Singualarities and Exact solutions

Recall the problem:

- Exact solutions to the Einstein equation can be obtained in practice only when restricting to highly symmetric cases.

- Examples, such as Schwarzschild black holes and FRW cosmological solutions, clearly exhibit the presence of a curvature singularity.

- This is highly significant because singularities imply the breakdown of GR at or close to these points.
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But, highly symmetric solutions to Newton's gravity can also exhibit "singularities":

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However, no such singularity occurs if the collapse...
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However: no such singularity occurs if the collapse is even slightly asymmetric.

Question:
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In GR, are conclusions drawn from the study of highly symmetric exact solutions, in particular the occurrence of singularities, in this sense robust?
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**Answer:**

Singularity theorems

Yes, the prediction of singularities is robust.

**Strategy for singularity theorems:**

a) Focus attention on singularities that can be identified by the existence of incomplete inextendible timelike (or null) geodesics.
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Why? It is clear that these are significant singularities because an observer travelling such a geodesic has his eigen-time bounded above and/or below.
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Other singularities?

May well exist in addition but the standard singularity theorems do not attempt to predict them too.

6.) Basic idea:

Singlarities can be in the way of geodesics.

⇒

The presence of singularities interferes with the property of geodesics of being
Other singularities? (e.g. singularities identified through incomplete spacelike geodesics or singularities identified by some other criterion.)

May well exist in addition but the standard singularity theorems do not attempt to predict them too.

b.) Basic idea:

Singualarities can lie in the way of geodesics.

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The presence of singularities interferes with the property of geodesics of being extremal length curves.
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c) Recall:

Euler-Lagrange equation

Extremizing curve length ⇒ geodesic equation

The geodesic equation is a differential equation.
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Extremizing curve length \( \Rightarrow \) geodesic equation

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Extremizing curve length \( \Rightarrow \) geodesic equation

The geodesic equation is a differential equation.

Thus:

At least locally, geodesics are paths of extremal length:

- Space-like geodesics are curves of
  shortest proper distance.
- Time-like geodesics are curves of
  maximal proper time (i.e., of maximal eigentime).

Why maximal?

If there is a timelike curve between two
events \( p, q \), then there are timelike curves
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Why maximal?

If there is a timelike curve between two events $p, q$, then there are timelike curves with shorter eigenvalues; just take a longer path and translate faster.
The geodesic equation is a differential equation. Thus:

At least locally, geodesics are paths of extremal length:

- Space-like geodesics are curves of shortest proper distance.
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The geodesic equation is a differential equation.

Thus:

At least locally, geodesics are paths of extremal length:
- Space-like geodesics are curves of maximal curvature.

Why maximal?

If there is a timelike curve between two events \( p, q \), then there are timelike curves with shorter proper time; just take a longer path and travel it faster.

d) Prove that, even in generic spacetimes:

There always exist curves of maximal length between two events.
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There always exist curves of maximal length between two events.

Assumptions needed?

Yes, e.g. the assumption that space-time is globally hyperbolic surfaces.

Note: Space-time may not be globally hyperbolic, let us for now disregard black holes. For the purpose, now, of studying the existence of a cosmological singularity, then space-time probably is well-described as globally hyperbolic.
e) Prove that these extremal length curves cannot be geodesics with eigenvalue larger than a certain finite amount either into the past or future.

Assumptions needed?

Yes: Matter must be assumed to obey an energy condition.

f) Conclude that there are incomplete geodesics, i.e., that we have a singularity in the past (or future):
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Assumptions needed?

Yes: Matter must be assumed to obey an energy condition.

f) Conclude that there are incomplete geodesics, i.e., that we have a singularity in the past (or future):
A singularity theorem:

Assume that: $(M, g)$ is a globally hyperbolic spacetime.

