partial \tilde{x}^3}
\end{pmatrix}
\Rightarrow \frac{\partial}{\partial x^i} = \tilde{\sigma}^0 \frac{\partial}{\partial \tilde{x}^0} + \tilde{\sigma}^1 \frac{\partial}{\partial \tilde{x}^1} + \tilde{\sigma}^2 \frac{\partial}{\partial \tilde{x}^2} + \tilde{\sigma}^3 \frac{\partial}{\partial \tilde{x}^3}
\]
Frames $\Theta, \Xi, \Theta_{-3}$:

Often, one uses as the bases of $T_p(M)$ and $T_p(M)'$, the canonical bases $\{dx^i\}$ and $\{\frac{\partial}{\partial x^i}\}$ respectively, which suggest themselves when one chooses coordinates, say $(x^0, ..., x^3)$. Thus, when changing coordinate system, $x \rightarrow \tilde{x}$, one also usually automatically changes basis in $T_p(M), T_p(M)'$.

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Important: The only reason why the components of a tensor can change when we change coordinates is that we can change basis in the (co-) tangent spaces, namely from one canonical basis to another

\[
\left( \frac{\partial \xi^r}{\partial x^s} = \frac{\partial \xi^s}{\partial x^r} = \frac{\partial^2 \xi^r}{\partial x^s \partial x^r} = \frac{\partial^2 \xi^s}{\partial x^r \partial x^s} \Rightarrow \xi^r = \xi^s \frac{\partial^2 \xi^r}{\partial x^s} \right) \Rightarrow \xi^r = \xi^s \frac{\partial \xi^r}{\partial x^s} = \xi^r \frac{\partial^2 \xi^s}{\partial x^s \partial x^r}
\]

We notice: If we choose a fixed basis, say \( \{ \Theta^r, \xi^s \} \), then the coefficients of tensors no longer depend on the choice of coordinates, i.e., they are scalars! E.g.: \( \text{the same scalar members in every coordinate system!} \)
Recall: a fixed vector has different coefficients in different bases:

\[
\begin{align*}
\xi^i \frac{\partial}{\partial x^i} &= \xi^{\bar{i}} \frac{\partial}{\partial x^{\bar{i}}} \\
\Rightarrow \bar{\xi}^i &= \frac{\partial x^i}{\partial x^{\bar{i}}} \xi^{\bar{i}}
\end{align*}
\]

\[
\xi = \xi^r \frac{\partial}{\partial x^r} = \bar{\xi}^\bar{r} \frac{\partial}{\partial x^{\bar{r}}}
\]

We notice: If we choose a fixed basis, say \{\bar{\theta}^1, \bar{\theta}^2, \bar{\theta}^3\}, then the coefficients of tensors no longer depend on the choice of coordinates, i.e., they are scalars! E.g.:

\[
\xi^r = \bar{\xi}^{\bar{r}} \bar{e}_r
\]

Conversely: Even staying with one coordinate system, we can freely change our choice of basis in the (co-) tangent spaces:

\[
\Theta^r = A^{r\bar{r}} \bar{\Theta}^{\bar{r}}
\]
A fixed vector has different coefficients in different bases:

\[
\frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial \xi^\nu} \frac{\partial \xi^\nu}{\partial x^\mu} = \frac{\partial}{\partial x^\mu}
\]

\[
\xi = \xi^\nu \frac{\partial \xi^\nu}{\partial x^\mu} = \xi^\nu \frac{\partial x^\mu}{\partial \xi^\nu}
\]

**We notice:** If we choose a fixed basis, say \(\{\theta^\nu\}, \{e_\nu\}\), then the coefficients of tensors no longer depend on the choice of coordinates, i.e. they are scalars! E.g.:

\[
\xi = \xi^\nu e_\nu
\]

**Conversely:** Even staying with one coordinate system, we can freely change our choice of basis in the (co-)tangent spaces:

\[
\Theta^\nu = A^\nu_\rho \Theta^\rho
\]

\[
e^\rho_\mu = (A^{-1})^\rho_\mu e_\nu
\]
Scalars: e.g.: $\xi = \xi^\nu e_\nu$

Conversely: Even staying with one coordinate system, we can freely change our choice of basis in the (co-)tangent spaces:

$\Theta^\nu = A^\nu_{\mu} \Theta^\mu$

$e'_\nu = (A^{-1})^\mu_{\nu} e_\mu$

So we have e.g.:

$\xi = \xi^\nu e_\nu = \xi^\nu A^\nu_{\mu} e'_\mu = \xi'^\mu e'_\mu$

I.e.:

$\xi'^\mu = A^\nu_{\mu} \xi^\nu$

Examples:

- The curvature form: $\Omega^\nu_{\mu} = A^\nu_{\alpha} (A^{-1})^\alpha_{\beta} \varepsilon_{\beta\delta}$
Conversely: Even staying with one coordinate system, we can freely change our choice of basis in the (co-)tangent spaces:

\[ \Theta^\nu = A^{\mu \nu} \Theta^\mu \]
\[ e'_\mu = (A^{\mu \nu})^{-1} e^\nu \]

So we have e.g.:

\[ \xi = \xi^\nu e_\nu = \xi^\nu A^{\mu \nu} e'_\mu = \xi^\mu e'_\mu \]

J.e.:

\[ \xi^{\nu \prime} = A^{\nu \mu} \xi^\mu \]

Examples:

- The curvature form:
  \[ \Omega^{\nu \mu} = A^\nu_{\rho} (A^{-1})^\rho_{\gamma} \Omega^\gamma_{\mu} \]

- But: the connection form
  \[ \omega^{\nu \mu} (s) = \xi^k \Gamma^\nu_{\mu \rho} \omega^{\rho} \]
  obeys:

\[ \omega^{\nu \mu} = A^\nu_{\rho} \omega^{\gamma \mu} (A^{-1})^\gamma_{\rho} - (dA)^\nu_{\mu} (A^{-1})^\rho_{\gamma} \xi^\gamma \]
\[ \Theta' = A \cdot \Theta \]

\[ e'_\mu = (A^{-1})^{\nu}_{\mu} e_\nu \]

So we have e.g.:

\[ \xi' = \xi^\nu e_\nu = \xi^\nu A^{-1 \mu}_\nu e_\nu = \xi^\nu e'_\nu \]

\[ \xi'' = A^{-1 \mu}_\nu \xi' \]

Examples:

- The curvature form:
  \[ \Omega''_{\nu} = A^{-1 \rho}_\nu (A^{-1})_{\nu}^{\beta} \Omega^\beta_b \]

- But: the connection form
  \[ \omega''_{\nu} = A^{-1 \rho}_\nu \omega''_{\beta} (A^{-1})_{\nu}^{\beta} - (dA)^{-1}_c (A^{-1})_{\nu}^c \]

  (Matrix notation: \( \omega' = A \omega A^{-1} - (dA) A^{-1} \))

How to specify frames?
J.e.: \( \xi^{\nu} = A_{\nu}^{\rho} \xi^{\rho} \)

Examples:

- The curvature form: \( \Omega^{\nu}_{\phantom{\nu}r} = A^{\rho}_{\nu}(A^{-1})_{\nu}^{\tau} \Omega_{\tau}^{\rho} \)
- But: the connection form \( \omega^{\nu}_{\phantom{\nu}r}(s) = \xi^{\tau} \Gamma_{\tau r}^{\nu} \) obeys:

\[
\omega^{\nu}_{\phantom{\nu}r} = A^{\rho}_{\nu} \omega^{\tau}_{\phantom{\tau}b}(A^{-1})_{\nu}^{\rho} - (dA)^{\tau}_{\phantom{\tau}c}(A^{-1})_{\nu}^{\rho}
\]

(Metric notation: \( \omega^i_{\nu} = A_{\nu}^{\rho} \omega^{\tau}_{\phantom{\tau}b} A^{\beta}_{\tau} - (dA)^{\tau}_{\phantom{\tau}c} A^{\beta}_{\tau} \))

---

How to specify frames?

In an arbitrary coordinate system, we may specify the bases in terms of the canonical bases:

\[
\Theta^i(x) = \lambda^i_{\ j}(x) \, dx^j
\]

(Another possibility: Take n scalar functions \( f^1, \ldots, f^n \))

and define \( \Theta^i = df^i \). For generic coordinate systems these \( \Theta^i \) will be linearly independent almost nowhere.)
How to specify frames?

In an arbitrary coordinate system, we may specify the bases in terms of the canonical bases:

$$\Theta^i(x) = \lambda^i_j(x) \, dx^j$$

(Another possibility: Take n scalar functions $f^1, \ldots, f^n$ and define $\Theta^i := df^{i\mu}$. For generic functions these $\Theta^i$ will be linearly independent almost everywhere.)

Note: the $\lambda^i_j(x)$ change nontrivially when changing the coordinate system!

Our choice now: orthonormal frames, i.e. "Tetrads"

We say that a frame $\{\Theta^i, \xi^a\}$ is orthonormal if in this frame, for all $p \in M$:

$$g(\xi^a, \xi^b) = (\delta^a_b)_p = \eta_{\mu\nu} \quad \text{i.e.} \quad g = -\Theta^i \Theta^i + \eta_{\mu\nu}$$
In an arbitrary coordinate system, we may specify the bases in terms of the canonical bases:
\[ \Theta^i(x) = \lambda^i_j(x) \, dx^j \]

Another possibility: Take n scalar functions \( f^1, \ldots, f^n \) and define \( \Theta^i := df^{i^*} \). For generic functions these \( \Theta^i \)'s will be linearly independent almost everywhere.

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- We say that a frame \( \{\Theta^1, \ldots, \Theta^n\} \) is orthonormal if in this frame, for all \( p \in M \):

\[
g(e^i, e^j) = \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{array} \right) = \eta_{ij} \text{ i.e. } g = -\Theta \otimes \Theta + \frac{1}{2} \Theta \otimes \Theta \}
Our choice now: orthonormal frames, i.e. "Tetrads":

- We say that a frame $\{\theta^1, \theta^2, \theta^3\}$ is orthonormal if in this frame, for all $p \in M$:

$$g(e_\mu, e_\nu) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & \nu \phi \\ 0 & \nu \phi & 0 \end{pmatrix}$$

i.e. $g = -\theta^0 \bigotimes \theta^0 + \frac{1}{2} \theta^\nu \bigotimes \theta^\nu$

- **Existence?** Always: At each $p \in M$ we may choose e.g. $\Theta^\mu = dx^\mu$ where $dx^\mu$ are canonical ON basis at centre of a geodesic disc.

- **Uniqueness?**

For a given space-time, $(M, g)$, any ON frame yields a new ON frame by transforming the bases through
Our choice now: orthonormal frames, i.e. "Tetrads":

- We say that a frame \( \{ \Theta^i \} \) is orthonormal if in this frame, for all \( p \in M \):
  \[
  g(e_i, e_j) = \begin{pmatrix} -1 & 0 \\ 0 & \gamma_{ij} \end{pmatrix} = \eta_{ij}, \quad \text{i.e. } g = -\Theta^j \Theta^j + \frac{1}{2} \Theta^0 \Theta^0
  \]

- **Existence?** Always: At each \( p \in M \) may choose e.g.
  \( \Theta^i = dx^i \) where \( dx^i \) are canonical ON basis at centre of a geodesic cds.

- **Uniqueness?**
  For a given space-time, \((M, g)\), any ON frame yields a new ON frame by transforming the bases through
  \[
  \Theta'^i(x) = \Lambda_i^j(x) \Theta^j(x)
  \]
We say that a frame \( \{ e^i \}, \{ e_i \} \) is orthonormal if in this frame, for all \( p \in \mathcal{M} \):

\[
 g(e_x, e_y) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{p,v} = \delta_{x,y} \text{ i.e. } g = -\theta^0 \theta^0 + \frac{\zeta}{\rho} \theta^1 \theta^1
\]

Existence? Always: At each \( p \in \mathcal{M} \) may choose \( e^i \)

\[
 \theta^i = dx^i \text{ where } dx^i \text{ are canonical ON basis at centre of a geodesic cds.}
\]

Uniqueness? For a given space-time, \((\mathcal{M}, g)\), any ON frame yields a new ON frame by transforming the bases through

\[
 \theta^\nu(x) = \Lambda(x)^i_\nu \theta^i(x),
\]

if the linear maps \( \Lambda(x) \) preserve the orthonormality.
Existence? Always: At each point $m$, may choose e.g. \( \Theta^r = dx^r \) where $dx^r$ are canonical ON basis at centre of a geodesic cd.s.

Uniqueness?

