Causal Structure & "Singularities"

Recall:

- The "chronological future" of a set $S$ is the set $I^+(S)$ of events that can be reached from $S$ on a future-directed timelike curve.

- The "causal future" of a set $S$ is the set $J^+(S)$ of events that can be reached from $S$ on a future-directed causal curve.

- In Minkowski space: $J^+(S) = I^+(S) + I^+(S)$ recall: includes null curves.
Properties of \( \mathcal{I}^+(s) \), in general?

**Definition:**

A subset \( Q \subseteq M \) is called "achronal" if no two points in \( Q \) can be connected by a future-directed timelike curve, i.e., by a curve that, e.g., a clock (with mass) could travel. Thus, \( Q \subseteq M \) is achronal if:

\[ T^+(Q) \cap Q = \emptyset \]
(with mass) could travel. Thus, $Q \in \mathcal{M}$

in achronal iff:

$$I^+(Q) \cap Q = \emptyset$$

---

**Theorem:**

For all $S \in \mathcal{M}$, the set

$$\dot{I}^+(S)$$

(if not empty) is an achronal 3-dimensional submanifold of $\mathcal{M}$. 
Theorem:

For all $S \subset M$, the set $\tilde{I}^+(S)$ (if not empty) is an achronal 3-dimensional submanifold of $M$.

Example: In Minkowski space, if $S$ is a point $p$, then $\tilde{I}^+(p)$ is the boundary of the light cone.

Indeed, no two points of the boundary of the light cone are connected by time-like paths.
Theorem:

For all $S \subseteq M$, the set $\hat{I}^+(S)$ (if not empty) is an achronal 3-dimensional submanifold of $M$.

Example: In Minkowski space, if $S$ is a point $p$, then $\hat{I}^+(p)$ is the boundary of the light cone.

Indeed, no two points of the boundary of the lightcone are connected by time-like paths.
Theorem:

For all \( S \subset M \), the set \( \overline{I^+(S)} \) (if not empty) is an achronal 3-dimensional submanifold of \( M \).

Example: In Minkowski space, if \( S \) is a point \( p \), then \( \overline{I^+(p)} \) is the boundary of the light cone.

Indeed, no two points of the boundary of the lightcone are connected by time-like paths.
Theorem:

For all $S \subset M$, the set

$$\hat{I}^+(S)$$

(if not empty) is an achronal 3-dimensional submanifold of $M$.

Example: In Minkowski space, if $S$ is a point $p$, then $\hat{I}^+(p)$ is the boundary of the light cone.

Indeed, no two points of the boundary of the lightcone are connected by time-like paths.
for all $S \subseteq M$, the set

$\mathcal{I}^+(S)$

(if not empty) is an achronal 3-dimensional submanifold of $M$.

Example: In Minkowski space, if $S$ is a point $p$, then $\mathcal{I}^+(p)$ is the boundary of the light cone.

Indeed, no two points of the boundary of the lightcone are connected by time-like paths.
I^+(p) is the boundary of the light cone.

Indeed, no two points of the boundary of the lightcone are connected by time-like paths.

- In general space times, however:

It is clear that  \( \overline{J^+(s)} = \overline{I^+(s)} \)

but we notice that, generally:

\( J^+(s) \subset \overline{I^+(s)} \)
In general space times, however:

It is clear that

\[ \overline{J^+(s)} = \overline{I^+(s)} \]

but we notice that, generally:

\[ J^+(s) \subseteq \overline{I^+(s)} \]

Example:

\[ I^+(p) \]

\[ q \in I^+(p) \quad \text{but} \quad q \notin J^+(p) \]

Assume we have removed a point from the manifold.

\[ p \quad \text{for simplicity, we choose} \quad S = \Sigma P \]

Here, \( q \in \overline{I^+(p)} \), but \( q \notin J^+(p) \) because there...
It is clear that

\[ \overline{J^+(s)} = \overline{I^+(s)} \]

but we notice that, generally:

\[ J^+(s) \subset \overline{I^+(s)} \]

**Example:**

Assume we have removed a point from the manifold for simplicity, we choose \( S = E_p^3 \).

\( p \)

\( q \in \overline{I^+(p)} \) but \( q \notin J^+(p) \)

\( \Rightarrow \) here, \( q \in \overline{I^+(p)} \) but \( q \notin J^+(p) \) because there is no nonspacelike curve between \( p \) and \( q \).
Idea:

Let us use the extendibility or nonextendibility of curves (or especially of geodesics) as indicator for the absence or existence of a singularity.

Definition:

We say that a point $p \in M$ is future (past) end point of a curve $\gamma$ if

A neighborhood $U$ of $p$ there exists a $t_0 \in \mathbb{R}$ so that

$\gamma(t) \in U \quad \forall \, t > \, t_0$ \hspace{1cm} ($t < t_0$ for past end point)

Note: $p$ need not be reached by $\gamma$!
Idea:

Let us use the extendibility or nonextendibility of curves (or especially of geodesics) as an indicator for the absence or existence of a singularity.

Definition:

1. We say that a point $p \in M$ is future (past) end point of a curve $\gamma$ if

   A neighborhood $U$ of $p$ there exists a $t_0 \in \mathbb{R}$ so that

   $\gamma(t) \in U \quad \forall \, t > t_0$ \hspace{1cm} (end point)

   or

   $\gamma(t) \not\in U \quad \forall \, t < t_0$ \hspace{1cm} (end point)

Note: $p$ need not be reached by $\gamma$!
Idea:

Let us use the extendibility or nonextendibility of curves (or especially of geodesics) as an indicator for the absence or existence of a singularity.

Definition:

We say that a point $p \in M$ is future (past) end point of a curve $\gamma$ if

A neighborhood $U$ of $p$ there exists a $t_0 \in \mathbb{R}$ so that

$\gamma(t) \in U \quad A \ t > t_0$ (future end point)

(\(t < t_0\) for past end point)

Note: $p$ need not be reached by $\gamma$!
Example:

\[ J^+(s) \subset I^+(s) \]

By assumption we have removed a point from the manifold for simplicity, we choose \( S = \Sigma p^i \).

\[ q \in I^+(p) \text{ but } q \notin J^+(p) \]

\( \Rightarrow \) here, \( q \in \overline{I^+(p)} \), but \( q \notin \overline{J^+(p)} \) because there is no nonspecial curve between \( p \) and \( q \).

\[ \Rightarrow \text{ Idea:} \]

Let us use the extendibility or nonextendibility of curves (or especially of geodesics) as indicator for the absence or existence of a singularity.
There is no nonspacelike curve between $p$ and $q$.

**Idea:**

Let us use the extendibility or nonextendibility of curves (or especially of geodesics) as an indicator for the absence or existence of a singularity.

**Definition:**

We say that a point $p \in M$ is future (past) and point of a curve $\gamma$ if for a neighborhood $U$ of $p$ there exists a $t_0 \in \mathbb{R}$ so that $\gamma(t) \in U$ for $t > t_0$ (or $t < t_0$).
Idea:

Let us use the extendibility or nonextendibility of curves (or especially of geodesics) as an indicator for the absence or existence of a singularity.

Definition:

We say that a point \( p \) is future (past) and point of a curve \( \gamma \) if

\[ \forall \text{ neighborhoods } U \text{ of } p \text{ there exists } t_0 \in \mathbb{R} \text{ so that } \gamma(t) \in U \quad (t > t_0) \quad (t < t_0) \text{ for past end point} \]

Note: \( p \) need not be reached by \( \gamma \)!
Idea:

Let us use the extendibility or nonextendibility of curves (or especially of geodesics) as indicator for the absence or existence of a singularity.

Definition:

- We say that a point \( p \in \mathbb{M} \) is future (past) end point of a curve \( \gamma \) if
  
A neighborhood \( U \) of \( p \) there exists a \( t_0 \in \mathbb{R} \) so that

\[ \gamma(t) \in U \quad \forall t > t_0 \] (future end point)

\[ \gamma(t) \in U \quad \forall t < t_0 \] (past end point)

Note: \( p \) need not be reached by \( \gamma \)!
Idea:

Let us use the extendibility or nonextendibility of curves (or especially of geodesics) as an indicator for the absence or existence of a singularity.

Definition:

We say that a point \( p \) \textbf{in future (past) end point of a curve} \( \gamma \) if

A neighborhoods \( U \) of \( p \) there exists a \( t_0 \in \mathbb{R} \) so that

\[ \gamma(t) \in U \quad A \ t > t_0, \quad (t < t_0 \text{ for past end point}) \]

Note: \( p \) need not be reached by \( \gamma \)!
Let us use the extendibility or nonextendibility of curves (or especially of geodesics) as an indicator for the absence or existence of a singularity.

**Definition:**

- We say that a point $p \in M$ is future (past) end point of a curve $\gamma$ if

  A neighborhood $U$ of $p$ there exists a $t_0 \in \mathbb{R}$ so that

  $$\gamma(t) \in U \quad A \quad t \geq t_0$$

  (in past)

- Note: $p$ need not be reached by $\gamma$. 
Libility of curves (or especially of geodesics) as indicator for the absence or existence of a singularity.