- The energy-momentum tensor of matter obeys the "Strong energy condition":
  
  \[ T_{\mu \nu} S^\mu S^\nu \geq -\frac{1}{2} T^g_{\mu \nu} \text{ for all timelike } S. \]

- There exists a $C^2$ spacelike Cauchy surface $\Sigma$, on which the trace of the extrinsic curvature, $K$, is bounded from above by a negative constant $C$.
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- There exists a \( C^2 \) spacelike Cauchy surface \( \Sigma \), on which the trace of the extrinsic curvature, \( K \), is bounded from above by a negative constant \( C \): 

\[ K(p) \leq C < 0 \text{ for all } p \in \Sigma \]
A singularity theorem:

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  \[ K(p) \leq C < 0 \] for all \( p \in \Sigma \).
Then:

No past-directed timelike curve from $\Sigma$ can have eigentime, i.e. proper length, larger than $\frac{3}{4}$.

All past-directed timelike geodesics are incomplete.

$\Rightarrow$ There is a cosmological singularity in the finite past!
A singularity theorem:

Assume that: \( (M, g) \) is a globally hyperbolic spacetime

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because all past-directed paths end on it.
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Extrinsic curvature?

later move on this
All past-directed timelike geodesics are incomplete.

⇒ There is a cosmological singularity in the finite past!

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later more on this

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Extrinsic curvature?

The extrinsic curvature of a spacelike hypersurface describes how...
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Extrinsic curvature?

The extrinsic curvature of a spacelike hypersurface describes how much curvature there is in between...
All past-directed timelike geodesics are incomplete.

\[ \Rightarrow \text{There is a cosmological singularity in the finite past!} \]

Extrinsic curvature?

[Later more on this]

- The extrinsic curvature of a spacelike hypersurface describes how much curvature there is in between the ambient space and the hypersurface.
There is a cosmological singularity in the finite past!

Extrinsic curvature?

The extrinsic curvature of a spacelike hypersurface describes how much curvature there is in between the spacelike hypersurface and the time...
There is a cosmological singularity in the finite past!

Extrinsic curvature?

The extrinsic curvature of a spacelike hypersurface describes how much curvature there is in between the spacelike hypersurface and the time dimension.

Later move on this.
Extrinsic curvature?

The extrinsic curvature of a spacelike hypersurface describes how much curvature there is in between the spacelike hypersurface and the time dimension.

Intuitively: it is the rate of the expansion of spacetime, more precisely its negative.

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Intuitively: it is the rate of the expansion of spacetime, more precisely its negative.

\[ \forall p \in \Sigma \]

Thus: Assuming \( K(p) \leq C < 0 \) meant that spacetime has a finite minimum expansion rate everywhere on \( \Sigma \).

The strong energy condition?

Recall: The “weak energy condition”:

\[ T^{\mu\nu}v_\mu v_\nu \geq 0 \text{ for all timelike } v : g(v,v) < 0 \]

Meaning? For an observer with unit tangent \( v \) the
Thus: Assuming $k(p) \leq \xi < 0$ meant that spacetime has a finite minimum expansion rate everywhere on $\Sigma$.

The strong energy condition?

Recall: The "weak energy condition":

$T^{\mu\nu}v_\mu v_\nu \geq 0$ for all timelike $v$: $g(v,v) < 0$

Meaning? For an observer with unit tangent $v$ the local energy density is: $T^{\mu\nu}v_\mu v_\nu \geq 0$

The "dominant energy condition":
The strong energy condition?

Recall:

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$$T_{\mu \nu} v^\mu v^\nu \geq 0$$ for all timelike $$v: g(v,v) < 0$$

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The "dominant energy condition":

$$T_{\mu \nu} v^\mu v^\nu \geq 0$$ and $$K^\mu K_\mu \leq 0$$

where $$v$$ is any timelike vector and $$K_\mu := T_{\mu \nu} v^\nu$$

Meaning? The local energy-momentum flow vector $$K$$ may not be conserved but has to be non-space-like: Flow should be into the tangent of the manifold.
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The "weak energy condition":

$$T_{\mu\nu}v^\mu v^\nu \geq 0$$ for all timelike \(v\): \(g(v,v) < 0\)