For a given space-time, \((M, g)\), any ON frame yields a new ON frame by transforming the bases through \( \Theta^r(x) = \Lambda(x)^r_\mu \Theta^\mu(x) \),

if the linear maps \( \Lambda(x) \) preserve the orthonormality:

\[ \eta_{\alpha\beta} \Theta^\alpha \Theta^\beta = \eta_{\alpha\beta} \Theta^\alpha \Theta^\beta \]

i.e. if:

\[ \Lambda^\alpha \Lambda^\beta \eta_{\alpha\beta} = \eta_{\alpha\beta} \]

\[ \Leftrightarrow \eta_{\alpha\beta} \Theta^\alpha \Theta^\beta = \eta_{\alpha\beta} \Theta^\alpha \Theta^\beta \]

recall: this is the defining equation for Lorentz transformations.
Existence? Always: At each point \( p \) of \( M \) may choose \( e.g. \)
\[ \theta^r = dx^r \] where \( dx^r \) are canonical ON basis at centre of a geodesic colls.

Uniqueness? For a given space-time, \((M, g)\), any ON frame yields a new ON frame by transforming the bases through
\[ \theta^r(x) = \Lambda(x)^{\nu} \circ \theta^\nu(x), \]
if the linear maps \( \Lambda(x) \) preserve the orthonormality:
\[ \eta_{\mu \nu} \theta^\mu \otimes \theta^\nu = \eta_{\mu \nu} \theta^\mu \otimes \theta^\nu \]
recalling this is the defining equation for Lorentz transformations.

i.e. if:
\[ \Lambda^a \Lambda^b \eta_{ab} = \eta_{ab} \] (X)
\[ \Rightarrow \] Frames are unique up to local Lorentz transformations.
We say that a frame $\{\theta^0, \theta^i, e_0, e_i\}$ is orthonormal if in this frame, for all $p \in M$:

$$g(e_i, e_j) = (\delta_{ij}^0, 0)_{p,\nu} = g_{\nu \nu} \quad \text{i.e.} \quad g = -\theta^0 \theta^0 + \frac{2}{3} \theta^i \theta^i$$

**Existence?** Always: At each $p \in M$ may choose e.g.

$$\theta^r = dx^r$$ where $dx^r$ are canonical ON basis at centre of a geodesic cds.

**Uniqueness?**

For a given space-time, $(M, g)$, any ON frame yields a new ON frame by transforming the bases through

$$\theta'^r(x) = \Lambda(x)^r_\nu \cdot \theta^\nu(x),$$

if the linear maps $\Lambda(x)$ preserve the orthonormality.
Existence? Always: At each point $p \in M$ may choose e.g.

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Uniqueness?

For a given space-time, $(M, g)$, any ON frame yields a new ON frame by transforming the bases through

$$\Theta'{}^r(x) = \Lambda(x)^{,r} \cdot \Theta^r(x),$$

if the linear maps $\Lambda(x)$ preserve the orthornormality:

$$\eta_{\mu \nu} \Theta^{\mu} \Theta^{\nu} = \eta_{\mu \nu} \Theta'^{\mu} \Theta'^{\nu} = \eta_{\mu \nu} \Theta^{\mu} \Theta^{\nu}$$

i.e., if:

$$\Lambda^a \Lambda^b \eta_{a b} = \eta_{a b}$$

($\times$)

Recall: This is the defining equation for Lorentz transformations.

$$\Rightarrow$$ Frames are unique up to local Lorentz transformations.
i.e. \( \frac{\partial \mathbf{n}}{\partial x_0} = \gamma_0 \mathbf{n} \) (×)

⇒ Frames are unique up to local Lorentz transformations.

Can now answer an old question:

Q: The connection 1-forms \( \omega^a \) are not, we know, tensor-valued 1-forms. Wherin do they take their values?

A: The connection 1-forms take values in the set of infinitesimal Lorentz transformations!

Intuition?

The connection yields the change under infinitesimal
Can now answer an old question:

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In tuition?

The connection yields the change under infinitesimal parallel transport - and parallel transport preserves the metric, i.e. it preserves the lengths of vectors, i.e. the change can only be an infinitesimal "rotation", i.e. an infinitesimal Lorentz transformation.
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**A:** The connection 1-forms take values in the set of infinitesimal Lorentz transformations!

Intuition?

The connection yields the change under infinitesimal parallel transport - and parallel transport preserves the metric, i.e. it preserves the lengths of vectors, i.e. the change can only be an infinitesimal "rotation", i.e. an infinitesimal Lorentz transformation.
Infinitesimal Lorentz transformations are of this form:

\[ A'^a = \delta^a_b + \epsilon_a^b \]  
\[ \epsilon^2 = 0 \text{ assumed} \]

(technically, we are going from the Lorentz group to the Lorentz Lie algebra)

Now, \((\star)\), i.e. \( A'^a A^b \eta_{ab} = \eta_{ab} \) reads:

\[ (\delta^a_b + \epsilon^a_b) (\delta^b_a + \epsilon^b_a) \eta_{ab} = \eta_{ab} \]

i.e.

\[ \epsilon_a^b \eta_{ab} + \epsilon^b_a \eta_{ab} = 0 \]

\[ \Rightarrow \text{Infinitesimal Lorentz transformations are given by all } A'^a = \delta^a_b + \epsilon^a_b \]

which obey:

\[ \epsilon_{ba} + \epsilon_{ab} = 0 \]
\[ A^\nu_a = \delta^\nu_a + \epsilon^\nu_a \]

(technically, we are going from the Lorentz group to the Lorentz Lie algebra)

Now, \((\ast)\), i.e. \(A^\nu_a A^\nu_b \eta_{\nu \mu} = 0\) reads:

\[ (\delta^\nu_a + \epsilon^\nu_a) (\delta^\nu_b + \epsilon^\nu_b) \eta_{\mu \nu} = \eta_{\nu \mu} \]

i.e.

\[ \epsilon^\nu_a \eta_{\nu \mu} + \epsilon^\nu_b \eta_{\mu \nu} = 0 \]

\[ \Rightarrow \text{Infinitesimal Lorentz transformations are given by all } A^\nu_a = \delta^\nu_a + \epsilon^\nu_a \]

which obey:

\[ \epsilon_{ba} + \epsilon_{ab} = 0 \]
Proposition:
In orthonormal frames, the $1$-form $\omega_{\mu\nu}$ obeys

$$\omega_{\mu\nu} + \omega_{\nu\mu} = 0$$

i.e. it takes values that are infinitesimal Lorentz transformations.

Proof:
Recall: Absolute exterior derivative: (an anti-derivative)

$$D t^{a^b} = dt^{a^b} + \omega_{\iota^{i\iota}}^i \wedge t^{a^b} + \ldots - \omega_{\iota^{i\iota}}^i \wedge t^{a^b} - \ldots$$

Play the role of the $\Omega_{\iota}$

Thus:
Recall that by using a tetrad, we achieved
that $g_{\mu\nu} = (0, 0, 0, 0) = g_{\nu\mu}$ everywhere!

$$0 = \nabla g_{\mu\nu} = D g_{\mu\nu} = d g_{\mu\nu} - \omega_{\iota^{i\iota}}^i \wedge g_{\iota^{i\iota}}^i - \ldots$$

i.e. $0 = \omega_{\mu\nu} + \omega_{\nu\mu} \checkmark$
Proposition:

In orthonormal frames, the 1-form \(\omega_{\mu}\) obeys

\[\omega_{\mu} + \omega_{\nu} = 0\]

i.e. it takes values that are infinitesimal Lorentz transformations.

Proof:

Recall: Absolute exterior derivative: (an anti-derivative)

\[D^{\alpha\beta} = \frac{d}{dt}^{\alpha\beta} + \omega^{\alpha\beta}_{\mu\nu} \cdots + \omega^{\alpha\beta}_{\mu\nu} \cdots - \omega^{\alpha\beta}_{\mu\nu} \cdots\]

Thus:

Recall that by using a tetrad, we achieved that \(\gamma^{\nu}_{\alpha\beta} = \gamma^{\nu}_{\beta\alpha} = \gamma^{\nu}_{\alpha\beta}\) every frame!

\[0 = \nabla g_{\mu\nu} = D g_{\mu\nu} = d g_{\mu\nu} - \omega_{\mu\nu} + \cdots \]

i.e. \(0 = \omega_{\mu\nu} + \cdots \)
Proposition:
In orthonormal frames, the 1-form $\omega_{\mu \nu}$ obeys

$$\omega_{\mu \nu} + \omega_{\nu \mu} = 0$$

i.e. it takes values that are infinitesimal Lorentz transformations.

Proof:
Recall: Absolute exterior derivative: (an anti-derivation)

$$Dt^{a_{-b}\cd} = dt^{a_{-b}\cd} + w_{int}^{a_{-b}\cd} + \ldots - w_{ext}^{a_{-b}\cd} + \ldots$$
which obey:

\[ E_{ba} + E_{ab} = 0 \]

**Proposition:**

In orthogonal frames, the 1-form \( w_{\mu \nu} \) obeys

\[ w_{\mu \nu} + w_{\nu \mu} = 0 \]

i.e. it takes values that are infinitesimal Lorentz transformations.

**Proof:**

Recall: Absolute exterior derivative: (an anti-derivation)

\[ D t^{a \cdot b} = d t^{a \cdot b} + w^{\cdot \mu} t^{i \cdot b} \cdot d \mu + \ldots - w^{\mu} t^{a \cdot b} \cdot d \mu + \ldots \]

Any tensor-valued differential form.

Play the role of the \( F_{\mu \nu} \)
\[ A^r_a = \delta^r_a + \varepsilon^r_a \]

\[ \varepsilon^2 = 0 \text{ assumed} \]

(technically, we are going from the Lorentz group to the Lorentz Lie algebra)

Now, \( \ast \), i.e. \[ A^r_a A^0_b \gamma^{r0} = \gamma a \]
reads:

\[ (\delta^r_a + \varepsilon^r_a) (\delta^0_b + \varepsilon^0_b) \gamma^{r0} = \gamma a \]

i.e.

\[ \varepsilon^r_a \eta_{r0} + \varepsilon^0_a \eta_{00} = 0 \]

\[ \Rightarrow \text{Infinitesimal Lorentz transformations are given by all } A^r_a = \delta^r_a + \varepsilon^r_a \]

which obey:

\[ \varepsilon_{ba} + \varepsilon_{ab} = 0 \]
Proposition:

In orthonormal frames, the 1-form $w_{\gamma\rho}$ obeys

$$w_{\gamma\rho} + w_{\rho\gamma} = 0$$

i.e. it takes values that are infinitesimal Lorentz transformations.

Proof:

Recall: Absolute exterior derivative: (an anti-derivative)

$$\text{Dt}^{a\cdots b}_{c\cdots d} = \text{dt}^{a\cdots b}_{c\cdots d} + w_{\gamma\int a\cdots b}^{c\cdots d} + \cdots - w_{\gamma\int a\cdots b}^{c\cdots d} - \cdots$$

Any tensor-valued differential form.

Thus:

Recall that by using a tetrads, we achieved that $g_{\gamma\rho} = (0,0) = g_{\mu\nu}$ everywhere!

$$\bigtriangledown g_{\gamma\rho} = D g_{\gamma\rho} = d g_{\gamma\rho} - w_{\gamma\rho} + w_{\gamma\rho} = 0$$

i.e. $0 = w_{\gamma\rho} + w_{\gamma\rho}$.
In orthonormal frames, the 1-form \( w^{\mu \rho} \) obeys

\[
w^{\mu \rho} + w^{\rho \mu} = 0
\]

i.e. it takes values that are infinitesimal Lorentz transformations.

**Proof:**

Recall: Absolute exterior derivative: (an anti-derivative)

\[
\mathcal{D}t_\alpha^{a..b} = dt_\alpha^{a..b} + \omega^{a..b} \wedge t_\alpha^{\cdots - b} + \cdots - \omega^{a..b} \wedge t_\alpha^{\cdots - b}
\]

Every tensor-valued differential form.

Thus:

Recall that by using a tetrad, we achieved that \( g^{\mu \sigma} = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \). For all \((0,1)\) tensor-valued \( g^{\mu \sigma} \) (since \( g^{\mu \sigma} = g^{\rho \sigma} = \text{const} \))

\[
0 = \nabla g^{\mu \sigma} = \mathcal{D}g^{\mu \sigma} = \partial g^{\rho \sigma} + \omega^{\rho \sigma} - \omega^{\rho \sigma}
\]

i.e. \( 0 = w^{\mu \rho} + w^{\rho \mu} \quad \checkmark \)
Proposition:

In orthonormal frames, the 1-form $\omega_{\mu}$ obeys

$$\omega_{\mu} + \omega_{\mu} = 0$$

i.e. it takes values that are infinitesimal Lorentz transformations.