**Definition:**

- We say that a point $p \in M$ is future (past) end point of a curve $\gamma$ if
  - A neighborhood $U$ of $p$ there exists a $t_0 \in \mathbb{R}$ so that $\gamma(t) \in U \quad t > t_0$ (future end point)
  - $\gamma(t) \in U \quad t < t_0$ (past end point)

**Note:** $p$ need not be reached by $\gamma$.

- Since $p \in M$, it is always possible to extend...
Definition:

- We say that a point \( p \in M \) is future (past) end point of a curve \( y \) if

  In neighborhoods \( U \) of \( p \) there exists a \( t_0 \in \mathbb{R} \) so that

  \[ y(t) \in U \quad A \quad t > t_0 \quad (t < t_0 \text{ for past end point}) \]

- Note: \( p \) need not be reached by \( y \! \).

Since \( p \in M \), it is always possible to extend a curve \( y \) continuously beyond \( p \). Note: \( p \in M \! \).

\[ \Rightarrow \text{ The existence of an endpoint here indicates not} \]

a singularity, or ’hole’ in the manifold, but merely that we chose
Since $p \in M$, it is always possible to extend a curve $\gamma$ continuously beyond $p$. Note: $p \in M$!

$\Rightarrow$ The existence of an endpoint here indicates not a singularity, or 'hole' in the manifold, but merely that we chose to end the curve before it is necessary!

**Definition:**

- We say that a curve $\gamma$ is future (past) in extendible if it does not possess a future (past) endpoint.

**Intuition:** If $\gamma$ future in extendible, then:

1) $\gamma$ runs to $\infty$, or
2) $\gamma$ runs around forever, or
3) $\gamma$ hits a hole i.e. singularity, thus can’t be extendible.
Since \( p \in M \), it is always possible to extend a curve \( \gamma \) continuously beyond \( p \). Note: \( p \in M \)!

\( \Rightarrow \) The existence of an endpoint here indicates not a singularity, or 'hole' in the manifold, but merely that we chose to end the curve before it is necessary!

**Definition:**

- We say that a curve \( \gamma \) is future (past) inextendible if it does not possess a future (past) endpoint.

**Intuition:** If \( \gamma \) future inextendible, then:

1. \( \gamma \) runs to \( \infty \), or
2. \( \gamma \) runs around forever, or
3. \( \gamma \) hits a hole i.e. singularity, thus can't be continuously extended:

\[ \text{hole in manifold} \]
Since $p \in M$, it is always possible to extend a curve $\gamma$ continuously beyond $p$. Note: $p \in M$!

⇒ The existence of an endpoint here indicates not a singularity, or 'hole' in the manifold, but merely that we chose to end the curve before it is necessary!

**Definition:**

- We say that a curve $\gamma$ is future (post) inextendible if it does not possess a future (past) endpoint.

**Intuition:** If $\gamma$ future inextendible, then:

a) $\gamma$ runs to $\infty$, or

b) $\gamma$ runs around forever, or

c) $\gamma$ hits a hole i.e. singularity, thus can't be continuously extended.
We say that a point p ∈ M is future (past) end point of a curve γ if a neighborhood U of p there exists a t ∈ \( R \) so that

\[ γ(t) \in U \quad \forall \ t > t_0 \quad (t < t_0 \text{ for past endpoint}) \]

Note: p need not be reached by γ!

Since p ∈ M, it is always possible to extend a curve γ continuously beyond p. Note: p ∈ M!

⇒ The existence of an endpoint here indicates not a singularity, a ‘hole’ in the manifold, but merely that we chose to end the curve before it is necessary!
of a curve \( \gamma \) if

A neighborhood \( U \) of \( p \) there exists a \( t_0 < R \) so that

\[ \gamma(t) \in U \quad \forall \ t > t_0 \quad (t < t_0 \text{ in past endpoint}) \]

\( \square \) Note: \( p \) need not be reached by \( \gamma \).

\( \square \) Since \( p \in M \), it is always possible to extend a curve \( \gamma \) continuously beyond \( p \). Note: \( p \in M \).

\( \Rightarrow \) The existence of an endpoint here indicates not a singularity, or 'hole' in the manifold, but merely that we chose to end the curve before it is necessary!

Definition:

We can that a curve \( \gamma \) in future (past)
Note: p need not be reached by \( y \).

Since \( p \in M \), it is always possible to extend a curve \( y \) continuously beyond \( p \). Note: \( p \in M \)!

\[ \Rightarrow \] The existence of an endpoint here indicates not a singularity, or 'hole' in the manifold, but merely that we chose to end the curve before it is necessary!

**Definition:**

We say that a curve \( y \) is future (past) inextendible if it does not possess a future (past) endpoint.

**Intuition:** If \( y \) future inextendible, then:

a) \( y \) runs to \( \infty \), or
Definition:

- We say that a curve $\gamma$ is future (past) inextendible if it does not possess a future (past) endpoint.

Intuition: If $\gamma$ future inextendible, then:

a) $\gamma$ runs to $\infty$, or
b) $\gamma$ runs around forever, or
c) $\gamma$ hits a hole, i.e., singularity, thus can't be continuously extended.

Theorem:

Assume $\mathcal{C}M$ is closed and assume:
**Definition:**

- We say that a curve $y$ is future (past) inextendible if it does not possess a future (past) endpoint.

- Intuition: If $y$ is future inextendible, then:
  
  a) $y$ runs to $\infty$, or

  b) $y$ runs around forever, or

  c) $y$ hits a hole i.e. singularity, thus can't be continuously extended.

**Theorem:**

Assume $\mathcal{C} \subset M$ is closed and assume:

$q \in \mathcal{I}^+(\mathcal{C}), \quad q \notin \mathcal{C}$
Theorem:

Assume \( C \subset M \) is closed and assume:

\[ q \in \mathring{I}^+(C), \ q \notin C \]

Then, \( q \) lies on a null geodesic \( \gamma \) inside \( \mathring{I}^+(C) \)
and the curve \( \gamma \) either:

a.) has past endpoint on \( C \)

or b.) is past inextendible (because meets hole)

Example for b:
a) has past endpoint on $G'$

or b) is past inextendible (because meets hole)

**Example for b:**

\[ I^+(G) \]

\[ I^+(G') \]

\[ \text{past inextendible geodesic } \gamma \]

\[ q \]

\[ \text{hole} \]

**Strategy:**

- Study inextendible curves!

- This includes these cases:
  
  a) $\gamma$ hits singularity - will be main interest!

  b) $\gamma$ remains off to $\partial M$.
or b) in past inextendible (because meets hole)

Example for b:

\[ I^+(C) \]
\[ I^-(C') \]

\[ q \]

hole

Strategy:

- Study inextendible curves!
- This includes these cases:
  a) \( y \) hits singularity - will be main interest!
  b) \( y \) running off to \( \infty \)
Strategy:

- Study inextendible curves!
- This includes these cases:
  a) $\gamma$ hits singularity - will be main interest!
  b) $\gamma$ running off to $\infty$.
  c) $\gamma$ going round and round forever.

Must address potential causality problems of case c)

Example:
or \( b \) is past inextendible (because meets hole)

Example for \( b \):

**Strategy:**

- Study inextendible curves!
- This includes these cases:
  - \( a \): \( y \) hits singularity - will be main interest!
  - \( b \): \( y \) running off to \( \infty \).
or b) in past inextendible (because meets hole)

Example for b:

\[ I^+ (C) \]

\[ I^- (C) \]

\[ \text{hole} \]

\[ q \]

Strategy:

- Study inextendible curves!
- This includes these cases:
  - a) \( \gamma \) hits singularity - will be main interest!
  - b) \( \gamma \) running off to \( \infty \)
  - c) \( \gamma \) going round and round forever.
Strategy:

- Study inextendible curves!
- This includes these cases:
  a) $y$ hits singularity - will be main interest!
  b) $y$ running off to $\infty$.
  c) $y$ going round and round forever.

- Must address potential causality problems of case c)
Strategy:

- Study inextendible curves!
- This includes these cases:
  a) \( \gamma \) hits singularity - will be main interest!
  b) \( \gamma \) running off to \( \infty \).
  c) \( \gamma \) going round and round forever.

\( \Rightarrow \) Must address potential causality problems of case c)

Example:

\[ \text{the forward light comes} \]
\[ \text{a closed timelike curve} \]
This includes these cases:

a) $\mu$ hits singularity — will be main interest!

b) $\mu$ running off to $\infty$.

c) $\mu$ going round and round forever.

Must address potential causality problems of case c)

Example:

Note: No problem with time-orientability here!

Causality conditions:
Causality conditions:

- We say that \((M, g)\) is "causal" if it does not contain closed causal (i.e. time-like) curves.

**Problem:** \((M, g)\) may nevertheless be arbitrarily close to being acausal:

\((M, g)\) is the boundary of a cylinder.

Those light cones could be arbitrarily close to the picture above, allowing a closed timelike curve.
Causality conditions:

We say that \((M, g)\) is "causal" if it does not contain closed causal (i.e. time or null) curves.

Problem: \((M, g)\) may nevertheless be arbitrarily close to being acausal:

\((M, g)\) is the boundary of a cylinder.