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The "dominant energy condition":

$$T_{\mu\nu}v^\mu v^\nu \geq 0 \quad \text{and} \quad K_{\mu}K^\mu \leq 0$$

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Meaning? The local energy-momentum flow vector \(K\) may not be conserved but has to be non-space-like: Flow should be into the future (need for causality).
\[ T_{\mu \nu} v^\mu v^\nu \geq 0 \text{ for all timelike } v : g(v,v) < 0 \]

Meaning? For an observer with unit tangent \( v \) the local energy density is: \( T_{\mu \nu} v^\mu v^\nu \geq 0 \)

The "dominant energy condition":

\[ T_{\mu \nu} v^\mu v^\nu \geq 0 \text{ and } K \cdot K \leq 0 \]

\[ \text{weak energy condition} \]

\[ \text{i.e. } T_{\mu \nu} v^\mu v^\nu \text{ is non-space-like.} \]

where \( v \) is any timelike vector and \( K = T_{\mu \nu} v^\mu v^\nu \)

Meaning? The local energy-momentum flow vector \( K \) may not be conserved but has to be non-space-like: Flow should be into the future needed for causality.

**Definition:**
The "dominant energy condition":

\[ T_{\mu \nu} v^\mu v^\nu \geq 0 \quad \text{and} \quad K_\mu K^\mu \leq 0 \]

weak energy condition

where \( v \) is any timelike vector and \( K_\mu := T_{\mu \nu} v^\nu \)

**Meaning:** The local energy-momentum flow vector \( K \) may not be conserved but has to be non-space-like; flow should be into the future \( \equiv \) need for causality.

**Definition:**

Matter is said to obey the strong energy condition if:

\[ T_{\mu \nu} K^\mu K^\nu \geq 0 \]
The "dominant energy condition":

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Meaning? The local energy-momentum flow vector \( K \) may not be conserved but has to be non-space-like: Flow should be into the future \( \langle \text{need for causality} \rangle \).

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Definition:

Matter is said to obey the strong energy condition iff:

\[ T_{\mu\nu}g^{\mu\nu} \geq -\frac{1}{2}T \]

for all timelike \( \mathbf{s} \).

Intuition? Nothing obvious

Plausible? Yes, obeyed by known matter.

Relationship? Independent of weak and dominant energy condition.
\[ T_{\mu\nu} \geq -\frac{1}{2} T \] for all time-like \( \sigma. \)

- **Intuition?** Nothing obvious
- **Plausible?** Yes, obeyed by known matter.
- **Relationship?** Independent of weak and dominant energy condition.

**Remarks:** For known matter, \( T_{\mu\nu} \) is diagonalizable to obtain:

\[ T_{\mu\nu} = \left( \begin{array}{cc} g & 0 \\ 0 & 0 \end{array} \right) \]
Remarks: For known matter, $T_{\mu\nu}$ is diagonalizable to obtain:

$$T_{\mu\nu} = \begin{pmatrix} g & 0 \\ 0 & \rho_i \end{pmatrix}$$

energy density observed by comoving observer

The energy conditions then read:

- Weak: $g > 0$ and $g + \rho_i > 0$ for $i \in \{1, 2, 3\}$
- Dominant: $g > |\rho_i|$ for $i \in \{1, 2, 3\}$

Note: A cosmological constant $\Lambda$ is to be viewed as a continuous
Remarks: For known matter, $T_{\mu\nu}$ is diagonalizable to obtain:

$$T_{\mu\nu} = \begin{pmatrix} \sigma & p_i & 0 \\ 0 & p_i & \rho_3 \\ \rho_3 & \end{pmatrix}$$

The energy density observed by comoving observer $\rho$ corresponds to principal pressures.