Proof:

Recall: Absolute exterior derivative: (an anti-derivative)

$$\operatorname{Dt}^{a-b}_{c-d} = \operatorname{dt}^{a-b}_{c-d} + w_{i}^{a-b} c_{d} + \cdots - w_{i}^{a-b} c_{d} - \cdots$$

Thus:

Recall that by using a tetrad, we achieved

$$\nabla g_{\mu\nu} = Dg_{\mu\nu} = dg_{\mu\nu} - \omega_{\mu}^{i} \gamma_{i}^{-\nu} - \omega_{\nu}^{i} \gamma_{i}^{-\mu}$$

i.e. $\nabla g_{\mu\nu} = 0$.

i.e. $$0 = \omega_{\mu} + \omega_{\mu} \checkmark$$
Recall: Absolute exterior derivative: \((\text{an anti-derivative})\)

\[ \text{Dt}^{a..b}_{\ c..d} = \text{dt}^{a..b}_{\ c..d} + \omega^c_{\ d} \Gamma^{i..b}_{\ c..d} + \ldots = \omega^c_{\ d} \Gamma^{i..b}_{\ c..d} + \ldots \]

Every tensor-valued differential form.

Play the role of the \(\Gamma^c_{\ bc}\)

Thus:

\[ 0 = \nabla g_{\mu\nu} = Dg_{\mu\nu} = dg_{\mu\nu} - \omega^i_{\ \mu} \Gamma^\nu_{\ i\ \rho} + g_{\mu\nu} \omega^\rho_{\ \nu} \]

\(\text{i.e.}\)

\[ 0 = \omega^\rho_{\ \nu} + \omega^\nu_{\ \rho} \]

**Tetrad formulation of GR:**

- Redefine the degrees of freedom:
  - We used to specify space-times
Tetrad formulation of GR:

- Redefine the degrees of freedom:
  - We used to specify space-times through these data: \((M, g)\)
  - Now, let us specify space-times, equivalently, through data \((M, \Theta^3)\):

Namely:

Assume the \(\Theta^3\) are given w. resp. to a basis \(dx^3\).

Through functions \(A^\omega\) so that:

\[
\Theta^\nu(x) = A^\mu_\nu(x) \, dx^\nu
\]

Then, \(g_{\mu\nu}(\Theta^3) = g_{\mu\nu}\) in the basis \(\Theta^3\).

But: knowing the \(A^\mu_\nu(x)\), can reconstruct \(g_{\mu\nu}(x)\) in basis \((dx^3)\).
The abstract $g$ is always the same, no matter which basis we express it in. 

Recall:

The abstract $g$ is always the same, no matter which basis we express it in. 

$g = g_{\mu \nu} \theta^\mu \theta^\nu = \eta_{\mu \nu} A^\mu_a A^\nu_b \, dx^a \otimes dx^b = g_{\mu \nu}(x) dx^\mu \otimes dx^\nu$

Namely:

Assume the $\Theta^i$ are given w. r. e. to a basis $\{dx^i\}$.

Through functions $A^\mu$, so that:

$\Theta^i(x) = A^\mu \, dx^\mu$

Then, $g_{\mu \nu}(\Theta^i, \Theta^j) = g_{\mu \nu}$ in the basis $\{dx^i\}$.

But: knowing the $A^\mu(x)$, can reconstruct $g_{\mu \nu}(x)$ in basis $\{dx^\mu\}$.
equivalently, through data \((M, \Theta^0)\):

Namely:

Assume the \(\Theta^0\) are given w. r. s. to a basis \(\{dx^0\}\).

through functions \(\Theta^0(x)\) so that:

\[ \Theta^0(x) = A^0(x) \, dx^0 \]

Then, \(g_{\alpha\beta}(\cdot, \Theta) = g_{\alpha\beta} \) in the basis \(\{dx^\alpha\}\).

But: knowing the \(A^\alpha(x)\), can reconstruct \(g_{\alpha\beta}(x)\) in basis \(\{dx^\alpha\}\):

Recall:

The abstract \(g\) is always the same, no matter which basis we express it in. \(\Rightarrow\)

\[ g = g_{\alpha\beta} \, dx^\alpha \otimes dx^\beta = g_{\alpha\beta}(x) \, dx^\alpha \otimes dx^\beta \]

\[ \Rightarrow \{\Theta^\alpha(x)\} \text{ indeed determines } g_{\alpha\beta}(x): \]
We used to specify space-times through these data: \((M, g)\).

Now, let us specify space-times, equivalently, through data \((M, \Theta^\flat)\):

**Namely:**

Assume the \(\Theta^\flat\) are given w. r. t. a basis \(\xi dx^\flat\).

Through functions \(A^\nu\) so that:

\[ \Theta^\nu(x) = A^\nu(x) dx^\nu \]

Then, \(g_{\mu\nu}(0,0)\) = \(g_{\mu\nu}\) in the basis \(\xi \Theta^\flat\).

But: knowing the \(A^\nu(x)\), can reconstruct \(g_{\mu\nu}(x)\) in basis \(\{dx^\flat\}\).

Recall:

The abstract \(g\) is always the same, no matter which because it's defined
Namely:

Assume the $\Theta^i_\alpha$ are given w. r. t. to a basis $\xi dx^\alpha$. Through functions $A^\nu_\mu$ so that:

$$\Theta^\nu_\mu(x) = A^\nu_\mu(x) dx^\nu$$

Then, $g_{\nu\mu} = (\Theta^i_\alpha, \Theta^j_\beta) = g_{\nu\mu}$ in the basis $\xi dx^\alpha$.

But: knowing the $A^\nu_\mu(x)$, can reconstruct $g_{\nu\mu}(x)$ in basis $\{dx^\alpha\}$.

Recall:
The abstract $g$ is always the same, no matter which basis we express it in. $\Rightarrow$

$$g = \eta_{\nu\mu} \Theta^\nu_\alpha \Theta^\mu_\beta = \eta_{\nu\mu} A^\nu_\alpha A^\mu_\beta dx^\alpha \otimes dx^\beta = g_{\nu\mu}(x) dx^\nu \otimes dx^\nu$$

$\Rightarrow \{\Theta^i_\alpha(x)\}$ indeed determines $g_{\nu\mu}(x)$:

$$g_{\nu\mu}(x) = \eta_{\nu\mu} A^\nu_\alpha(x) A^\mu_\beta(x)$$
Assume the $\theta^{i\alpha}$ are given w. r. e. to a basis $\{dx^\nu\}$ through functions $A^{\nu}_{\alpha}$ so that:

$$\theta^{i\alpha}(x) = A^{\nu}_{\alpha}(x) dx^\nu$$

Then, $g_{\nu\alpha}(\theta^{i\alpha}) = g_{\nu\alpha}$ in the basis $\{dx^\nu\}$.

But: knowing the $A^{\nu}_{\alpha}(x)$, can reconstruct $g_{\nu\alpha}(x)$ in basis $\{dx^\nu\}$.

Recall:

The abstract $g$ is always the same, no matter which basis we express it in. $\Rightarrow$

$$g = g_{\nu\alpha} \theta^{i\nu} \theta^{j\alpha} = g_{\nu\alpha} A^{\nu}_{\alpha} A^{\mu}_{\beta} dx^\nu \otimes dx^\mu = g_{\nu\alpha}(x) dx^\nu \otimes dx^\mu$$

$$\Rightarrow \{\theta^{i\alpha}(x)\} \text{ indeed determines } g_{\nu\alpha}(x):$$

$$g_{\nu\alpha}(x) = \eta_{\nu\mu} A^{\mu}_{\alpha}(x) A^{\nu}_{\beta}(x)$$
But knowing the $A^\nu_a(x)$, can reconstruct $g_{\mu\nu}(x)$ in basis $\{dx^\nu\}$.

Recall:
The abstract $g$ is always the same, no matter which basis we express it in.

\[ g = \eta_{\mu\nu} A^\nu_a(x) A^\mu_b(x) \, dx^a \otimes dx^b = g_{\mu\nu}(x) \, dx^\nu \otimes dx^\mu \]

\[ \Rightarrow \{\Theta^\nu(x)\} \text{ indeed determines } g_{\mu\nu}(x) : \]

\[ g_{ab}(x) = \eta_{\mu\nu} A^\nu_{a}(x) A^\mu_{b}(x) \]

Thus:

- It should be possible to formulate the action principle with variations with respect to $\{\Theta^\nu\}$.
- GR can be viewed as a particular case of this.
Recall:
The abstract $g$ is always the same, no matter which basis we express it in. $\Rightarrow$

$g = \eta_{\mu\nu} \Theta^\nu \Theta^\mu = \eta_{\mu\nu} A^\mu_a(x) A^\nu_b(x) \, dx^a \Theta^\mu \, dx^b = g_{\mu\nu}(x) \, dx^\mu \otimes dx^\nu$

$\Rightarrow \{\Theta^\mu(x)\}$ indeed determines $g_{\mu\nu}(x)$:

$g_{\mu\nu}(x) = \eta_{\mu\nu} A^\mu_a(x) A^\nu_b(x)$

Thus:

0 It should be possible to formulate the action principle with variations with respect to $\{\Theta^\mu\}$.

$\Rightarrow$ GR can be viewed as generalizing SR by
Recall:
The abstr Y is always the same, no matter which basis we express it in. \( g = \eta_{\mu \nu} \Theta^\mu \Theta^\nu = \eta_{\mu \nu} A^\mu_a A^\nu_b \, dx^a \otimes dx^b = g_{\mu \nu}(x) \, dx^\mu \otimes dx^\nu \)

\[\Rightarrow \{ \Theta^\mu(x) \} \text{ indeed determines } g_{\mu \nu}(x): \]

\[g_{\mu \nu}(x) = \eta_{\mu \nu} A^\mu_a(x) A^\nu_b(x)\]

Thus:

\( \Rightarrow \text{GR can be viewed as generalizing SR by allowing local, not just global, Lorentz transformations.} \)
Namely:

Assume the $\Theta_i^j$ are given w. r. s. to a basis $\{dx^i\}$ through functions $A^u_i$ so that:

$$\Theta^i_j(x) = A^u_i(x) dx^u$$

Then, $g_{uv} = (\Theta^i_j, \Theta^j_i) = g_{uv}$ in the basis $\{dx^i\}$.

But: knowing the $A^u_i(x)$, can reconstruct $g_{uv}(x)$ in basis $\{dx^i\}$.

Recall:

The abstract $g$ is always the same, no matter which basis we express it in. $\Rightarrow$ $g = g_{\mu \nu} \Theta^\mu \Theta^\nu = g_{\mu \nu} A^\mu_a A^\nu_b dx^a \otimes dx^b = g_{\mu \nu}(x) dx^\mu \otimes dx^\nu$

$$\Rightarrow \{\Theta^i_j(x)\} \text{ indeed determines } g_{\mu \nu}(x):$$

$$g_{\mu \nu}(x) = \eta_{\mu \nu} A^\mu_i(x) A^\nu_j(x)$$
But: knowing the $A^\mu(x)$, can reconstruct $g_{\mu\nu}(x)$ in basis $\{dx^\mu\}$.

Recall:
The abstract $g$ is always the same, no matter which basis we express it in. $\Rightarrow$

$$g = \eta_{\mu\nu} \Theta^\mu \otimes \Theta^\nu = \eta_{\mu\nu} A^\mu_a A^\nu_b \ dx^a \otimes dx^b = g_{\mu\nu}(x) \ dx^\mu \otimes dx^\nu$$

$\Rightarrow$ $\{\Theta^\mu(x)\}$ indeed determines $g_{\mu\nu}(x)$:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} A^\mu_a(x) A^\nu_b(x)$$

Thus:

- It should be possible to formulate the action principle with variations with respect to $\{\Theta^\mu\}$.
- GP can be viewed as some choice SP had.
Recall:
The abstract $g$ is always the same, no matter which base we express it in. ⇒

\[ g = \eta_{\mu \nu} \Theta^\mu \Theta^\nu = \eta_{\mu \nu} A^\mu_a A^\nu_b \, dx^a \otimes dx^b = g_{\mu \nu}(x) \, dx^\mu \otimes dx^\nu \]

⇒ $\{\Theta^\mu(x)\}$ indeed determines $g_{\mu \nu}(x)$:

\[ g_{\mu \nu}(x) = \eta_{\mu \nu} A^\mu_a(x) A^\nu_b(x) \]

Thus:

- It should be possible to formulate the action principle with variations with respect to $\{\Theta^\mu\}$.

$\Rightarrow$ GR can be viewed as generalizing SR by
Recall:
The abstrad $g$ is always the same, no matter which basis we express it in. 

\[ g = \eta_{\mu \nu} \Theta^\mu \otimes \Theta^\nu = \eta_{\mu \nu} A^\mu_a A^\nu_b \, dx^a \otimes dx^b = g_{\mu \nu}(x) \, dx^\mu \otimes dx^\nu \]

\[ g_{\mu \nu}(x) = \eta_{\mu \nu} A^\mu_a(x) A^\nu_b(x) \]

\[ \Rightarrow \{ \Theta^\mu(x) \} \text{ indeed determines } g_{\mu \nu}(x) : \]

Thus:

\[ \Rightarrow \text{GR can be viewed as generalizing SR by allowing local, not just global, Lorentz transformations} \]

\[ \text{Gauge principle of GR:} \]

"It should be possible to formulate the action principle with variations with respect to $\{ \Theta^\mu \}'\]
Namely:

Assume the $\Theta^i_j$ are given w. r. t. a basis $\{\text{d}x^\alpha\}$. through functions $A^\alpha_\nu$ so that:

$$\Theta^i_j(x) = A^\alpha_\nu(x) \text{d}x^\nu$$

Then, $g_{\nu\mu} = (\Theta^i_j; \Theta^j_i) = g_{\nu\mu}$ in the basis $\{\Theta^i_j\}$.