Those light cones could be arbitrarily close to the picture above i.e. to allowing a closed tornado-like curve!
We say that a spacetime $(\mathcal{M},g)$ is "strongly causal", if

\[ \forall p \text{ and } \forall \text{ neighborhoods } \mathcal{U} \text{ of } p \text{ there is a neighborhood } \mathcal{V} \subset \mathcal{U} \text{ so that:} \]

No causal curve $\gamma$ intersects $\mathcal{V}$ more than once.

Indeed:

If $(\mathcal{M},g)$ is not strongly causal $\Rightarrow$ there exists a causal curve $\gamma$ which comes arbitrarily close to intersecting itself.
We say that a spacetime \((M, g)\) is "strongly causal", if

\[ \forall p \text{ and } \forall \text{ neighborhoods } U \text{ of } p \text{ there is a neighborhood } V \subset U \text{ so that:} \]

\[ \text{No causal curve } \gamma \text{ intersects } V \text{ more than once.} \]

Indeed:

\[ (M, g) \text{ is not strongly causal } \Rightarrow \text{ there exists a causal curve } \gamma \text{ which comes arbitrarily close to intersecting itself.} \]

We require strong causality to keep causal curves at
We say that a spacetime \((M, g)\) is "strongly causal", if

\[\forall p \text{ and } \forall \text{ neighborhoods } U \text{ of } p \text{ there is a neighborhood } V \subset U \text{ so that:} \]

\[\text{No causal curve } \gamma \text{ intersects } V \text{ more than once.} \]

Indeed:

If \((M, g)\) is not strongly causal \(\Rightarrow\) there exists a causal curve \(\gamma\) which comes arbitrarily close to intersecting itself.

We require strong causality to keep causal curves at least a finite distance from intersecting themselves.
contain closed causal (i.e. time or null) curves.

Problem: \((M, g)\) may nevertheless be arbitrarily close to being acausal:

\((M, g)\) is the boundary of a cylinder.

Those light cones could be arbitrarily close to the picture above i.e. to allowing a closed timelike curve!

We say that a spacetime \((M, g)\) is "strongly..."
We say that a spacetime \((M, g)\) is "\text{strongly causal}\), if

\[ \forall p \text{ and } \forall \text{ neighborhoods } U \text{ of } p \therefore \text{ there is a neighborhood } V \subseteq U \text{ so that:} \]

\[ \text{No causal curve } \gamma \text{ intersects } V \text{ more than once.} \]
We say that a spacetime \((M, g)\) is "strongly causal", if

\[\forall p \text{ and } \forall \text{ neighborhoods } U \text{ of } p \text{ there is a neighborhood } V < U \text{ so that:}\]

No causal curve \(p\) intersects \(V\) more than once.

Indeed:

If \((M, g)\) is not strongly causal \(\Rightarrow\) there exists
We say that a spacetime \((M, g)\) is "strongly causal", if

\[
\forall p \text{ and } \forall \text{ neighborhoods } U \text{ of } p \text{ there is a neighborhood } V \subset U \text{ so that:}
\]

No causal curve \(\gamma\) intersects \(V\) more than once.

Indeed:\n
\[
\text{If } (M, g) \text{ is not strongly causal } \Rightarrow \text{ there exists}
\]
We say that a spacetime \((M, g)\) is "**strongly causal**, if

\[ V_p \] and \( U \) neighborhoods \( U \) of \( p \) there is a neighborhood \( \mathcal{V} \subset U \) so that:

No causal curve \( \gamma \) intersects \( \mathcal{V} \) more than once.

Indeed:

If \((M, g)\) is not strongly causal \(\implies\) there exists a causal curve \( \gamma \) which comes arbitrarily close to intersecting itself.

We require strong causality to keep causal curves at least a finite distance from intersecting themselves.
We say that a spacetime \((M, g)\) is **strongly causal** if

\[ A_p \text{ and } A \text{ neighborhoods } U \text{ of } p \text{ there is a neighborhood } V \subset U \text{ so that:}
\]

No causal curve \(\gamma\) intersects \(V\) more than once.

Indeed:

If \((M, g)\) is not strongly causal \(\Rightarrow\) there exists a causal curve \(\gamma\) which comes arbitrarily close to intersecting itself.

\[ \Rightarrow \text{ We require strong causality to keep causal curves at least a finite distance from intersecting themselves.} \]
Indeed:

If \((M, g)\) is not strongly causal \(\Rightarrow\) there exists a causal curve \(\gamma\) which comes arbitrarily close to intersecting itself.

We require strong causality to keep causal curves at least a finite distance from intersecting themselves.

Problem:

Still, arbitrarily small perturbations in the metric, somewhere, could allow causal curves to self-intersect!
Indeed:

If \((M, g)\) is not strongly causal \(\Rightarrow\) there exists a causal curve \(\gamma\) which comes arbitrarily close to intersecting itself.

\[ \Rightarrow \text{We require strong causality to keep causal curves at least a finite distance from intersecting themselves.} \]

Problem:

Still, arbitrarily small perturbations in the metric, somewhere, could allow causal curves to self-intersect!

Solution:
If \((M,g)\) is not strongly causal \(\Rightarrow\) there exists a causal curve \(\gamma\) which comes arbitrarily close to intersecting itself.

\[\text{We require strong causality to keep causal curves at least a finite distance from intersecting themselves.}\]

- **Problem:**

  Still, arbitrarily small perturbations in the metric, somewhere, could allow causal curves to self-intersect!

- **Solution:**
Problem:

Still, arbitrarily small perturbations in the metric, somewhere, could allow causal curves to self-intersect!

Solution:

a) Consider perturbing the metric $g$ through

$$g_{\mu \nu} \rightarrow \tilde{g}_{\mu \nu} = g_{\mu \nu} - \omega_{\mu \nu}$$

with a time-like cotangent vector field.
Problem:

Still, arbitrarily small perturbations in the metric, somewhere, could allow causal curves to self-intersect!

Solution:

a) Consider perturbing the metric $g$ through

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = g_{\mu\nu} + \omega_{\mu\nu}$$

with a time-like cotangent vector field.

b) Notice: $\tilde{g}_{\mu\nu}$ still has same signature.
Solution:

a) Consider perturbing the metric $g$ through

$$g_{\mu\nu} \rightarrow \mathring{g}_{\mu\nu} = g_{\mu\nu} - \omega_{\mu\nu}$$

with a time-like cotangent vector field.

b) Notice: \( \mathring{g}_{\mu\nu} \) still has same signature

but light cones are now "wider":

If \( \mathring{v}^\mu \mathring{v}^\nu g_{\mu\nu} < 0 \)

then \( \mathring{v}^\mu \mathring{v}^\nu \mathring{g}_{\mu\nu} = \mathring{v}^\mu \mathring{v}^\nu g_{\mu\nu} - \mathring{v}^\mu \omega_{\mu\nu} \mathring{v}^\nu < 0 \)
Solution:

a) Consider perturbing the metric $g$ through

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = g_{\mu\nu} - w_{\mu\nu}$$

with a time-like cotangent vector field.

b) Notice: $\tilde{g}_{\mu\nu}$ still has same signature

but light cones are now "wider":

If $v^\mu v^\nu g_{\mu\nu} < 0$

then $v^\mu v^\nu \tilde{g}_{\mu\nu} = v^\mu v^\nu g_{\mu\nu} - v^\mu v^\nu w_{\mu\nu} < 0$

always $< 0$
b) Notice: $\tilde{g}_{\mu\nu}$ still has same signature but light cones are now "wider":

If $\nabla^\nu \tilde{g}_{\mu\nu} < 0$

then $\sqrt{-\tilde{g}} \nabla^\nu \tilde{g}_{\mu\nu} = \sqrt{-\tilde{g}} \tilde{g}^\nu_{\mu\nu} \nabla^\nu \sqrt{-\tilde{g}} < 0$

always $< 0$

Thus, it is easier for vectors $\nabla^\nu$ to be timelike or null for $\tilde{g}$ than for $g$.

$\Rightarrow$ $(M, \tilde{g})$ has all the causal curves of $(M, g)$ and more!
1. **Solution:**

   a) Consider perturbing the metric $g$ through

   
   
   \[ g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = g_{\mu\nu} - \xi_{\mu\nu} \]

   with a time-like cotangent vector field.

   b) Notice: $\tilde{g}_{\mu\nu}$ still has same signature but light cones are now "wider":

   If $\tilde{v}^{\mu} \tilde{v}^{\nu} \tilde{g}_{\mu\nu} < 0$

   then $\tilde{v}^{\mu} \tilde{v}^{\nu} \tilde{g}_{\mu\nu} = \tilde{v}^{\mu} \tilde{v}^{\nu} \tilde{g}_{\mu\nu} - \tilde{v}^{\mu} \tilde{v}^{\nu} \xi_{\mu\nu}$ always $< 0$
**Solution:**

a) Consider perturbing the metric $g$ through

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = g_{\mu\nu} - \omega_{\mu\nu},$$

with a time-like cotangent vector field.

b) Notice: $\tilde{g}_{\mu\nu}$ still has same signature

but light cones are now "wider":

If $v^\nu v_\nu < 0$

then $v^\nu v_\nu \tilde{g}_{\mu\nu} = v^\nu v_\nu g_{\mu\nu} - v^\nu v_\nu \omega_{\mu\nu} < 0$

always < 0
\[ \psi \psi' g_{\mu \nu} < 0 \]

then \[ \quad \psi \psi' g_{\mu \nu} = \psi \psi' g_{\mu \nu} - \psi \psi' \psi \psi' \psi \psi' g_{\mu \nu} \]

always < 0

Thus, it is easier for vectors \( \psi \) to be timelike or null for \( \psi' \) than for \( \psi \).

\[ \Rightarrow \]

\((M, \gamma)\) has all the causal curves of \((M, g)\), and more!

c) Define:

\((M, g)\) is called "stably causal", if there exists a \( \psi \) so that even \((M, \gamma)\) is causal.
then $\nabla^2 \eta = \nabla^2 \phi = -\nabla^2 \psi < 0$

always $< 0$

Thus, it is easier for vectors $\nu$ to be timelike or null for $\tilde{\eta}$ than for $\phi$.