But knowing the $A^\alpha_\nu(x)$, can reconstruct $g_{\nu\mu}(x)$ in basis $\{\text{d}x^\alpha\}$.

Recall:
The abstract $g$ is always the same, no matter which basis we express it in. $\Rightarrow$

$$g = g_{\nu\mu} \Theta^\nu_\alpha \Theta^\mu_\beta = g_{\nu\mu} A^\alpha_a A^\beta_b \text{d}x^a \Theta \text{d}x^b = g_{\nu\mu}(x) \text{d}x^\nu \Theta \text{d}x^\mu$$

$\Rightarrow \{\Theta^i_j(x)\}$ indeed determines $g_{\nu\mu}(x)$:
\[ \Rightarrow \{\Theta^\mu(x)\} \text{ indeed determines } g_{\mu\nu}(x): \]

\[ g_{\mu\nu}(x) = \eta_{\mu\nu} A_{\alpha}^\mu(x) A_{\beta}^\nu(x) \]

Thus:

- It should be possible to formulate the action principle with variations with respect to \[ \{\Theta^\mu\} \].

- GR can be viewed as generalizing SR by allowing local, not just global, Lorentz transformations at the cost of introducing curvature \[ R_{\mu\nu}^{\alpha\beta} \].

"Gauge principle of GR":

Remark:
principle with variations with respect to $\theta$. GR can be viewed as generalizing SR by allowing local, not just global, coordinate transformations at the "cost" of introducing curvature $\mathcal{R}$. "Gauge principle of GR":

Remark:

The Schrödinger and Dirac equations allow

\[ \psi(x) \rightarrow e^{i\theta} \psi(x) \]  

"global" phase transformation

trivially. Indeed, also local transformations

\[ \psi(x) \rightarrow e^{i\varphi(x)} \psi(x) \]  

work. At the cost of introducing curvature $\mathcal{R}$.
Remark:

The Schrödinger and Dirac equations allow

\[ \psi(x) \rightarrow e^{i\lambda} \psi(x) \]

"global" phase transformation

trivially. Indeed, also local transformations

\[ \psi(x) \rightarrow e^{i\lambda(x)} \psi(x) \]

work: At the cost of introducing a field \( F_{\mu\nu}(x) \). Thus, the electromagnetic field is derivable this way. Similar for weak and strong force. Main difference to gravity: Connection yes, but no...
to gravity: connection yes, but no metric.

The action principle: (in terms of $\Theta^i$ and $\Theta^{*i}$)

Consider the action, for now, without cosmological constant and without matter:

$$S_{\text{grav}} = \frac{1}{16\pi G} \int \mathcal{R} \sqrt{g} \, d^4x$$

Recall Hodge $\star$: If

$$\nu = \frac{1}{p!} \nu_{i_1 \ldots i_p} \Theta^{i_1} \wedge \ldots \wedge \Theta^{i_p}$$

then

$$\star \nu = \frac{1}{p!} \sqrt{\mathcal{R}} \epsilon_{i_1 \ldots i_{p+1}} \nu_{i_1 \ldots i_p} \Theta^{i_{p+1}} \wedge \ldots \wedge \Theta^{i_{p+1}}$$

i.e. $\star : \Lambda^p \rightarrow \Lambda^{n-p}$
The action principle: (in terms of \( \Theta^i \) and \( \Omega^i \))

Consider the action, for now, without cosmological constant and without matter:

\[
S'_{\text{grav}} = \frac{1}{16\pi G} \int_B R \, V^g \, d^4x
\]

Recall Hodge \( \ast \): \[ \ast \Phi = \frac{1}{p!} \, V^g \, \varepsilon_{i_1 \ldots i_p} \, \Theta^{i_1 \ldots i_p} \]

then \( \ast V = \frac{1}{p!} \, V^g \, \varepsilon_{i_1 \ldots i_p} \, \Theta^{i_1 \ldots i_p} \ldots \Theta^{i_p} \)

i.e. \( \ast : \Lambda^p \rightarrow \Lambda^{m-p} \)

Thus:

\[
S'_{\gamma_{\mu\nu}} = \frac{1}{16\pi G} \int \ast R
\]
Consider the action, for now, without cosmological constant and without matter:

\[ S_{\text{act}} = \frac{1}{16\pi G} \int_B R \, \sqrt{g} \, d^4x \]

Recall Hodge \( \star \): If

\[ \nu = \frac{1}{p!} \nu_{i_1 \ldots i_p} \Theta^{i_1 \ldots i_p} \]

then

\[ \star \nu = \frac{1}{p!} \sqrt{g} \, \varepsilon_{i_1 \ldots i_p} \nu^{i_1 \ldots i_p} \Theta^{i_{p+1} \ldots i_n} \]

i.e.

\[ \star : \Lambda^p \rightarrow \Lambda^{n-p} \]

Thus:

\[ S' = \frac{1}{16\pi G} \int_B \star R \]

 Aim now: Re-express \( S' \) in terms of \( \Theta^{i_1} \) and \( \Theta_{i_1} \).
and without matter:

$$S_{\text{pol}} = \frac{1}{16\pi G} \int_B R \, \Omega^2 \, d^4x$$

Recall Hodge $\star$:

If

$$\nu = \frac{1}{p!} \nu_{i_1 \ldots i_p} \theta^{i_1} \wedge \ldots \wedge \theta^{i_p}$$

then

$$\star \nu = \frac{1}{p!} \sqrt{g} \, v^*_{i_1 \ldots i_p} \theta^{i_1} \wedge \ldots \wedge \theta^{i_p}$$

i.e.

$$\star : \Lambda^p \rightarrow \Lambda^{n-p}$$

Thus:

$$S'_{\text{pol}} = \frac{1}{16\pi G} \int_B \star R$$

 Aim now: Re-express $S'_{\text{pol}}$ in terms of $\{ \theta^i \}$ and $\Omega^i$. 
\[ S'_{\text{grav}} = \frac{1}{16\pi G} \int_B R \, V_g \, d^4x \]

Recall Hodge \( \ast \):

\[ Jf \quad \psi = \frac{1}{p!} \psi_{i_1 \cdots i_p} \theta^{i_1} \cdots \theta^{i_p} \]

then \( \ast \psi = \frac{1}{p!} \sqrt{g} \, \epsilon_{i_1 \cdots i_p} \psi^{j_1 \cdots j_p} \theta^{i_1} \cdots \theta^{i_p} \]

i.e. \( \ast : \Lambda^p \rightarrow \Lambda^{n-p} \)

Thus:

\[ S'_{\text{grav}} = \frac{1}{16\pi G} \int_B \ast R \]

Aim now: Re-express \( S'_{\text{grav}} \) in terms of \( \{\Theta^\alpha\} \) and \( \Omega^\alpha \).

Define:

- \( \text{"capital" } n \) is a \((0,2)\) tensor-valued 2-form.
Recall Hodge $\ast$: \[ J \quad \ast \psi \equiv \frac{1}{p!} \psi_{i_1 \ldots i_p} \theta^{i_1} \wedge \ldots \wedge \theta^{i_p} \]

then \[ \ast \psi = \frac{1}{p!} \sqrt{g} \epsilon_{i_1 \ldots i_p} \psi^{i_1 \ldots i_p} \theta^{i_1} \wedge \ldots \wedge \theta^{i_p} \]
in other words, \[ \ast : \Lambda^p \rightarrow \Lambda^{n-p} \]

Thus: \[ S'_{\mu
u} = \frac{1}{16\pi G} \int_B \ast R \]

Aim now: Re-express $S'_{\mu\nu}$ in terms of $\{\Theta^{i_j}\}$ and $\Omega^{i_1 \ldots i_j}$. 
Recall Hodge $\ast$: 
\[ \nu = \frac{1}{p!} \nu_{i_1 \ldots i_p} \Theta^{i_1} \wedge \ldots \wedge \Theta^{i_p} \]

then \[ \ast \nu = \frac{1}{p!} \sqrt{g} \epsilon_{i_1 \ldots i_p} \nu^{i_1 \ldots i_p} \Theta^{i_{p+1}} \wedge \ldots \wedge \Theta^{i_m} \]
i.e. \[ \ast: \Lambda^p \to \Lambda^{m-p} \]

Thus:
\[ S'_{\rho\omega} = \frac{1}{16\pi G} \int_B \ast R \]

Aim now: Re-express $S'_{\rho\omega}$ in terms of $\Theta^{i\omega}$ and $\Omega^{\mu\nu}$.

Define: \"capital $\eta$" is a $(0,2)$ tensor-valued 2-form
Aim now: Re-express $S'_\mu$ in terms of $\Theta^\nu$ and $\Omega_\mu^\nu$.

**Define:** "capital $\Omega$" is a $(0,2)$ tensor-valued 2-form

$$H_{\mu\nu} := \star (\Theta^\kappa \wedge \Theta^\sigma) = \frac{1}{2} \sqrt{g} \varepsilon_{\rho\sigma\mu\nu} \Theta^\rho \wedge \Theta^\sigma$$

$$H_{\mu\nu\rho} := \star (\Theta^\kappa \wedge \Theta^\xi \wedge \Theta^\omega) = \frac{1}{2} \sqrt{g} \varepsilon_{\rho\sigma\mu\nu\xi} \Theta^\xi \wedge \Theta^\omega$$

^a $(0,3)$ tensor-valued 1-form.

**Proposition:**

$$\star R = H_{\mu\nu} \wedge \Omega_\mu^\nu$$  

(it is a $(0,3)$ tensor-valued 4-form)
Aim now: Re-express $\mathcal{F}$ in terms of $\Theta^\mu$ and $\Omega^{\nu\alpha}$.

Define: "capital $\eta$" is a $(0, 2)$ tensor-valued 2-form

$$H_{\mu\nu} := \ast (\Theta^\mu \wedge \Theta^\nu) = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \Theta^\rho \wedge \Theta^\sigma$$

$$H_{\mu\rho\nu} := \ast (\Theta^\mu \wedge \Theta^\rho \wedge \Theta^\nu) = \frac{1}{2} \epsilon_{\mu\rho\nu\sigma\tau} \Theta^\sigma \wedge \Theta^\tau$$

a $(0, 3)$ tensor-valued 1-form.

Proposition:

$$\ast R = H_{\mu\nu} \wedge \Omega^{\mu\nu}$$

(it is a $(0, 0)$ tensor-valued 4-form)
Define: 

\[ H_{\mu \nu} := \star (\theta^\mu \wedge \theta^\nu) = \frac{1}{2} \sqrt{g} \epsilon_{\mu \nu \rho \sigma} \theta^\rho \wedge \theta^\sigma \]

\[ H_{\mu \nu \rho} := \star (\theta^\mu \wedge \theta^\nu \wedge \theta^\rho) = \frac{1}{2} \sqrt{g} \epsilon_{\mu \nu \rho \sigma} \theta^\sigma \]

\[ \text{a (0,3) tensor-valued 1-form.} \]

Proposition:

\[ \star R = H_{\mu \nu} \wedge \Omega^{\mu \nu} \]

(it is a (0,0) tensor-valued 4-form)
Then \[ *u = \frac{1}{P!} \sqrt{g} \epsilon_{\ldots}^\mu \ldots \cdot \theta^\nu \wedge \ldots \wedge \theta^p \]
i.e. \[ *: \Lambda^p \to \Lambda^{n-p} \]

**Thus:**

\[ S' = \frac{1}{16\pi G} \int_B *R \]

**Aim now:** Re-express \( S' \) in terms of \( \Theta^{\mu\nu} \) and \( \Omega^{\mu\nu} \).

**Define:**

- \( H_{\mu\nu} := * (\Theta^{\sigma\nu} \wedge \Theta^\sigma) = \frac{1}{2} \sqrt{g} \epsilon_{\rho\nu} \Theta^\rho \wedge \Theta^\sigma \)
- \( H_{\mu\nu\rho} := * (\Theta^{\sigma\rho} \wedge \Theta^\sigma \wedge \Theta^\sigma) = \frac{1}{2} \sqrt{g} \epsilon_{\mu\nu\rho} \Theta^\sigma \wedge \Theta^\sigma \wedge \Theta^\sigma \)

\( \Theta \) is a \((0,2)\) tensor-valued 2-form.
Define: "capital η" is a (0,2) tensor-valued 2-form

$$H_{\mu\nu} := \star (\theta^\mu \wedge \theta^\nu) = \frac{1}{2} \sqrt{g} \varepsilon_{\mu\nu\rho\sigma} \theta^\rho \wedge \theta^\sigma$$

$$H_{\mu\rho\nu} := \star (\theta^\mu \wedge \theta^\rho \wedge \theta^\nu) = \frac{1}{2} \sqrt{g} \varepsilon_{\mu\rho\nu\sigma} \theta^\sigma$$

↑ a (0,3) tensor-valued 1-form.