$\Rightarrow$

$(M, \tilde{\eta})$ has all the causal curves of $(M, \eta)$, and more!

c) Define:

$(M, \eta)$ is called "stably causal", if there exists a $\omega$ so that even $(M, \tilde{\eta})$ is causal.
null for \( \tilde{g} \) than for \( g \).

\[ \Rightarrow \]

\((M, \tilde{g})\) has all the causal curves of \((M, g)\), and more!

c) Define:

\((M, g)\) is called "stably causal", if there exists a \( w \) so that even \((M, \tilde{g})\) is causal.

**Theorem 1:**

\((M, g)\) stably causal \( \Rightarrow \) \((M, g)\) strongly causal.

**Theorem 2:**
null for g then for \( g \).

\[ \Rightarrow \]

\( (M, \tilde{g}) \) has all the causal curves of \( (M, g) \), and more!

c) Define:

\( (M, g) \) is called "stably causal", if there exists a \( \omega \) so that even \( (M, \tilde{g}) \) is causal.

**Theorem 1:**

\( (M, g) \) stably causal \( \Rightarrow \) \( (M, g) \) strongly causal.

**Theorem 2:**

\( (M, g) \) stably causal
Theorem 1: \((M, g)\) stably causal \(\Rightarrow\) \((M, g)\) strongly causal.

Theorem 2: \((M, g)\) stably causal

\[ \exists f \in \mathcal{F}(M) \]
so that \(\nabla f\) is a past-directed time-like vector field.

Remark: This means that \(f\) can be viewed as a cosmic "clock." (It is not unique, however)
**Theorem 1:**

\((M, g)\) stably causal \(\Rightarrow\) \((M, g)\) strongly causal.

**Theorem 2:**

\((M, g)\) stably causal

\[\iff\]

There exists a differentiable function \(f \in \mathcal{F}(\sigma)\) so that \(\partial f\) is a past-directed time-like vector field.

**Remark:**

This means that \(f\) can be viewed as a cosmic "clock." (It is not unique, however)

**Recall:** Time-orientability means that there exists a past-pointing smooth vector field but not that it is a gradient field.
c) Define:

\((M, g)\) is called "stably causal", if there exists a \(w\) so that even \((M, \tilde{g})\) is causal.

**Theorem 1:**

\((M, g)\) stably causal \(\Rightarrow\) \((M, g)\) strongly causal.

**Theorem 2:**

\((M, g)\) stably causal

\[\iff\]

There exists a differentiable function \(f \in \mathcal{C}\).
c) Define:

\((M, g)\) is called "stably causal", if there exists a \(w\) so that even \((M, \tilde{g})\) is causal.

**Theorem 1:**

\((M, g)\) stably causal \(\Rightarrow\) \((M, g)\) strongly causal.

**Theorem 2:**

\((M, g)\) stably causal

\[\text{There exists a differentiable function } f \in F(M)\]

so that \(\partial f\) is a past-directed time-like vector.
c) Define:

$(M, g)$ is called "stably causal", if there exists a $w$ so that even $(M, \tilde{g})$ is causal.

Theorem 1:

$(M, g)$ stably causal $\Rightarrow (M, g)$ strongly causal.

Theorem 2:

$(M, g)$ stably causal $\iff$ There exists a differentiable function $f \in \mathcal{F}(M)$
c) Define:

\((M, g)\) is called "stably causal", if there exists a \(w\) so that even \((M, \tilde{g})\) is causal.

Theorem 1:

\((M, g)\) stably causal \(\Rightarrow\) \((M, g)\) strongly causal.

Theorem 2:

\((M, g)\) stably causal

\[\Updownarrow\]

There exists a differentiable function \(f \in F(M)\) so that \(Df\) is a past-directed time-like vector field.
c) Define:

\((M, g)\) is called "stably causal", if there exists a \(\omega\) so that even \((M, \tilde{g})\) is causal.

**Theorem 1:**

\((M, g)\) stably causal \(\Rightarrow\) \((M, g)\) strongly causal.

**Theorem 2:**

\((M, g)\) stably causal \(\iff\) There exists a differentiable function \(f \in \mathcal{C}(M)\).
Theorem 1:

\((M, g)\) stably causal \(\Rightarrow\) \((M, g)\) strongly causal.

Theorem 2:

\((M, g)\) stably causal

There exists a differentiable function \(f \in F(M)\)
so that \(\nabla f\) is a past-directed time-like vector field.

Remark:

This means that \(f\) can be viewed as a cosmic "clock." (It is not unique, however)

Recall: Time-orientability means that there exists a past-
Theorem 1:

\((M, g)\) stably causal \(\Rightarrow\) \((M, g)\) strongly causal.

Theorem 2:

\((M, g)\) stably causal

\[\Uparrow\]

There exists a differentiable function \(f \in \mathcal{F}(M)\)

so that \(\nabla f\) is a past-directed time-like vector field.

Remark:

This means that \(f\) can be viewed as a cosmic "clock." (It is not unique, however)

Recall: Time-orientability means that there exists a past-pointing smooth vector field but not that it is a gradient field.
Theorem 2: 

\((M,g)\) stably causal

\[\uparrow\]

There exists a differentiable function \(f \in \mathcal{F}(M)\) so that \(\partial f\) is a past-directed time-like vector field.

Remark: This means that \(f\) can be viewed as a cosmic "clock". (It is not unique, however)

Recall: Time-orientability means that there exists a past-pointing smooth vector field but not that it is a gradient field.

We assume that spacetime is stably causal.
Theorem 2: \( (\mathcal{M}, g) \) is causal

There exists a differentiable function \( f \in \mathcal{F}(\mathcal{M}) \) so that \( o_f \) is a past-directed time-like vector field.

Remark:

This means that \( f \) can be viewed as a cosmic "clock." (It is not unique, however)

Recall: Time-orientability means that there exists a past-pointing smooth vector field but not that it is a gradient field.

We assume that spacetime is stably causal.
Theorem 2:

\[(M, g)\] stably causal

\[\Updownarrow\]

There exists a differentiable function \( f \in \mathcal{F}(M) \)
so that \( \partial f \) is a past-directed time-like vector field.

Remark:

This means that \( f \) can be viewed as a cosmic "clock". (It is not unique, however)

Recall: Time-orientability means that there exists a past-pointing smooth vector field but not that it is a gradient field.

We assume that spacetime is stably causal.
Remark: This means that \( f \) can be viewed as a cosmic "clock". (It is not unique, however)

Recall: Time-orientability means that there exists a past-timelike pointing smooth vector field but not that it is a gradient field.

\[ \Rightarrow \text{We assume that spacetime is stably causal.} \]

**Intuition:**

Therefore, inextendible paths either:

a) go to \( \infty \), or

b) end in a singularity

\[ \Rightarrow \text{Continue to study inextendible curves:} \]
Remark: This means that $f$ can be viewed as a cosmic "clock." (It is not unique, however)

Recall: Time-orientability means that there exists a past-timelike pointing smooth vector field but not that it is a gradient field.

We assume that spacetime is stably causal.

Intuition: Therefore, inextendible paths either:

a) go to $\infty$, or
b) end in a singularity

Continue to study inextendible curves:
Remark: This means that $f$ can be viewed as a cosmic "clock." (It is not unique, however)

Recall: Time-orientability means that there exists a past-pointing smooth vector field but not that it is a gradient field.

We assume that spacetime is stably causal.

Intuition: Therefore, inextendible paths either:

a) go to $\infty$, or

b) end in a singularity

Continue to study inextendible curves:
This means that $f$ can be viewed as a cosmic "clock". (It is not unique, however.)

Recall: Time-orientability means that there exists a past-timelike pointing smooth vector field but not that it is a gradient field.

We assume that spacetime is stably causal.

Intuition: Therefore, inextendible paths either:

a) go to $\infty$, or

b) end in a singularity

Continue to study inextendible curves:
as a cosmic "clock". (It is not unique, however)

Recall: Time-orientability means that there exists a past-pointing smooth vector field but not that it is a gradient field.

We assume that spacetime is stably causal.

Intuition:

Therefore, inextendible paths either:

a.) go to \( \infty \), or

b.) end in a singularity

继续 to study inextendible curves:

Recall:
a) go to \( \infty \), or
b) end in a singularity

Continue to study inextendible curves:

\[ \text{Recall:} \]

- We considered the set of points \( J^+(S) \) that can somehow be reached from a set \( S \). (i.e. the set of points that are affected by \( S' \))
- Now consider set of points that can only be reached from \( S \): (i.e. the set of events that depend on \( S \) and only \( S' \))

\[ \text{Definition:} \]

Assume \( S_{CRM} \) is a closed achronal set.
Continue to study inextendible curves:

Recall:

- We considered the set of points $J^+(S)$ that can somehow be reached from a set $S$. (i.e. the set of points that are affected by $S$)
- Now consider set of points that can only be reached from $S$: (i.e. the set of events that depend on $S$ and only $S$)

Definition:

Assume $S$ is a closed achronal set. Then, the "future domain of dependence of $S" is defined as:

$D^+(S) = \{ x \in M \mid x \text{ can be joined to } S \}$

Every past inextendible causal
**Definition:**

Assume $S \subset M$ is a closed achronal set. Then, the "future domain of dependence of $S$" is defined as:

$$D^+(S) := \{ p \in M \mid \text{every past inextendible causal curve through } p \text{ intersects } S \}$$

**Example:**

A past inextendible smooth curve that does not intersect $S$.

**Question:** Why $q \notin D^+(S)$? Some of its past inextendible curves intersect $S$. 
In defined as:

$$D^+(S') := \{ p \in \mathbb{N} \mid \text{Every past inextendible causal curve through } p \text{ intersects } S \}$$

**Example:**

![Diagram of a spacetime slice with a past inextendible smooth curve that does not intersect \( S \)]

Why \( q \notin D^+(S) \)? Some of its past inextendible causal curves do not intersect \( S \) because they get stuck at the hole!

(\( q \) is affected by events in the "shadow" of the hole)
\[ D^+(S) := \{ p \in M \mid \text{Every past inextendible causal curve through } p \text{ intersects } S \} \]

**Example:**

Why \( q \notin D^+(S) \)? Some of its past inextendible causal curves do not intersect \( S \) because they get stuck at the hole!

\( q \) is affected by events in the "shadow" of the hole.

**Definition:**

Analogously, the "past domain of dependence of \( q \)" is:
**Example:**

A past inextendible smooth curve that does not intersect $S$.

Why $q \notin D^+(S)$? Some of its past inextendible causal curves do not intersect $S$ because they get stuck at the hole!

$q$ is affected by events in the "shadow" of the hole.

**Definition:**

Analogously, the "past domain of dependence of $S'" is:

$$D^-(S') := \{ p \in M \mid \text{Every future inextendible causal curve through } p \text{ intersects } S' \}$$
Why \( q \notin D^+(S) \)? Some of its past inextendible causal curves do not intersect \( S \) because they get stuck at the hole!

(\( q \) is affected by events in the "shadow" of the hole)

**Definition:**

Analogously, the "past domain of dependence of \( S \)" is:

\[
D^-(S) := \{ p \in M \mid \text{Every future inextendible causal curve through } p \text{ intersects } S \}
\]

(The set of events \( p \) that affect only \( S \))

**Definition:**

The "full domain of dependence of \( S \)" is:

\[
D(S) := D^+(S) \cup D^-(S)
\]
is defined as:

\[ D^+(s') := \{ p \in \mathbb{R} \mid \text{Every past inextendible causal curve through } p \text{ intersects } s \} \]

**Example:**

Why \( q \notin D^+(s) \)? Some of its past inextendible causal curves do not intersect \( S \) because they get stuck at the hole!

\( q \) is affected by events in the "shadow" of the hole.

**Definition:**

Analogously, the "past domain of dependence of \( s' \)"...
Definition:

Analogously, the "past domain of dependence of $S'$" is:

$$D^-(S') := \left\{ p \in M \mid \text{Every future inextendible causal curve through } p \text{ intersects } S \right\}$$

(the set of events $p$ that affect only $S'$)

Definition:

The "full domain of dependence of $S'$" is:

$$D(S') := D^+(S') \cup D^-(S')$$

Definition:

(set of latest events that are affected only by $S'$? How far have initial conditions on $S$ field predictive power?)

The "future Cauchy horizon of $S'$", denoted $H^+(S')$ is:

$$H^+(S') := \overline{D^+(S)} - I^{-}(D^+(S))$$

(Note: $H^+(S)$ is achronal. Why?)
Definition:

Analogously to the "past domain of dependence of \( S' \),"
The past domain of dependence of $S$ is defined as:

$$D^+(S') := \{ p \in M \mid \text{Every past inextendible causal curve through } p \text{ intersects } S \}$$

**Example:**

- $q^*$ is a past inextendible smooth curve that does not intersect $S$.
- $D^+(S)$: Hole in the diagram.
- $D^+(S')$: Arrow indicates past inextendible causal curve which intersects $S$.
- $\{ S' \}$: Indicates the set of all past inextendible causal curves that intersect $S$.

**Why $q \notin D^+(S)$?** Some of its past inextendible causal curves do not intersect $S$ because they get stuck at the hole.

$q$ is affected by events in the "shadow" of the hole.

**Definition:**
**Definition:**

Analogously, the "past domain of dependence of $S'$" is:

$$D^{-}(S') := \{ p \in M \mid \text{Every future inextendible causal curve through } p \text{ intersects } S' \}$$

(the set of events $p$ that affect only $S$)

**Definition:**

The "full domain of dependence of $S'$" is:

$$D(S) := D^{+}(S') \cup D^{-}(S')$$

**Definition:**

The "future Cauchy horizon of $S'$", denoted $H^{+}(S')$.
Analogously, the "past domain of dependence of $S'$" is:

$$D^-(S') := \{ p \in M \mid \text{Every future inextendible causal curve through } p \text{ intersects } S \}$$

(the set of events $p$ that affect only $S$)

**Definition:**

The "full domain of dependence of $S'$" is:

$$D(S') := D^+(S') \cup D^-(S')$$

**Definition:** (set of latest events that are affected only by $S'$? How far have initial conditions on $S$ full predictive power?)

The "future Cauchy horizon of $S'$", denoted $H^+(S)$, is:

$$H^+(S') := \overline{D^+(S')} - I^-(D^+(S'))$$

(Note: $H^+(S')$ is achronal. Why?)
Analogously, the past domain of dependence of $S'$ is:

$$D^{-}(S') := \{ p \in \mathcal{M} \mid \text{Every future inextendible causal curve through } p \text{ intersects } S \}$$

(the set of events $p$ that affect only $S$)

**Definition:**

The "full domain of dependence of $S'$" is:

$$D(S) := D^{+}(S') \cup D^{-}(S')$$

**Definition:**

(set of latest events that are affected only by $S$? How far have initial conditions on $S'$ full predictive power?)

The "future Cauchy horizon of $S'$", denoted $H^{+}(S')$ is:

$$H^{+}(S') := \overline{D^{+}(S')} - I^{-}(D^{+}(S))$$

(Note: $H^{+}(S)$ is achronal Why?)

**Example:**

$$H^{+}(S)$$
\[ D^-(S) := \{ p \in U \mid \text{Every future inextendible causal curve through } p \text{ intersects } S \} \]

**Definition:**

The "full domain of dependence of } S' \text{" is:

\[ D(S) := D^+(S') \cup D^-(S) \]

**Definition:**

(set of latest events that are affected only by } S' \text{. How far have initial conditions on } S \text{ full predictive power?)

The "future Cauchy horizon of } S' \text{", denoted } H^+(S) \text{, is:

\[ H^+(S') := \overline{D^+(S)} - \overline{I^-(D^+(S))} \]

(Note: } H^+(S) \text{ is achronal. Why?)

**Example:**

\[ H^+(S) \]
\[ D^{-}(S') := \{ p \in M \mid \text{curve through } p \text{ intersects } S \} \]

**Definition:**

The "full domain of dependence of } S' \text{" is:

\[ D(S) := D^+(S) \cup D^-(S') \]

**Definition:**

(set of latest events that are affected only by } S \text{. How far have initial conditions on } S \text{ field predictive power?}

The "future Cauchy horizon of } S' \text{", denoted } H^+(S') \text{ is:

\[ H^+(S') := \overline{D^+(S)} - I^-(D^+(S)) \]

(achronal. Why?)

**Example:**

\[ H^+(S') \]
\[ D(S) := \{ p \in M \mid \text{curve through } p \text{ intersects } S \} \]

**Definition:**

The "full domain of dependence of \( S' \)" is:

\[ D(S) := D^+(S') \cup D^-(S') \]

**Definition:**

(set of latest events that are affected only by \( S' \). How far have initial conditions on \( S \) field predictive power?)

The "future Cauchy horizon of \( S' \)," denoted \( H^+(S') \)

\[ H^+(S') := \overline{D^+(S)} - I^-(D^+(S)) \]

*Note:* \( H^+(S') \) is

(acchronal. Why?)

**Example:**

![Diagram of \( H^+(S') \)](image)
The "full domain of dependence of \( S' \)" is:

\[
D(S) := D^+(S) \cup D^-(S)
\]

**Definition:** (set of latest events that are affected only by \( S' \). How far have initial conditions on \( S \) full predictive power?)