Proposition:

$$\star R = H_{\mu\nu} \wedge \Omega^{\mu\nu}$$

(it is a (0,0) tensor-valued 4-form)
\[ H_{\mu \nu} := \star \left( \Theta^\mu \wedge \Theta^\nu \wedge \Theta^s \right) = \frac{1}{2} \epsilon_{\mu \nu \lambda \delta} \Theta^\lambda \Theta^\delta \]

A \text{ (0,3) tensor-valued 1-form.}

\[ \star R = H_{\mu \nu} \wedge \Omega^{\mu \nu} \]

(it is a (0,0) tensor-valued 4-form)

A \text{ Proof:}

Use \[ \Omega^{\mu \nu} = \frac{1}{2} R^{\mu \nu \lambda \kappa} \Theta^\lambda \wedge \Theta^\kappa \Rightarrow \]

\[ H_{\mu \nu} \wedge \Omega^{\mu \nu} \]
\[ H_{\mu
u} := \star (T^\alpha \wedge \theta^i \wedge \psi) = \frac{1}{2} \varepsilon_{\lambda \mu \nu \rho} \theta^\alpha \]

\[ \text{a (0,3) tensor-valued 1-form.} \]

**Proposition:**

\[ \star R = H_{\mu\nu} \wedge \Omega^{\mu\nu} \]

(it is a (0,4) tensor-valued 4-form)

**Proof:**

Use \[ \Omega^{\mu\nu} = \frac{1}{2} R^{\mu\nu\kappa\lambda} \theta^\kappa \wedge \theta^\lambda \Rightarrow \]

\[ H_{\mu\nu} \wedge \Omega^{\mu\nu} = \frac{1}{2 \cdot 2} \varepsilon_{\lambda \mu \nu \rho} R^{\mu\nu\kappa\lambda} \theta^\kappa \wedge \theta^\lambda \wedge \theta^\alpha \wedge \theta^\beta \]
**Proof:**

Use \( \Omega^\nu = \frac{1}{2} R^\nu_{\nu \kappa \lambda} \Theta^\kappa \wedge \Theta^\lambda \Rightarrow \)

\[
H_{\mu \nu} \wedge \Omega^\nu = \frac{1}{2} \sqrt{g} \epsilon_{\mu \nu \rho \sigma} R^\nu_{\nu \kappa \lambda} \Theta^\kappa \wedge \Theta^\sigma \wedge \Theta^\lambda \wedge \Theta^\rho
\]

Use also: \( \epsilon_{\mu \nu \rho \sigma} = 2 \left( \delta_{\nu \rho} \delta_{\mu \sigma} - \delta_{\nu \sigma} \delta_{\mu \rho} \right) \Rightarrow \)

\[
H_{\mu \nu} \wedge \Omega^\nu = \frac{1}{4} R^{\mu \nu \rho \sigma} \sqrt{g} \Theta^\rho \wedge \Theta^\sigma \wedge \Theta^\nu \wedge \Theta^\mu = \ast R
\]

\( \square \) **Proposition:** \( DH_{\mu \nu} = 0 \)

Recall the "first structure equation": \( D\Theta = 0 \)

\( \square \) **Proof:** \( DH_{\mu \nu} = D \left( \frac{1}{2} \sqrt{g} \epsilon_{\mu \nu \rho \sigma} \Theta^\rho \wedge \Theta^\sigma \right) = \frac{1}{2} \sqrt{g} \epsilon_{\mu \nu \rho \sigma} \left( \Theta^\rho \wedge \Theta^\sigma \right) \)
Proof:

Use \( \Omega_{\mu \nu} = \frac{1}{2} R_{\nu \kappa \lambda}^{\mu} \Theta^\kappa \wedge \Theta^\lambda \) \( \Rightarrow \)

\[ H_{\mu \nu} \wedge \Omega_{\mu \nu} = \frac{1}{2} \nabla^\gamma \epsilon_{\mu \nu \rho \sigma} R_{\nu \kappa \lambda}^{\mu} \Theta^\kappa \wedge \Theta^\sigma \wedge \Theta^\lambda \]

Use also: \( \epsilon_{\gamma \kappa \lambda} \epsilon_{\mu \nu \sigma} = 2 \left( \delta_{\kappa \rho} \delta_{\lambda \sigma} - \delta_{\kappa \sigma} \delta_{\lambda \rho} \right) \) \( \Rightarrow \)

\[ H_{\mu \nu} \wedge \Omega_{\mu \nu} = \frac{4}{5} R_{\mu \nu} \nabla^\gamma \Theta^\gamma \wedge \Theta^\sigma \wedge \Theta^\lambda \wedge \Theta^\sigma = \kappa R \]

(need data for derivation of the Einstein equation)

\[ \square \) Proposition: \( D H_{\mu \nu} = 0 \)

Recall the "first structure equation": \( D \theta = 0 \)

\[ \square \) Proof: \( D H_{\mu \nu} = \nabla_{(\gamma} H_{\mu \nu \sigma)} = \frac{1}{2} \nabla^\lambda \epsilon_{\mu \nu \rho \sigma} (D \Theta^\rho \wedge \Theta^\sigma + \Theta^\sigma \wedge \Theta^\rho) \]
\[ *R = H_{\mu\nu} \wedge \Omega^\mu_{\nu} \]

(it is a (0,3) tensor-valued) 4-form

**Proof:**

Use \[ \Omega^\mu_{\nu} = \frac{1}{2} R^\mu_{\nu\kappa\lambda} \Theta^\kappa \wedge \Theta^\lambda \] \[ \Rightarrow \]

\[ H_{\mu\nu} \wedge \Omega^\mu_{\nu} = \frac{1}{2 \cdot 2} \varepsilon_{\mu\nu\rho\sigma} R^\mu_{\nu\kappa\lambda} \Theta^\kappa \wedge \Theta^\sigma \wedge \Theta^\lambda \wedge \Theta^\lambda \]

Use also: \[ \varepsilon_{\mu\nu\rho\sigma} = 2 \left( \delta_{\nu\rho} \delta_{\sigma\mu} - \delta_{\nu\mu} \delta_{\sigma\rho} \right) \] \[ \Rightarrow \]
Define: The capital $\eta$ is a $(0,2)$ tensor-valued 2-form

$$H_{\mu\nu} := \star (\theta^\mu \wedge \theta^\nu) = \frac{1}{2} \sqrt{g} \epsilon_{\mu\nu\rho\sigma} \theta^\rho \wedge \theta^\sigma$$

$$H_{\mu\nu\rho} := \star (\theta^\nu \wedge \theta^\rho \wedge \theta^\mu) = \frac{1}{2} \sqrt{g} \epsilon_{\mu\nu\rho\sigma} \theta^\sigma$$

...and is a $(0,3)$ tensor-valued 1-form.

Proposition:

$$\star R = H_{\mu\nu} \wedge \Omega^{\mu\nu}$$

(it is a $(0,0)$ tensor-valued 4-form)
Proof:

Use $\Omega_{\mu}^\nu = \frac{1}{2} R_{\nu \kappa \lambda}^\nu \Theta^\kappa \Theta^\lambda \Rightarrow$

$$H_{\mu \nu} \wedge \Omega_{\mu}^\nu = \frac{1}{2} \sqrt{g} \varepsilon_{\mu \nu \rho \sigma} R_{\rho \kappa \lambda}^\nu \Theta^\kappa \Theta^\omega \Theta^\lambda \Theta^\nu$$

Use also: $\varepsilon_{\rho \kappa \lambda} \varepsilon_{\mu \nu \rho \sigma} = 2 (\delta_{\kappa \nu} \delta_{\lambda \mu} - \delta_{\kappa \mu} \delta_{\lambda \nu}) \Rightarrow$

$$H_{\mu \nu} \wedge \Omega_{\mu}^\nu = \frac{1}{2} R_{\mu \nu} \sqrt{g} \Theta^\kappa \Theta^\omega \Theta^\lambda \Theta^\rho \Rightarrow x R \checkmark$$

Proposition: $D H_{\mu \nu} = 0$

Proof: $D H_{\mu \nu} = D (\frac{1}{2} \sqrt{g} \varepsilon_{\mu \nu \rho \sigma} \Theta^\rho \Theta^\sigma) = \frac{1}{2} \sqrt{g} \varepsilon_{\mu \nu \rho \sigma} (\Theta^\rho \Theta^\sigma \Theta^\rho \Theta^\sigma + \Theta^\rho \Theta^\sigma \Theta^\rho \Theta^\sigma)$

constant because the basis

Recall the first structure equation: $D \Theta = 0$
Proof:

\[ \Omega_{\mu\nu} = \frac{1}{2} R_{\nu\mu\kappa\lambda} \Theta^\kappa \wedge \Theta^\lambda \implies \]

\[ H_{\mu\nu} \wedge \Omega_{\mu\nu} = \frac{1}{2} \nabla^\sigma \epsilon_{\mu\nu\rho\sigma} R_{\nu\mu\kappa\lambda} \Theta^\kappa \wedge \Theta^\lambda \wedge \Theta^\sigma \wedge \Theta^\lambda \]

Use also: \[ \epsilon_{\mu\nu\rho\sigma} = 2(\delta_{\nu\rho} - \delta_{\nu\sigma} \delta_{\lambda\rho}) \implies \]

\[ H_{\mu\nu} \wedge \Omega_{\mu\nu} = \frac{1}{4} R_{\mu\nu} \nabla^\lambda \Theta^0 \wedge \Theta^0 \wedge \Theta^1 \wedge \Theta^0 = \lambda R \]

Proposition: \[ D H_{\mu\nu} = 0 \]

Recall the first structure equation: \[ D \Theta^0 = 0 \]

Proof: \[ D H_{\mu\nu} = D \left( \frac{1}{2} \nabla^\sigma \epsilon_{\mu\nu\rho\sigma} \Theta^0 \wedge \Theta^0 \right) = \frac{1}{2} \nabla^\sigma \epsilon_{\mu\nu\rho\sigma} \left( \Theta^0 \wedge \Theta^0 \right) \]

\[]
**Proof:**

Use \( \Omega_{\nu} = \frac{1}{2} R_{\nu, kl} \theta^k \wedge \theta^l \implies \)

\[
H_{\mu \nu} \wedge \Omega_{\nu} = \frac{1}{2} \sqrt{g} \varepsilon_{\mu \nu \rho \sigma} R_{\rho \sigma, kl} \theta^k \wedge \theta^l
\]

Use also: \( \varepsilon_{\mu \nu \rho \sigma} = 2(\delta_{\nu \rho} \delta_{\mu} - \delta_{\nu \mu} \delta_{\rho}) \implies \)

\[
H_{\mu \nu} \wedge \Omega_{\nu} = \frac{1}{4} R^{\mu \nu} \sqrt{g} \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 = x R
\]

(need data for derivation of Einstein equation)

\( \implies \)

**Proposition:** \( \text{D} H_{\mu \nu} = 0 \)

Recall the “first structure equation”: \( \text{D} \theta = 0 \)

\( \text{D} H_{\mu \nu} = \text{D} \left( \frac{1}{2} \sqrt{g} \varepsilon_{\mu \nu \rho \sigma} \theta^\rho \wedge \theta^\sigma \right) = \frac{1}{2} \sqrt{g} \varepsilon_{\mu \nu \rho \sigma} \left( D \theta^\rho \wedge \theta^\sigma + \theta^\rho \wedge D \theta^\sigma \right) \)
**Proof:**

Use \( \Omega^{\nu} = \frac{1}{2} R^{\nu}_{\mu \lambda \kappa} \theta^\lambda \theta^\kappa \Rightarrow \)

\[ H_{\mu \nu} \wedge \Omega^{\nu} = \frac{1}{2} \sum_{\gamma} \varepsilon_{\mu \nu \rho \sigma} R^{\rho \sigma}_{\mu \nu \kappa \lambda} \theta^\kappa \theta^\lambda \Rightarrow \]

Use also: \( \varepsilon_{\gamma \delta \kappa \lambda} \varepsilon_{\mu \nu \rho \sigma} = 2 \left( \delta_{\kappa \mu} \delta_{\gamma \nu} - \delta_{\kappa \nu} \delta_{\gamma \mu} \right) \Rightarrow \)

\[ H_{\mu \nu} \wedge \Omega^{\nu} = \frac{1}{4} R^{\rho \sigma}_{\mu \nu} \sum_{\gamma} \varepsilon_{\gamma \delta \kappa \lambda} \theta^\kappa \theta^\lambda \varepsilon_{\mu \nu \rho \sigma} = x R \]

**Proposition:** \( D H_{\mu \nu} = 0 \)

Recall the "first structure equation": \( \delta \Omega = 0 \)

**constant because on base's**
Proof:

Use $\Omega_{\mu\nu} = \frac{1}{2} R_{\mu\nu}^{\lambda \kappa} \Theta^\lambda \wedge \Theta^\kappa \Rightarrow$

$H_{\mu\nu} \wedge \Omega_{\mu\nu} = \frac{1}{2} \nu g_{\nu\mu} R_{\mu\nu}^{\lambda \kappa} \Theta^\lambda \wedge \Theta^\kappa \wedge \Theta^\sigma \wedge \Theta^\tau$

Use also: $\varepsilon_{\mu \nu \lambda \kappa} = 2 (\delta_{\mu \nu} \delta_{\lambda \kappa} - \delta_{\mu \kappa} \delta_{\nu \lambda}) \Rightarrow$

$H_{\mu\nu} \wedge \Omega_{\mu\nu} = \frac{1}{2} R_{\mu\nu}^{\mu \nu} \nu g_{\nu\mu} \Theta^\nu \wedge \Theta^\nu \wedge \Theta^\kappa \wedge \Theta^\lambda \Rightarrow \nabla H_{\mu\nu} = 0$

Proposition: $D H_{\mu\nu} = 0$

Recall the first structure equation: $D \theta = 0$

Proof: $D H_{\mu\nu} = D (\frac{1}{2} \nu g_{\nu\mu} \varepsilon_{\mu \nu \lambda \kappa} \Theta^\lambda \wedge \Theta^\kappa) = \frac{1}{2} \nu g_{\nu\mu} (\Theta^\lambda \wedge \Theta^\lambda \wedge \Theta^\nu \wedge \Theta^\nu)$
\[ H_{\mu \nu} \wedge \Omega^{\nu} = \frac{1}{2} \nabla \varepsilon_{\mu \nu \sigma \tau} R^{\sigma \tau}_{\ \theta \lambda} \Theta^\lambda \Theta^\theta \Theta^\nu \Theta^\mu \]

Use also: \[ \varepsilon_{\mu \nu \sigma \tau} = 2 \left( \delta_{\mu \lambda} \delta_{\nu \tau} - \delta_{\mu \tau} \delta_{\nu \lambda} \right) \Rightarrow \]

\[ H_{\mu \nu} \wedge \Omega^{\nu} = \frac{\mu}{4} R^{\mu \nu}_{\ \rho \lambda} \nabla \varepsilon_{\mu \nu \sigma \tau} \Theta^\lambda \Theta^\theta \Theta^\nu \Theta^\mu = \mu R \checkmark \]

\[ \square \text{Proposition: } \text{DH}_{\mu \nu} = 0 \]

Proof: \[ \text{DH}_{\mu \nu} = D \left( \frac{1}{2} \nabla \varepsilon_{\mu \nu \sigma \tau} \Theta^\theta \Theta^\lambda \Theta^\sigma \Theta^\tau \right) = \frac{1}{2} \nabla \varepsilon_{\mu \nu \sigma \tau} \left( \Theta^\theta \Theta^\lambda \Theta^\sigma \Theta^\tau + \Theta^\tau \Theta^\sigma \Theta^\lambda \Theta^\theta \right) \]

Recall important identities: (torsionless case)
\[ H_{\mu\nu} \Lambda^{\nu\rho} = \frac{1}{2} \nabla_{\gamma} \epsilon_{\mu\nu\rho\sigma} R^{\rho\gamma}_{\kappa\lambda} \Theta^\sigma \Theta^\lambda \Theta^\kappa \Theta^\lambda \]

Use also: \( \epsilon_{\gamma\delta\kappa\lambda} \epsilon_{\mu\nu\rho\sigma} = 2(\delta_{\nu\rho} \delta_{\kappa\lambda} - \delta_{\nu\kappa} \delta_{\rho\lambda}) \Rightarrow \)

\[ H_{\mu\nu} \Lambda^{\nu\rho} = \frac{4}{\mathcal{E}} R^{\mu\nu}_{\rho\sigma} \nabla_{\gamma} \Theta^\sigma \Theta^\lambda \Theta^\kappa \Theta^\lambda = x R \checkmark \]

**Proposition:** \( \text{D} H_{\mu\nu} = 0 \)

Recall the "first structure equation": \( \text{D} \Theta = 0 \)

**Proof:** \( \text{D} H_{\mu\nu} = \text{D} \left( \frac{1}{2} \nabla_{\gamma} \epsilon_{\mu\nu\rho\sigma} \Theta^\sigma \Theta^\rho \right) = \frac{1}{2} \nabla_{\gamma} \epsilon_{\mu\nu\rho\sigma} (\Theta^\sigma \Theta^\rho + \Theta^\rho \Theta^\sigma) \)

Recall important identities: (torsionless case)
Use also: $\varepsilon^{\mu \nu \rho \sigma} = 2 \left( \delta_{\mu \rho} \delta_{\nu \sigma} - \delta_{\mu \sigma} \delta_{\nu \rho} \right) \Rightarrow$

$H_{\mu \nu} \Lambda^{\lambda \mu} = \frac{2}{3} R^\mu_{\lambda \nu} \nabla^\gamma \theta^\lambda \theta^\sigma \theta^\sigma \theta^\rho = x \mathbf{R}$

Proposition: $D H_{\mu \nu} = 0$

Proof: $D H_{\mu \nu} = D \left( \frac{1}{2} \nabla^\gamma \varepsilon_{\mu \nu \sigma} \theta^\sigma \theta^\rho \right) = \frac{1}{2} \nabla^\gamma \varepsilon_{\mu \nu \sigma} \left( D \theta^\rho \theta^\sigma + \theta^\rho \theta^\sigma \right)$

Recall important identities: (torsionless case)

Structure eqn. 1: used it just now
Use also: $\epsilon^\delta_{\gamma\lambda} \epsilon_{\mu\nu\rho} = 2(\delta_{\nu\rho} \delta_{\lambda\mu} - \delta_{\lambda\mu} \delta_{\nu\rho}) = \Rightarrow$

$(\text{need data for})$

(derivation of the Einstein equation)

$H_{\mu\nu\lambda\rho} \tau^\lambda \tau^\rho = \frac{1}{4} R_{\mu\nu} \tau^\gamma \theta^\delta \theta^\epsilon \theta^\theta \theta^\theta \theta^\theta \theta^\theta = R \checkmark$

Proposition: $D H_{\mu\nu} = 0$

Recall the first structure equation: $D \theta = 0$

Proof: $D H_{\mu\nu} = D(\frac{1}{2} \nabla^\gamma \epsilon_{\mu\nu\gamma\delta} \theta^\delta \theta^\delta) = \frac{1}{2} \nabla^\gamma \epsilon_{\mu\nu\gamma\delta} (D \theta^\delta \theta^\delta - \theta^\gamma \nabla^\gamma \theta^\delta)$

Recall important identities: (torsionless case)

Structure eqn. I:

$D \theta^i = d \theta^i + \omega^i_{\ j} \theta^j = 0$ used it just now
\[
H_{\mu \nu} \wedge \Omega^{\mu \nu} = \frac{\kappa}{4} R_{\mu \nu} \Omega^\lambda \wedge \theta^\lambda \wedge \theta^3 \wedge \theta^4 = \kappa R
\]

\[\square\] Proposition: \( D H_{\mu \nu} = 0 \)

\[\square\] Proof: \( D H_{\mu \nu} = D \left( \frac{1}{2} \sqrt{g} \varepsilon_{\mu \nu \rho \sigma} \Theta^\rho \wedge \Theta^\sigma \right) = \frac{1}{2} \sqrt{g} \varepsilon_{\mu \nu \rho \sigma} (D \Theta^\rho \wedge \Theta^\sigma + \Theta^\rho \wedge D \Theta^\sigma) \)

Recall important identities: (torsionless case)

\[\square\] Structure eqn. I:
\[
D \Theta^i = d \Theta^i + \omega^i_j \wedge \theta^j = 0
\]

\[\square\] Structure eqn. II:
\[
\Omega^i_j = d \omega^i_j + \omega^k_i \wedge \omega^i_j
\]
Proposition: $D H_{\mu \nu} = 0$

Proof: $D H_{\mu \nu} = D \left( \frac{1}{2} \varepsilon_{\mu \nu \rho \sigma} \Theta^\rho \wedge \Theta^\sigma \right) = \frac{1}{2} \varepsilon_{\mu \nu \rho \sigma} \left( D \Theta^\rho \wedge \Theta^\sigma + \Theta^\rho \wedge D \Theta^\sigma \right)$

Recall important identities: (torsionless case)

- Structure eqn. I:
  
  $D \Theta^i = d \Theta^i + \omega^i_j \wedge \Theta^j = 0$

- Structure eqn. II:
  
  $\Omega^i_j = d \omega^i_j + \omega^i_k \wedge \omega^k_j$

  (Recall: $R^i_{\cdots} = \Gamma^i_{\cdots} + \Gamma^i_{\cdots} + \Gamma^i_{\cdots}$)

- Bianchi identity I:
  
  $\Omega^i_j \wedge \Theta^j = 0$
Proof: $D H_{\mu \nu} = D \left( \frac{1}{2} \sqrt{g} \varepsilon_{\mu \nu \sigma \rho} \Theta^\sigma \wedge \Theta^\rho \right) = \frac{1}{2} \sqrt{g} \varepsilon_{\mu \nu \sigma \rho} (D \Theta^\sigma \wedge \Theta^\rho + \Theta^\rho \wedge D \Theta^\sigma)$

Recall important identities: (torsionless case)

- Structure eqn. I: $D \Theta^i = d \Theta^i + \omega^i_j \wedge \Theta^j = 0$

- Structure eqn. II: $\Omega^i_j = d \omega^i_j + \omega^k_i \wedge \omega^j_k$

- Bianchi identity I: $\Omega^i_j \wedge \Theta^j = 0$

- Bianchi identity II: (Recall: $R^i ... = \Gamma^i_{j \rho} + \Gamma^i_{\rho j} + \Gamma^j_{\rho k} \Gamma^k_{i \rho}$)
Recall important identities: (torsionless case)

- Structure eqn. I:
  \[ \Theta^i = d\Theta^i + \omega^i_j \wedge \Theta^j = 0 \]

- Structure eqn. II:
  \[ \Omega^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j \]
  \[ R \cdot \Omega^i_j \wedge \Theta^j = 0 \]

- Bianchi identity II:
  \[ D\Omega^i_j = 0 \]

And, in the case of ON frames:

\[ \omega^i_0 + \omega^i_1 = 0 \]
Recall important identities: (torsionless case)

- **Structure eqn. I**: 
  \[ D\theta^i = d\theta^i + \omega^i_j \wedge \theta^j = 0 \]
  used it just now

- **Structure eqn. II**: 
  \[ \Omega^i_j = d\omega^i_j + \omega^k_i \wedge \omega^i_j \]
  (Recall: \( R^i_{\ldots} = \Gamma^i_{\ldots} + \Gamma_j^{\ldots} + \omega_j \wedge \omega^i_{\ldots} \))

- **Bianchi identity I**: 
  \[ \Omega^i_j \wedge \theta^j = 0 \]

- **Bianchi identity II**: 
  \[ D\Omega^i_j = 0 \]

And, in the case of ON frames:

\[ \omega_{\mu
u} + \omega_{\nu\mu} = 0 \]
Structure eqn. I:
\[ D\theta^i = d\theta^i + \omega^i \wedge \theta^j = 0 \]

Structure eqn. II:
\[ \Omega^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j \]

Bianchi identity I:
\[ \Omega^i_j \wedge \theta^j = 0 \]

Bianchi identity II:
\[ D\Omega^i_j = 0 \]

And, in the case of ON frames:
\[ \omega_{\mu\nu} + \omega_{\nu\mu} = 0 \]

The main proposition:

variation, not co-derivative
The main proposition:

Variation of the action with respect to $\delta \Theta^i$ yields:

$$S(\ast R) = (\delta \Theta^i) \ast H_{\mu\nu\rho} \ast \Omega^{\nu\rho} + d(\text{something})$$

It implies:

$$16\pi G \mathcal{E} = \int_{\mathcal{B}} \delta \Theta^i \ast H_{\mu\nu\rho} \ast \Omega^{\nu\rho} + \int_{\partial \mathcal{B}} (\text{something})$$

Definition: The "energy-momentum 1-form" $T_\mu$ is defined as the solution to:
\[ \Omega' \wedge \Theta' = 0 \]

Bianchi identity II:
\[ D \Omega' = 0 \]

And, in the case of ON frames:
\[ \omega_{\mu \nu} + \omega_{\nu \mu} = 0 \]

The main proposition:

Variation of the action with respect to \( \Theta' \) yields:
\[ S(\star R) = (\delta \Theta') \wedge H_{\mu \nu \sigma} \wedge \Omega^{\mu \nu \sigma} + d(\text{something}) \]

It implies:
\[ \int_{S'} S = \left( \delta \Theta' \wedge H_{\mu \nu \sigma} \wedge \Omega^{\mu \nu \sigma} + d(\text{something}) \right) \int_{\partial S} \]
The main proposition: Variation of the action with respect to $\delta \Theta^\mu$ yields:

$$S(\ast R) = (\delta \Theta^\mu) \wedge H_{\mu\nu\sigma} \wedge \Omega^{\ast \sigma} + \alpha(\text{something})$$

It implies:

$$16\pi \bullet \delta S_{\text{grav}} = \int_B (\delta \Theta^\mu) \wedge H_{\mu\nu\sigma} \wedge \Omega^{\ast \sigma} + \int_B (\text{something})$$

Definition: The "energy-momentum 1-form" $T^\mu$ is defined as the solution to:

$$\delta S_{\text{matter}} = \int_B (\delta \Theta^\mu \wedge (\ast T^\mu))$$
The main proposition:

Variation of the action with respect to $\delta \Theta^\alpha$ yields:

$$S(\ast R) = (\delta \Theta^\alpha) \wedge H_{\mu\nu\sigma} \wedge \Omega^{\ast \sigma} + d(\text{something})$$

It implies:

$$16\pi G \delta S_{\text{grav}} = \int_B \delta \Theta^\alpha \wedge H_{\mu\nu\sigma} \wedge \Omega^{\ast \sigma} + \int_{\partial B} (\text{something})$$

Stokes:
$$\int_B df = \int_{\partial B} f$$

Definition: The "energy-momentum 1-form" $T_\mu$ is defined as the solution to:

$$\delta S_{\text{matter}} = \int_B \delta \Theta^\alpha \wedge (\ast T_\mu)$$
Variation of the action with respect to $\delta \Theta^\nu$ yields:

$$S(\ast R) = (\delta \Theta^\nu) \wedge H_{\mu \nu \sigma} \wedge \Omega^{\sigma \nu} + d(\text{something})$$

It implies:

$$16\pi G \delta S'_{\text{grav}} = \int_B (\delta \Theta^\nu \wedge H_{\mu \nu \sigma} \wedge \Omega^{\sigma \nu} + \int_{\partial B} (\text{something})$$

Definition: The "energy-momentum 1-form" $T_\mu$ is defined as the solution to:

$$\delta S_{\text{matter}} = \int_B (\delta \Theta^\nu \wedge (\ast T_\mu))$$
Variation of the action with respect to $\delta \Theta^\nu$ yields:

$$S(\star R) = (\delta \Theta^\nu) \lrcorner H_{\mu\nu\sigma} \wedge \Omega^{\nu\sigma} + d(\text{something})$$

It implies:

$$16\pi G \delta S_{\text{grav}} = \int_B \delta \Theta^\nu \wedge H_{\mu\nu\sigma} \wedge \Omega^{\nu\sigma} + \int_{\partial B} (\text{something})$$

Definition: The "energy-momentum 1-form" $T_\mu$ is defined as the solution to:

$$\delta S_{\text{matter}} = \int_B \delta \Theta^\nu \wedge (\star T_\mu)$$
Variation of the action with respect to $\theta^i$ yields:

$$S(\ast R) = (\delta \theta^i) \wedge H_{\mu \nu \rho} \wedge \Omega^{\ast \sigma} + d(\text{something})$$

It implies:

$$16\pi G \delta S_{_{\text{grav}}} = \int_B \delta \theta^i \wedge H_{\mu \nu \rho} \wedge \Omega^{\ast \sigma} + \int_{\partial B} (\text{something})$$

Definition: The "energy-momentum 1-form" $T_{\mu}$ is defined as the solution to:

$$\delta S_{\text{matter}} = \int_B \delta \theta^i \wedge (\ast T_{\mu})$$

⇒ The equation of motion, i.e., the Einstein equation...
Definition: The "energy-momentum 1-form" $T^\mu_r$ is defined as the solution to:

$$\delta S_{\text{matter}} = \int_B \delta \theta^r \wedge (\ast T^\mu_r)$$

$\Rightarrow$ The equation of motion, i.e., the Einstein equation, becomes:

$$\frac{\delta (S_{\text{grav}} + S_{\text{matter}})}{\delta \theta^r} = 0$$

$$-\frac{1}{2} H_{\mu
u\sigma} \wedge \Omega^{\nu\sigma} = 8\pi G \ast T^\mu_r$$
Definition: The "energy-momentum 1-form" $T_{\mu}$ is defined as the solution to:

$$\delta S_{\text{matter}} = \left( \int_B \delta \theta^r \wedge (\ast T_{\mu}) \right)$$

$\Rightarrow$ The equation of motion, i.e., the Einstein equation,

$$\delta \left( S_{\text{grav}} + S_{\text{matter}} \right) = 0$$

becomes:

$$-\frac{1}{2} H_{\mu\nu\sigma} \wedge \Omega^{\nu\sigma} = 8\pi G \ast T_{\mu}$$
\[ \delta S_{\text{matter}} = \int_B \delta \Theta^r \wedge (\ast T_r) \]

\[ \Rightarrow \text{The equation of motion, i.e., the Einstein equation,} \]

\[ \frac{\delta (S_{\text{grav}} + S_{\text{matter}})}{\delta \Theta^r} = 0 \]

becomes:

\[ -\frac{1}{2} \Theta_{\mu\nu} \wedge \Omega^{\nu\sigma} = 8\pi G \ast T_{\mu} \]

**Exercise:** add the cosmological constant.
The equation of motion, i.e., the Einstein equation,

\[ 8 \left( S_{\text{grav}} + S'_{\text{matter}} \right) = 0 \]

becomes:

\[ -\frac{1}{2} H_{\mu\nu\rho} \wedge \Omega^{\nu\rho} = 8\pi G \ast T_{\mu} \]

Exercise: add the cosmological constant.

Remark: The Einstein form \( G_{\mu} := G_{\mu\nu} \Theta^{\nu} \) obeys

\[ \ast G_{\mu} = -\frac{1}{2} H_{\mu\nu\rho} \wedge \Omega^{\nu\rho} \]

\[ G = 8\pi G \ast T \]
becomes:
\[-\frac{1}{2} H_{\mu\nu\sigma} \wedge \Omega^{\nu\sigma} = 8\pi G \ast T_{\mu}\]

**Exercise:** add the cosmological constant.

**Remark:** The Einstein form \( G_{\mu} = G_{\mu\nu} \Theta^\nu \) obeys
\[
\ast G_{\mu} = -\frac{1}{2} H_{\mu\nu\sigma} \wedge \Omega^{\nu\sigma}
\]
\[\Rightarrow\]
\[G_{\mu} = 8\pi G T_{\mu}\]
becomes:
\[-\frac{1}{2} \, H_{\mu\nu\gamma} \wedge \Omega^{\gamma} = 8\pi G \ast T_\mu\]

**Exercise:** add the cosmological constant.

**Remark:** The Einstein form \( G_\mu := G_\mu \wedge \Omega \) obeys
\[*G_\mu = -\frac{1}{2} \, H_{\mu\nu\gamma} \wedge \Omega^{\gamma} \]

\[\Rightarrow \quad G_\mu = 8\pi G \ast T_\mu\]

**Proof of the main proposition:**
\[S(\star R) = (\star \Theta^\mu) \wedge H_{\mu\nu\gamma} \wedge \Omega^{\gamma} + d(\text{something})\]
\[-\frac{1}{2} H_{\mu\nu\rho} \wedge \Omega^{\nu\rho} = 8\pi G \ast T_{\mu}\]

**Exercise:** add the cosmological constant.

**Remark:** The Einstein form \( G_{\mu} := G_{\mu\nu} \Theta^{\nu}\) obeys

\[ \ast G_{\mu} = -\frac{1}{2} H_{\mu\nu\rho} \wedge \Omega^{\nu\rho}\]

\[ \Rightarrow \quad G_{\mu} = 8\pi G T_{\mu}\]

**Proof of the main proposition:**

\[ S(\ast R) = (\ast \Theta^{\mu}) \wedge H_{\mu\nu\rho} \wedge \Omega^{\nu\rho} + d(\text{something}) \]
\[
\delta (S_{\text{gravity}} + S_{\text{matter}}) = 0
\]

becomes:

\[
-\frac{1}{2} H_{\mu \nu \sigma} \wedge \Omega^{\nu \sigma} = 8\pi G \ast T_{\mu}
\]

**Exercise:** add the cosmological constant.

**Remark:** The Einstein form \( G_{\mu} := G_{\mu \nu} \Theta^{\nu} \) obeys

\[
\ast G_{\mu} = -\frac{1}{2} H_{\mu \nu \sigma} \wedge \Omega^{\nu \sigma}
\]

\[
\Rightarrow \quad G_{\mu} = 8\pi G \ast T_{\mu}
\]
\[
\frac{\partial}{\partial \theta^r} = 0
\]

becomes:

\[-\frac{1}{2} H_{\mu\nu\gamma} \wedge \Omega^{\nu\gamma} = 8\pi G \ast T_{\mu}\]

Exercise: add the cosmological constant.

 Remark: The Einstein form \( G_{\mu} = G_{\mu\nu} \theta^{\nu} \) obeys

\[\ast G_{\mu} = -\frac{1}{2} H_{\mu\nu\gamma} \wedge \Omega^{\nu\gamma}\]

\[\Rightarrow \]

\[G_{\mu} = 8\pi G T_{\mu}\]

Proof of the main proposition:
Exercise: add the cosmological constant.

Remark: The Einstein form \( G_\mu := G_\mu^\nu \Theta^\nu \) obeys
\[
* G_\mu = -\frac{i}{2} H_{\mu \nu \sigma} \wedge \Omega^{\nu \sigma}
\]
\[
\Rightarrow \quad G_\mu = 8\pi G T_\mu
\]

Proof of the main proposition:
\[
S(*R) = (S\Theta^\nu) \wedge H_{\mu \nu \sigma} \wedge \Omega^{\nu \sigma} + d(something)
\]

Indeed:
\[
S(\Theta) = (S\Theta^\nu) \wedge H_{\mu \nu \sigma} \wedge \Omega^{\nu \sigma} + d(something)
\]
\[ \frac{\kappa}{\rho} \left(S_{\text{grav}} + S_{\text{matter}}\right) = 0 \]

becomes:

\[ -\frac{1}{2} H_{\mu\nu\rho} \wedge \Omega^{\nu\rho} = 8\pi G * T_{\nu} \]

**Exercise:** Add the cosmological constant.

**Remark:** The Einstein form \( G_{\mu} := \epsilon_{\mu\nu\rho} \Theta^\nu \) obeys

\[ *G_{\mu} = -\frac{1}{2} H_{\mu\nu\rho} \wedge \Omega^{\nu\rho} \]

\[ \Rightarrow \quad G_{\mu} = 8\pi G T_{\mu} \]

**Proof of the main proposition:**
becomes:
\[- \frac{1}{2} H_{\mu\nu\sigma} \wedge \Omega^\nu^\sigma = 8\pi G \ast T_{\mu}\]

**Exercise:** add the cosmological constant.

**Remark:** The Einstein form \( G_{\mu} := G_{\mu\nu} \Theta^\nu \) obeys
\[ \ast G_{\mu} = -\frac{1}{2} H_{\mu\nu\sigma} \wedge \Omega^\nu^\sigma \]

\[ \Rightarrow \quad G_{\mu} = 8\pi G T_{\mu} \]

**Proof of the main proposition:**
\[ S(\ast R) = (S \Theta^\mu) \wedge H_{\mu
u\sigma} \wedge \Omega^\nu^\sigma + d(\text{something}) \]
Becomes:

\[-\frac{1}{2} H_{\mu\nu\sigma} \land \Omega^{\sigma^g} = 8\pi G \ast T_\mu\]

Exercise: add the cosmological constant.