The "future Cauchy horizon of \( S' \), denoted \( H^+(S) \)" is:

\[
H^+(S') := \overline{D^+(S) - I^{-}(D^+(S))}
\]

(Note: \( H^+(S) \) is achronal. Why?)

**Example:**

![Diagram showing \( H^+(S') \) and \( D^+(S) \) with a hole and arrows indicating time and space.]

*Analogously:*
The full domain of dependence of \( S' \) is:

\[
D(S) := D^+(S) \cup D^-(S)
\]

**Definition:** set of latest events that are affected only by \( S \). How far have initial conditions on \( S \) field predictive power?

The "future Cauchy horizon of \( S' \)", denoted \( H^+(S') \) is:

\[
H^+(S') := \overline{D^+(S)} \cap I^-(D^+(S))
\]

(Note: \( \Rightarrow H^+(S') \) in)

(achronal. Why?)

**Example:**

![Diagram showing \( H^+(S') \) and \( D^+(S) \)]

analogously:

**Definition:**
**Definition:** (set of latest events that are affected only by S) How far have initial conditions on S field predictive power?

The "future Cauchy horizon of S", denoted $H^+(S)$ is:

$$H^+(S) := \overline{D^+(S)} - \overline{I^-(D^+(S))}$$

(Note: $H^-(S)$ is achronal. Why?)

**Example:**

![Diagram of a spacetime diagram with $H^+(S)$, $D^+(S)$, and $I^-(D^+(S))$ labeled.]

*Analagously*

**Definition:** The "past Cauchy horizon of S", denoted $H^-(S)$
**Definition:**
The "past Cauchy horizon of $S'$", denoted $H^-(S')$ is:

$$H^-(S') := \overline{D^-(S)} \cap I^+(D^-(S))$$

(set of earliest events that affect only $S'$)

**Definition:**
The "full Cauchy horizon of $S'$" is defined as:

$$H(S') := H^-(S) \cup H^+(S)$$
**Example:**

\[ H(S) \]

**Definition:**

The "past Cauchy horizon of \( S' \)", denoted \( H^{-}(S) \), is:

\[ H^{-}(S') := \overline{D^{-}}(S) - I^{+}(D^{-}(S)) \]  
(set of earliest events that affect only \( S') \)

**Definition:**

The "full Cauchy horizon of \( S' \)" is defined as:

\[ H(S') := H^{+}(S) \cup H^{-}(S) \]
Definition:
The "past Cauchy horizon of $S'$", denoted $H^-(S)$, is defined as:
$$H^-(S') := \overline{D^-(S)} - I^+(D^-(S))$$
(set of earliest events that affect only $S'$)

Definition:
The "full Cauchy horizon of $S'$" is defined as:
$$H(S') := H^+(S) \cup H^-(S)$$

Proposition:
$$H(S') = D(S')$$

Definition:
A closed causal set $S'$ is called a
\[ H^+(S') := \overline{D^+(s)} - I^- (D^+(s)) \]

Example:

The "past Cauchy horizon of \( S' \)\), denoted \( H^-(s) \), is defined as:

\[ H^-(S') := \overline{D^-(s)} - I^+ (D^-(s)) \]

(Note: \( H^+(S') \) is achronal. Why?)

Definition:

The "future Cauchy horizon of \( S' \)\) is defined as.
$H(S) := D'(S) - I^-(D'(S))$ (chronal)

Example:

Analogously:

Definition:

The "past Cauchy horizon of $S'$", denoted $H^-(S)$,

is:

$H^-(S) := \overline{D^-(S)} - I^+(D^-(S))$ (set of earliest events that affect only $S$)

Definition:

The "full Cauchy horizon of $S'$" is defined as:
Example:

Definition:
The "past Cauchy horizon of $S'$", denoted $H^-(S)$, is:

$$H^-(S') := D^-(S) - \overline{I^+(D^-(S))}$$

(set of earliest events that affect only $S$)

Definition:
The "full Cauchy horizon of $S'$" is defined as:

$$H(S') := H^+(S) \cup H^-(S)$$
**Example:**

\[ H(S) \]

**analogously:**

**Definition:**

The "past Cauchy horizon of \( S' \)" is denoted \( H^-(S) \).

\[ H^-(S') := D^-(S) - I^+(D^-(S)) \] (set of earliest events that affect only \( S \))

**Definition:**

The "full Cauchy horizon of \( S' \)" is defined as:

\[ H(S) := H^+(S) \cup H^-(S) \]
Definition:
The "past Cauchy horizon of $S'$", denoted $H^-(S)$, is:

$$H^-(S') := \overline{D^-(S)} - I^+(D^-(S))$$

(set of earliest events that affect only $S'$)

Definition:
The "full Cauchy horizon of $S'$" is defined as:

$$H(S') := H^+(S) \cup H^-(S)$$

Proposition:

$$H(S') = \partial(S')$$
\[ H^+(S') := \overline{D^+(s)} - I^-(D^+(s)) \]

(Note: \( \overline{H^+(S')} \) is achronal. Why?)

Example:

\[ H^+(S') \]

analogously:

**Definition:**

The "past Cauchy horizon of \( S' \)" is denoted \( H^-(s) \)

\[ H^-(S') := \overline{D^-(s)} - I^+(D^-(s)) \]

(set of earliest events that affect only \( S' \))

**Definition:**

The "future Cauchy horizon of \( S' \)" is defined as...
**Definition:**

The "past Cauchy horizon of $S'$", denoted $H^-(S')$

\[ H^-(S') := \overline{D^+(S)} - I^+(\overline{D^-(S)}) \]  
(set of earliest events that affect only $S$)

**Definition:**

The "full Cauchy horizon of $S'$" is defined as:

\[ H(S') := H^+(S) \cup H^-(S) \]

**Proposition:**

\[ H(S') = \partial(S') \]

**Definition:**

A closed, achronal set $S'$ is called a
The "full Cauchy horizon of $S'$" is defined as:

$$H(S') := H^+(S) \cup H^-(S)$$

**Proposition:**

$$H(S') = \mathcal{D}(S')$$

**Definition:**

A closed, achronal set $S'$ is called a "Cauchy surface", if its full Cauchy horizon vanishes, i.e. if

a) $H(S) = \emptyset$ or equivalently if

b) $\mathcal{D}(S) = \emptyset$
Proposition:

\[ H(S') = \dot{D}(S') \]

Definition:

A closed, achronal set \( S' \) is called a "Cauchy surface", if its full Cauchy horizon vanishes, i.e. if

1. \( H(S) = \emptyset \) or equivalently if
2. \( \dot{D}(S) = \emptyset \) or equivalently if
3. \( D(S') = M \)

Note: This follows Wald. The definitions by others are equivalent.
Proposition:

\[ H(S') = \dot{D}(S') \]

Definition:

A closed, achronal set \( S' \) is called a "Cauchy surface", if its full Cauchy horizon vanishes, i.e. if

\[ a) \quad H(S) = \emptyset \quad \text{or equivalently if} \]

\[ b) \quad \dot{D}(S) = \emptyset \quad \text{or equivalently if} \]

\[ c) \quad \dot{D}(S') = \mathbb{M} \]

Note: This follows Wald. The definitions by others are equivalent, but more tedious.
b.) \( \mathcal{D}(S) = \emptyset \)

or equivalently if

c.) \( \mathcal{D}(S') = M \)

**Note:** This follows Wald. The definitions by others are equivalent, but more tedious.

**Remarks:**
- Cauchy surfaces are important because if the conditions on a Cauchy surface are known, then everything on \( M \) can be predicted and retrodicted.
- Since a Cauchy surface is achronal, it can be viewed as an "instant in time."
- The term "surface" is motivated by a theorem.
c) \( \mathcal{D} (S') = M \)

**Note:** This follows Wald. The definitions by others are equivalent, but more tedious.

**Remarks:**
- Cauchy surfaces are important because if the conditions on a Cauchy surface are known, then everything on \( M \) can be predicted and retrodicted.
- Since a Cauchy surface is achronal, it can be viewed as an "instant in time".
- The term "surface" is motivated by a theorem:
  
  Every Cauchy surface, \( \Sigma \), is a 3-dimensional \( \mathbb{C}^0 \) submanifold of \( M \).
Remarks:

- Cauchy surfaces are important because if the conditions on a Cauchy surface are known, then everything on $\Sigma$ can be predicted and retrodicted.

- Since a Cauchy surface is achronal, it can be viewed as an “instant in time”.

- The term “surface” is motivated by a theorem:

  Every Cauchy surface, $\Sigma$, is a $3$-dimensional $C^\infty$ submanifold of $M$.

**Definition:**

If $(M,g)$ possesses a Cauchy surface then
Remarks:

- Cauchy surfaces are important because if the conditions on a Cauchy surface are known, then everything on $M$ can be predicted and reintroduced.

- Since a Cauchy surface is achronal, it can be viewed as an "instant in time".

- The term "surface" is motivated by a theorem:

  Every Cauchy surface, $\Sigma$, is a 3-dimensional $C^0$ submanifold of $M$.

Definition:

If $(M,g)$ possesses a Cauchy surface then it is called "globally hyperbolic".
\[ H^+(S') := \overline{D^-(S)} - I^+(D^-(S)) \]

**Example:**

\[ H^+(S') \]

**Definition:**

The "past Cauchy horizon of \( S' \)" is denoted \( H^-(S') \).

\[ H^-(S') := \overline{D^-(S)} - I^+(D^-(S)) \]

(set of earliest events that affect only \( S' \))

**Definition:**

The "light Cauchy horizon of \( S' \)" is defined as...
The future Cauchy horizon of $S$, denoted $H^+(S)$, is:

$$H^+(S) := \overline{D^+(S)} - I^-(D^+(S))$$

(Not: $H^+(S)$ is achronal. Why?)

**Example:**

![Diagram of a causal structure with $H^+(S)$, $D^+(S)$, and $I^-(D^+(S))$]

Analogously:

**Definition:** The "past Cauchy horizon of $S'$", denoted $H^-(S)$, is:

$$H^-(S') := \overline{D^-(S')} - I^+(D^-(S'))$$

(set of earliest events that affect only $S'$)
Proposition:

\[ H(S') = D(S') \]

Definition:

A closed, achronal set \( S \) is called a "Cauchy surface", if its full Cauchy horizon vanishes, i.e. if:

a) \( H(S) = \emptyset \) or equivalently if

b) \( D(S) = \emptyset \) or equivalently if

c) \( D(S') = M \)

Note: This follows Wald. The definitions by others are equivalent, but more tedious.
Remarks:

- Cauchy surfaces are important because if the conditions on a Cauchy surface are known, then everything on $\mathcal{M}$ can be predicted and retrodicted.

- Since a Cauchy surface is achronal, it can be viewed as an "instant in time".

- The term "surface" is motivated by a theorem:

  Every Cauchy surface, $\Sigma$, is a 3-dimensional $C^\infty$ submanifold of $\mathcal{M}$.

Definition:

If $(\mathcal{M}, g)$ possesses a Cauchy surface then
everything on \( M \) can be predicted and retrodicted.

- Since a Cauchy surface is achronal, it can be viewed as an "instant in time".
- The term "surface" is motivated by a theorem:

Every Cauchy surface, \( \Sigma \), is a 3-dimensional \( C^0 \) submanifold of \( M \).

**Definition:**

If \((M, g)\) possesses a Cauchy surface then it is called "globally hyperbolic".
The term "surface" is motivated by a theorem:

Every Cauchy surface, $\Sigma$, is a 3-dimensional $C^0$ submanifold of $M$.

**Definition:**

If $(M, g)$ possesses a Cauchy surface then it is called "globally hyperbolic".

**Proposition:**

If $(M, g)$ is globally hyperbolic, then:

There exists a "global time function $f$" so that every
The term "surface" is motivated by a theorem:

Every Cauchy surface, $\Sigma$, is a 3-dimensional $C^0$ submanifold of $M$.

**Definition:**

If $(M, g)$ possesses a Cauchy surface then it is called "globally hyperbolic".

**Proposition:**

If $(M, g)$ is globally hyperbolic, then:

There exists a "global time function $\phi$" so that every surface of constant $\phi$ is a Cauchy surface.
Definition:
If \((M, g)\) possesses a Cauchy surface then it is called "globally hyperbolic".

Proposition:
If \((M, g)\) is globally hyperbolic, then:

- There exists a "global time function \(f\)" so that every surface of constant \(f\) is a Cauchy surface.
- \((M, g)\) is stably (and therefore also strongly) causal.

Remark: It is clear that any globally hyperbolic \((M, g)\) is causal.
Proposition:

If $(M, g)$ is globally hyperbolic, then:

- There exists a "global time function $f" so that every surface of constant $f$ is a Cauchy surface.

- $(M, g)$ is strongly (and therefore also strongly) causal.

Remark: It is clear that any globally hyperbolic $(M, g)$ is causal:

- Each of its Cauchy surfaces $\Sigma$ is intersected by every causal curve $\gamma$.

- Any causal curve intersects $\Sigma$ but if $\gamma$ is closed this contradicts the achronicity of $\Sigma$. 
Proposition:

1. If \((M, g)\) is globally hyperbolic, then:
   - There exists a "global time function \(f\)" so that every surface of constant \(f\) is a Cauchy surface.
   - \((M, g)\) is stably (and therefore also strongly) causal.

Remark: It is clear that any globally hyperbolic \((M, g)\) is causal:
   - Each of its Cauchy surfaces \(\Sigma\) is intersected by every causal curve \(\gamma\).
   - Any causal curve intersects \(\Sigma\) and if \(\gamma\) is closed this contradicts the achronicity of \(\Sigma\).
If \((M, g)\) is globally hyperbolic, then:

- There exists a "global time function \(f\)" so that every surface \(\Sigma\) of constant \(f\) is a Cauchy surface.
- \((M, g)\) is stably (and therefore also strongly) causal.

Remark: It is clear that any globally hyperbolic \((M, g)\) is causal:

- Each of its Cauchy surfaces \(\Sigma\) is intersected by every causal curve \(\gamma\).
- Any causal curve intersects \(\Sigma\) limit if \(\gamma\) is closed, this contradicts the achronicity of \(\Sigma\).

Recall:
Remark: It is clear that any globally hyperbolic \((M, g)\) is causal:

- Each of its Cauchy surfaces \(\Sigma\) is intersected by every causal curve \(\gamma\).
- Any causal curve intersects \(\Sigma\) limit if \(\gamma\) is closed this contradicts the achronicity of \(\Sigma\).

Recall:

Plan is to study inextendible geodesics in order to detect singularities.

Now: How to identify these geodesics which are inextendible because they end at a hole in the manifold?
Remark: It is clear that any globally hyperbolic \((M, g)\) is causal:

- Each of its Cauchy surfaces \(\Sigma\) is intersected by every causal curve \(\gamma\).

- Any causal curve intersects \(\Sigma\) limit if \(\gamma\) is closed, this contradicts the achronicity of \(\Sigma\).

Recall:

Plan is to study inextendible geodesics in order to detect singularities.

Now: How to identify these geodesics which are inextendible because they end at a hole in the manifold?
There exists a "global time function \( f \)" so that every surface of constant \( f \) is a Cauchy surface.

\( (M, g) \) is stably (and therefore also strongly) causal.

**Remark:** It is clear that any globally hyperbolic \( (M, g) \) is causal:

- Each of its Cauchy surfaces \( \Sigma \) is intersected by every causal curve \( \gamma \).
- Any causal curve intersects \( \Sigma \) and if \( \gamma \)
  in closed thin contradicts the achronicity of \( \Sigma \).

**Recall:**
Recall:

Plan is to study inextendible geodesics in order to detect singularities.

Now: How to identify these geodesics which are inextendible because they end at a hole in the manifold?

First: Avoid trivial cases

Definition:

We say that \((M, g)\) is inextendible, if it is not isometric to a proper subset of another spacetime \((N', g')\).

We will always assume that \((M, g)\) is inextendible.
are inextendible because they end at a hole in the manifold?

**First:** Avoid trivial cases

**Definition:**

We say that \((M, g)\) is inextendible, if it is not isometric to a proper subset of another spacetime \((M', g')\).

\[\Rightarrow\] We will always assume that \((M, g)\) is inextendible.

**Definition:**

A geodesic which is inextendible but possesses
First: Avoid trivial cases

Definition:

We say that \((M, g)\) is inextendible, if it is not isometric to a proper subset of another spacetime \((M', g')\).

We will always assume that \((M, g)\) is inextendible.

Definition:

A geodesic which is inextendible but possesses a finite range of its affine parameter is called "incomplete".

Recall: For space- or timelike geodesics the affine
Definition:

We say that \((M, g)\) is inextendible, if it is not isometric to a proper subset of another spacetime \((M', g')\).

→ We will always assume that \((M, g)\) is inextendible.

Definition:

A geodesic which is inextendible but possesses a finite range of its affine parameter is called "incomplete".

Recall: For space- or timelike geodesics the affine parameter can be chosen to be the proper distance.
We will always assume that \((M,g)\) is inextendible.

**Definition:**

A geodesic which is inextendible but possesses a finite range of its affine parameter is called "incomplete".

**Recall:** For space- or timelike geodesics the affine parameter can be chosen to be the proper distance or proper time respectively.

**Definition:**

We say that \((M,g)\) possesses a singularity...
Definition:

A geodesic which is inextendible but possesses a limit range of its affine parameter is called "incomplete".

Recall: For space- or timelike geodesics the affine parameter can be chosen to be the proper distance or proper time respectively.

Definition:

We say that \((M, g)\) possesses a singularity if it possesses an incomplete geodesic. (i.e. there are time-like, null and spacelike singularities)
A geodesic which is inextendible but possesses a limit range of its affine parameter is called "incomplete".

Recall: For space- or timelike geodesics the affine parameter can be chosen to be the proper distance or proper time respectively.

**Definition:**

We say that \((M, g)\) possesses a singularity if it possesses an incomplete geodesic. (i.e. there are time-like, null and spacelike singularities)
Parameters can be chosen to be the proper distance or proper time respectively.