Remark: The Einstein form \(G_\mu := G_{\mu\nu} \Theta^\nu\) obeys

\(\ast G_\mu = -\frac{1}{2} H_{\mu\nu\sigma} \land \Omega^{\sigma^g}\)

\(\Rightarrow G_\mu = 8\pi G T_\mu\)

Proof of the main proposition:

\(S(\ast R) = (\ast \Theta^\nu) \land H_{\mu\nu\sigma} \land \Omega^{\sigma^g} + d(\text{something})\)
Remark: The Einstein form \( G_\mu := G_{\mu\nu} \Theta \) obeys

\[
*G_\mu = -\frac{i}{2} H_{\mu\nu\sigma} \wedge \Omega^{\nu\sigma}
\]

\[
\Rightarrow 
G_\mu = 8\pi G T_\mu
\]

Proof of the main proposition:

\[
S(*R) = (S\Theta^\nu) \wedge H_{\mu\nu\sigma} \wedge \Omega^{\nu\sigma} + d(\text{something})
\]

Indeed:

\[
S(*R) = (SH_{\mu\nu}) \wedge \Omega^{\mu\nu} + H_{\mu\nu} \wedge S\Omega^{\mu\nu}
\]

Consider the first term:
\[ \ast G_\mu = -\frac{i}{2} H_{\mu}^{\nu\sigma} \wedge \Omega^{\nu\sigma} \]

\[ \Rightarrow \quad G_\mu = 8\pi G T_\mu \]

**Proof of the main proposition:**

\[ \delta(\ast R) = (\delta \Theta^\nu) \wedge H_{\mu\nu\sigma} \wedge \Omega^{\nu\sigma} + d(\text{something}) \]

**Indeed:**

\[ \delta(\ast R) = (\delta H_{\mu\nu}) \wedge \Omega^{\nu\rho} + H_{\mu\nu} \wedge \delta \Omega^{\nu\rho} \]

Consider the first term:

\[ \delta H_{\mu\nu} = \delta \frac{1}{2} \sqrt{g} \varepsilon_{\mu\nu\rho\sigma} \Theta^\rho \wedge \Theta^\sigma \]
Proof of the main proposition:

\[ S(R) = (\delta \Theta^\gamma) \land H_{\mu\nu\delta} \land \Omega^{\nu\gamma} + \text{d(something)} \]

Indeed:

\[ S(R) = (\delta H_{\mu\nu}) \land \Omega^{\nu\gamma} + H_{\mu\nu} \land \delta \Omega^{\nu\gamma} \]

Consider the first term:

\[ \delta H_{\mu\nu} = \delta \left( \frac{1}{2} \nabla^2 \varepsilon_{\rho\mu\nu\theta} \Theta^\rho \Theta^\nu \right) \]

by definition of \( H_{\mu\nu} \) above:

\[ H_{\mu\nu} = \frac{1}{2} \nabla^2 \varepsilon_{\rho\mu\nu\theta} \Theta^\rho \Theta^\nu \]
Proof of the main proposition:

\[ S(\star R) = (S \Theta^r) \wedge H_{\mu\nu} \wedge \Omega^{\mu\nu} + d(\text{something}) \]

Indeed:

\[ S(\star R) = (S H_{\mu\nu}) \wedge \Omega^{\mu\nu} + H_{\mu\nu} \wedge \delta \Omega^{\mu\nu} \]

Consider the first term:

\[ \delta H_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} \Theta^\rho \wedge \Theta^\sigma \]

by definition of \( H_{\mu\nu} \) above:

\[ H_{\mu\nu} : = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} \Theta^\rho \wedge \Theta^\sigma \]

\[ \Rightarrow \quad S(\star R) = (S \Theta^r) \wedge H_{\mu\nu} \wedge \Omega^{\mu\nu} + H_{\mu\nu} \wedge \delta \Omega^{\mu\nu} \]
Proof of the main proposition:

\[ S(R) = (S \Theta^r) \wedge H_{\mu\nu} \wedge \Omega^{\nu r} + d(\text{something}) \]

Indeed:

\[ S(R) = (S H_{\mu\nu}) \wedge \Omega^{\nu r} + H_{\mu\nu} \wedge \delta \Omega^{\nu r} \]

Consider the first term:

\[ \delta H_{\mu\nu} = \delta \frac{1}{2} \sqrt{g} \epsilon_{\alpha\beta\mu\nu} \Theta^\alpha \wedge \Theta^\beta \]

by definition of $H_{\mu\nu}$ above:

\[ H_{\mu\nu} = \frac{1}{2} \sqrt{g} \epsilon_{\alpha\beta\mu\nu} \Theta^\alpha \]

\[ \Rightarrow \quad S(R) = (S \Theta^r) \wedge H_{\mu\nu} \wedge \Omega^{\nu r} + H_{\mu\nu} \wedge \delta \Omega^{\nu r} \]

examine this term
Consider the first term:

\[
\delta H_{\mu\nu} = \delta \frac{1}{2} \sqrt{g} \varepsilon_{\mu\nu\rho\sigma} \Theta^\rho \Theta^\sigma
\]

by definition of \( H_{\mu\nu} \) above:

\[
H_{\mu\nu} = \frac{1}{2} \sqrt{g} \varepsilon_{\mu\nu\rho\sigma} \Theta^\rho
\]

\[\Rightarrow\]

\[
\delta (\ast R) = (\delta \Theta^\mu) \wedge H_{\mu\nu} \wedge \Omega^{\nu^\rho} + H_{\mu\nu} \wedge \delta \Omega^{\nu^\rho}
\]

examine this term:

\[
\Sigma^{\mu^\nu} = \delta \left( dw^{\mu^\nu} + \omega^{\mu^\nu} \wedge \omega^{\rho^\sigma} \right)
\]

\[
= d\delta w^{\mu^\nu} + \delta w^{\mu^\nu} \wedge \omega^{\rho^\sigma} + \omega^{\rho^\sigma} \wedge \delta w^{\mu^\nu}
\]

\[\Rightarrow\]

\[
H_{\mu\nu} \delta \Sigma^{\mu^\nu} = d(H_{\mu\nu} \delta w^{\mu^\nu}) - (dH_{\mu\nu}) \wedge \delta w^{\mu^\nu}
\]
Consider the first term:

\[ \delta H_{\mu\nu} = \delta \left( \frac{1}{2} \nabla^\gamma \epsilon_{\mu\nu\rho\sigma} \Theta^\rho \Theta^\sigma \right) \]

by definition of \( H_{\mu\nu} \) above:

\[ H_{\mu\nu} = \frac{1}{2} \nabla^\gamma \epsilon_{\mu\nu\rho\sigma} \Theta^\rho \Theta^\sigma \]

\[ \Rightarrow \delta (\star R) = (\delta \Theta^\mu) \wedge H_{\mu\nu} \wedge \Omega^{\nu\rho} + H_{\mu\nu} \wedge \delta \Omega^{\nu\rho} \]

examine this term:

\[ \Omega^{\mu\nu} = \delta \left( d\omega^{\mu\nu} + \omega^{\mu}_{\rho\sigma} \wedge \omega^{\rho\sigma}_\nu \right) \]

\[ = d\delta \omega^{\mu\nu} + \delta \omega^{\mu}_{\rho\sigma} \wedge \omega^{\rho\sigma}_\nu + \omega^{\mu}_{\rho\sigma} \wedge \delta \omega^{\rho\sigma}_\nu \]

\[ \Rightarrow H_{\mu\nu} \wedge \delta \Omega^{\mu\nu} = d(H_{\mu\nu} \wedge \delta \omega^{\mu\nu}) - (dH_{\mu\nu}) \wedge \delta \omega^{\mu\nu} \]
\[
\delta \Sigma^\mu_\nu = \delta (d\omega^\mu_\nu + \omega^\mu_\nu \wedge \omega^8) \\
= d\delta \omega^\mu_\nu + \delta \omega^\mu_\nu \wedge \omega^8 \\
= H^\mu_\nu \delta \omega^\mu_\nu \wedge \omega^8 + \omega^\mu_\nu \wedge \omega^8 \wedge \omega^\alpha_\beta \wedge \omega^8_\gamma \\
\Rightarrow \text{Indeed:} \\
S^{(R)} = (\delta \Theta^\mu_\nu) \wedge H^\mu_\nu \delta \omega^\mu_\nu \wedge \omega^8 + d(H^\mu_\nu \delta \omega^\mu_\nu)
\]

\Rightarrow \text{The Einstein eqn. indeed follows from local Lorentz invariance.}
\[ \alpha_{\mu\nu} = \alpha \wedge \Omega_{\mu\nu} \]

by definition of \( \alpha_{\mu\nu} \) above:

\[ H_{\mu
u} = \frac{i}{2} \sqrt{g} \varepsilon_{\mu
u\rho\sigma} \Theta^\rho \]

\[ \Rightarrow \quad \delta(R) = (\delta \Theta^\rho) \wedge H_{\mu\nu} \wedge \delta \Omega_{\mu\nu} + H_{\mu\nu} \wedge \delta \Omega_{\mu\nu} \]

examine this term:

\[ 2\text{nd structure equation} \]

\[ \delta \Omega^{\mu\nu} = \delta \left( d\omega^{\mu\nu} + \omega^\rho \wedge \omega^{\mu\nu} \right) \]

\[ = d\delta \omega^{\mu\nu} + \delta \omega^\rho \wedge \omega^{\mu\nu} + \omega^\rho \wedge \delta \omega^{\mu\nu} \]

\[ \Rightarrow \quad H_{\mu\nu} \wedge \delta \Omega^{\mu\nu} = d(H_{\mu\nu} \wedge \delta \omega^{\mu\nu}) - (dH_{\mu\nu}) \wedge \delta \omega^{\mu\nu} \]

+ \( H_{\mu\nu} \wedge \delta \omega^\rho \wedge \omega^{\mu\nu} + H_{\mu\nu} \wedge \delta \omega^\rho \wedge \delta \omega^{\mu\nu} \)

by def. of \( D \):

\[ = (\delta \omega^{\mu\nu}) \wedge DH_{\mu\nu} + d(H_{\mu\nu} \wedge \delta \omega^{\mu\nu}) \]
\[ \Omega^{\nu} = \Omega^{\nu} + \omega^{\nu}_{\rho} \wedge \omega^{\rho}_{\nu} \]

\[ = d(\delta \omega^{\nu}) + \delta \omega^{\nu} \wedge \omega^{\rho}_{\nu} + \omega^{\nu}_{\rho} \delta \omega^{\rho} \wedge \omega^{\rho}_{\nu} \]

\[ \Rightarrow H_{\mu \nu} \delta \Omega^{\nu} = d(H_{\mu \nu} \delta \omega^{\nu}) - (\partial H_{\mu \nu}) \wedge \delta \omega^{\nu} + H_{\mu \nu} \delta \omega^{\nu} \wedge \omega^{\rho}_{\nu} + H_{\mu \nu} \omega^{\nu}_{\rho} \delta \omega^{\rho} \wedge \omega^{\rho}_{\nu} \]

by Def. of D:

\[ = (\delta \omega^{\nu}) \wedge D H_{\mu \nu} + d(H_{\mu \nu} \delta \omega^{\nu}) \]

recall: \( = 0 \)

by Prop. above.

\[ \Rightarrow \text{Indeed:} \]

\[ S(R) = (\delta \Theta^{\nu}) \wedge H_{\mu \nu} \delta \omega^{\nu} + d(H_{\mu \nu} \delta \omega^{\nu}) \]
\[ \omega_{\mu\nu} = \sigma_\mu^{\nu\rho} + \omega_\nu^\rho \land \omega_\rho^\mu + \omega_\rho^\nu \land \delta\omega_\rho^\mu \]

\[ \Rightarrow H_{\mu\nu} \delta\omega_\rho^{\nu\rho} = d(H_{\mu\nu} \delta\omega_\rho^{\nu\rho}) - (dH_{\mu\nu}) \land \delta\omega_\rho^{\nu\rho} \]

\[ + H_{\mu\nu} \delta\omega_\rho^{\nu\rho} \land \omega_\rho^\mu + H_{\mu\nu} \omega_\rho^\nu \land \delta\omega_\rho^\mu \]

by def. of D:

\[ = (\delta\omega_\rho^{\nu\rho}) \land D H_{\mu\nu} + d(H_{\mu\nu} \delta\omega_\rho^{\nu\rho}) \]

recall: \[ = 0 \]
by Prop. above.

\[ \Rightarrow \text{Indeed:} \]

\[ S(\ast R) = (\delta\Theta^\nu) \land H_{\mu\nu} \land \Omega_\rho^{\nu\rho} + d(H_{\mu\nu} \delta\omega_\rho^{\nu\rho}) \checkmark \]

\[ \Rightarrow \text{The Einstein eqn. indeed follows from local Lorentz invariance.} \]
\[
\Rightarrow H_{\mu \nu} \delta \Omega^{\mu \nu} = d(H_{\mu \nu} \delta \omega^{\mu \nu}) - (dH_{\mu \nu}) \wedge \delta \omega^{\mu \nu} + H_{\mu \nu} \delta \omega^{\mu \nu} \wedge \delta \omega^{\mu \nu} + H_{\mu \nu} \wedge \delta \omega^{\mu \nu} \\
\text{by def. of } D \\
= (\delta \omega^{\mu \nu}) \wedge D H_{\mu \nu} + d(H_{\mu \nu} \delta \omega^{\mu \nu})
\]

\[\Rightarrow \text{Indeed:}\]

\[
S(\star R) = (S \Theta^\prime) \wedge H_{\mu \nu} \wedge \delta \omega^{\mu \nu} + d(H_{\mu \nu} \delta \omega^{\mu \nu})
\]

\[\Rightarrow \text{The Einstein eqn. indeed follows from local Lorentz invariance.}\]