**Definition:**

1. We say that \((M, g)\) possesses a singularity if it possesses an incomplete geodesic. (i.e. there are time-like, null and space-like singularities)

   a “singularity”, 3 things can happen:

   1. A scalar constructed from \(R_{\Gamma^\alpha}^\beta\),
      
      e.g. \(\tilde{R}, R^{\tau\tau}, R_{\mu\nu}, \) etc diverges.
Definition:

- We say that \((M, g)\) possesses a singularity if it possesses an incomplete geodesic. (i.e. there are time-like, null and space-like singularities)

- When going along an incomplete geodesic towards a "singularity", 3 things can happen:
  1) A scalar constructed from \(R^{\mu}_{\nu\rho\sigma}\), e.g. \(R\), \(R^{\mu
\nu\rho\sigma}R_{\mu\nu\rho\sigma}\), etc diverges.
A geodesic which is inextendible but possesses a finite range of its affine parameter is called "incomplete".

Recall: For space- or timelike geodesics the affine parameter can be chosen to be the proper distance or proper time respectively.

Definition:

1. We say that \((M, g)\) possesses a singularity if it possesses an incomplete geodesic. (I.e. there are time-like, null and spacelike singularities)
a finite range of its affine parameter is called "incomplete."

Recall: For space- or timelike geodesics the affine parameter can be chosen to be the proper distance or proper time respectively.

**Definition:**

- We say that $(M, g)$ possesses a singularity if it possesses an incomplete geodesic. (i.e. there are time-like, null and spacelike singularities)

- When going along an incomplete geodesic towards
Definition:

1. We say that \((M, g)\) possesses a singularity if it possesses an incomplete geodesic.
   (i.e. there are time-like, null and space-like singularities)

2. When going along an incomplete geodesic towards a "singularity", 3 things can happen:

   1) A scalar constructed from \(R^{ij\cdots}_{\phantom{ij\cdots}k}\),
      e.g. \(R\), \(R^{ij\cdots}_{\phantom{ij\cdots}k}\), etc. diverges.
      → We say it is a "scalar curvature singularity"
We say that \((M,g)\) possesses a singularity if it possesses an incomplete geodesic. (i.e. there are time-like, null and space-like singularities)

When going along an incomplete geodesic towards a "singularity", 3 things can happen:

I) A scalar constructed from \(R, \Gamma, R_{\mu
\nu} \), etc diverges.

\[ \Rightarrow \text{We say it is a "scalar curvature singularity".} \]

II) The parallel transported tensor diverges.
When going along an incomplete geodesic towards a "singularity", 3 things can happen:

I) A scalar constructed from $R^{\mu\nu}_{\kappa\lambda}$, e.g. $R$, $R^{\mu\nu\rho\sigma}$, etc diverges. 
→ We say it is a "scalar curvature singularity".

II) In a parallel transported tetrad frame, a scalar component of $R^{\mu\nu}_{\kappa\lambda}$ or its covariant derivatives diverge.
→ We say it is a "parallel-transported curvature singularity".
When going along an incomplete geodesic towards a "singularity", 3 things can happen:

I) A scalar constructed from $R^{rs}_ss$, e.g. $R$, $R^{rs}_rr$, etc diverges.
   $\Rightarrow$ We say it is a "scalar curvature singularity".

II) In a parallel transported tetrad frame, a scalar component of $R^{rs}_ss$ or its covariant derivatives diverge.
   $\Rightarrow$ We say it is a "parallel-propagated curvature singularity".
When going along an incomplete geodesic towards a "singularity", 3 things can happen:

I) A scalar constructed from $R^{\mu\nu\rho\sigma}$, e.g. $R$, $R^{\mu\nu\rho\sigma}$, etc. diverges.
   ➔ We say it is a "scalar curvature singularity".

II) In a parallel transported tetrad frame, a scalar component of $R^{\mu\nu\rho\sigma}$ or its covariant derivatives diverge.
   ➔ We say it is a "parallel-propagated curvature singularity".
When going along an incomplete geodesic towards a “singularity”, 3 things can happen:

I) A scalar constructed from $R_{s^s}$, e.g. $R$, $R_{\mu\nu}$, etc diverges.

→ We say it is a “scalar curvature singularity”.

II) In a parallel transported tetrad frame, a scalar component of $R_{s^s}$ or its covariant derivatives diverges.

→ We say it is a “parallel-propagated curvature singularity”.
a singularity, 3 things can happen:

I) A scalar constructed from $R^s_{ss}$, e.g. $R$, $R^s_{r0}$, etc diverges.

$\rightarrow$ We say it is a "scalar curvature singularity".

II) In a parallel transported tetrad frame, a scalar component of $\tilde{R}^s_{ss}$ or its covariant derivatives diverge.

$\rightarrow$ We say it is a "parallel-propagated curvature singularity".

III) No additional scalar curvature singularities.
II) In a parallel transported tetrad frame, a scalar component of $\mathcal{R}$ or its covariant derivatives diverge.

$\Rightarrow$ We say it is a "parallel-propagated curvature singularity".

III) None of the above. Example: "Conical singularity".

(Write out a suitable piece and identify the boundaries of the cut)

$\Rightarrow$ We say it is a non-curvature singularity.

Fundamental problem:

In concrete solutions, such as Schwarzschild,...
II) In a parallel transported tetrad frame, a scalar component of $R_{ss}$ or its covariant derivatives diverge.

→ We say it is a "parallel-propagated curvature singularity".

III) None of the above. Example: "Conical singularity".

(This way manifold can be differentiable while some paths cannot.

→ We say it is a non-curvature singularity.

Fundamental problem:

In concrete solutions, such as Schwarzschild...
1) A scalar conserved from \( g_{ab} \),
e.g. \( R, R^{ab}, R_{ab} \), etc diverges.

\[ \rightarrow \] We say it is a "scalar curvature singularity".

2) In a parallel transported tetrad frame,
a scalar component of \( R_{ab} \) or its covariant derivatives diverge.

\[ \rightarrow \] We say it is a "parallel-propagated curvature singularity".

3) None of the above. Example: "Conical singularity".

(If my script can be
difficult while some
cut out a suitable piece and identify the boundaries of the cut)
a scalar component of $R_{ss}$ or its covariant derivatives diverge.

\[ \Rightarrow \text{We say it is a "parallel-propagated curvature singularity"} \]

(\begin{itemize}
\item This way manifold can be\item diffable while some\item paths cannot.
\end{itemize})

\[ \Rightarrow \text{We say it is a non-curvature singularity}\]

**Fundamental problem:**

- In concrete solutions, such as Schwarzschild or FRW cosmologies, curvature singularities are
III) None of the above. Example: “Conical singularity.”

(we can cut out a suitable piece and identify the boundaries of the cut)

We say it is a non-curvature singularity.

Fundamental problem:

- In concrete solutions, such as Schwarzschild or FRW cosmologies, curvature singularities are obviously present.

- But these spacetimes are highly symmetric.
singularity

III) None of the above. Example: "Conical singularity". (cut out a suitable piece and identify the boundaries of the cut)

We say it is a non-curvature singularity.

Fundamental problem:

In concrete solutions, such as Schwarzschild or FRW cosmologies, curvature singularities are obviously present.

But these spacetimes are highly symmetric.
Fundamental problem:

- In concrete solutions, such as Schwarzschild or FRW cosmologies, curvature singularities are obviously present.

- But these spacetimes are highly symmetric. Do more realistic, i.e. perturbed spacetimes also show these singularities?

Example:

Spherically symmetric dust shell infall.
Fundamental problem:

- In concrete solutions, such as Schwarzschild or FRW cosmologies, curvature singularities are obviously present.

- But these spacetimes are highly symmetric. Do more realistic, i.e. perturbed spacetimes also show these singularities?

Example:

Spherically symmetric dust shell in fall
Example:

Spherically symmetric dust shell in fall.

In Newtonian gravity:

$\rightarrow$ predict $\infty$ mass density to occur, but not if symmetry perturbed!

In Einstein gravity:

$\rightarrow$ predict black hole singularity to occur, even if symmetry is perturbed,

Remark:
Black holes provide finite energy endpoint of grav. collapse, thus contradicting GR cosmologically.

Note: In QM, charge driven
Example:

Spherically symmetric dust shell infall.

In Newton gravity:

→ predict ∞ mass density to occur, but not if symmetry perturbed!

In Einstein gravity:

Remark:

Black holes provide finite energy endpoint of star collapse, thus stabilizing GR energetically.

Note: In QM, charge driven collapse is bounded at finite energy by uncertainty principle.

→ predict black hole singularity to occur, even if symmetry is perturbed, (if assuming e.g. dominant energy cond., etc.)
Spherically symmetric dust shell in flat

In Newton gravity:

→ predict ∞ mass density to occur,

but not if symmetry perturbed!

In Einstein gravity:

→ predict black hole singularity to occur,

even if symmetry is perturbed,

(i.e. assuming e.g. dominant energy cond. etc.)

Remark:
Black holes provide finite energy
and point of view: collapse, thus
stabilizing CR energetically
Note: In QM, charge shrink
collapse is bounded at finite energy
by uncertainty principle.

Next: "Singularity theorems"