The energy conditions then read:

- Weak: $\sigma > 0$ and $\sigma + p_i > 0$ for $i \in \{1, 2, 3\}$
- Dominant: $\sigma > \frac{1}{3} p_i$ for $i \in \{1, 2, 3\}$
- Strong: $\sigma + \frac{3}{2} p_i > 0$ and $\sigma + p_i > 0$ for $i \in \{1, 2, 3\}$

Note: A cosmological constant $\Lambda$ is to be viewed as a contribution to $T_{\mu\nu}$. Exercise: $\Lambda$ contributes a positive $\sigma$. Implications?
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**Note:** A cosmological constant $\Lambda$ is to be viewed as a contribution to $T_{\mu \nu}$. Exercise: $\Lambda$ contributes a positive $g_\Lambda$. Implications?
obtain:

\[ T_{\mu\nu} = \begin{pmatrix} \sigma & 0 \\ 0 & p_i & p_3 \end{pmatrix} \]

\( \sigma, p_i, p_3 \) principal pressures

The energy conditions then read:

- **Weak**: \( \sigma \geq 0 \) and \( \sigma + p_i \geq 0 \) for \( i \in \{1,2,3\} \)
- **Dominant**: \( \sigma \geq |p_i| \) for \( i \in \{1,2,3\} \)
- **Strong**: \( \sigma + \sum_{i=1}^{3} p_i \geq 0 \) and \( \sigma + p_3 \geq 0 \) for \( i \in \{1,2,3\} \)

**Note:** A cosmological constant \( \Lambda \) is to be viewed as a contribution to \( T_{\mu\nu} \). Exercise: \( \Lambda \) contributes a positive \( \sigma \). Implications?
Essence of point e):

Given, in particular, the strong energy condition, one can show that geodesics meet a divergence of a quantity called expansion, $\Theta$, in finite proper time:

The "expansion", $\Theta$:

Consider a "congruence of timelike geodesics" (e.g., freely falling dust) through $\Sigma$, i.e., a smooth family of timelike geodesics, exactly one through each $p \in \Sigma$. If parametrized by proper time, their two spacelike $\Sigma$-surfaces

\[\text{[Equation]}\]

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Consider a "congruence of timelike geodesics" e.g., freely falling dust. through $\Sigma$, i.e., a smooth family of timelike geodesics, exactly one through each $p \in \Sigma$. If parametrized by proper time, their tangent vector field $\xi$, namely
Intention? Nothing obvious.

Plausible? Yes, obeyed by known matter.

Relationship? Independent of weak and dominant energy condition.

Remarks: For known matter, $\mathbf{\bar{T}}_{\mu\nu}$ is diagonalizable to obtain:

$$T_{\mu\nu} = \left( \begin{array}{cc} g & p' \\ -p' & 0 \end{array} \right)$$
The strong energy condition?

Recall: The "weak energy condition":

\[ T_{\mu\nu} v^\mu v^\nu \geq 0 \quad \text{for all timelike } v : g(v,v) < 0 \]

Meaning? For an observer with unit tangent \( v \) the local energy density is: \( T_{\mu\nu} v^\mu v^\nu \geq 0 \)

The "dominant energy condition":

\[ T_{\mu\nu} v^\mu v^\nu \geq 0 \quad \text{and} \quad K_{\mu} K^{\mu} \leq 0 \]

\( \text{weak energy condition} \)

where \( v \) is any timelike vector and \( K_{\mu} = T_{\mu\nu} v^\nu \)

Meaning? The local energy-momentum flow vector \( K \) may not be conserved but...
Then:

No past-directed timelike curve from $\Sigma$ can have eigentime, i.e. proper length, larger than $\frac{3}{c}$. All past-directed timelike geodesics are incomplete.

$\implies$ There is a cosmological singularity in the finite past!
be solutions with eigenvalue from the geodesics with eigenvalue larger than a certain finite amount either into the past or future.

Assumptions needed?

Yes: Matter must be assumed to obey an energy condition.

f) Conclude that there are incomplete geodesics, i.e., that we have a singularity in the past (or future):
then space-time probably is well-described as globally hyperbolic.

e) Prove that these extremal length curves cannot be geodesics with eigentime longer than a certain finite amount either into the past or future.

Assumptions needed?

Yes: Matter must be assumed to obey an energy condition.

f) Conclude that there are incomplete geodesics, i.e., that we have a singularity in the past (or future):
curvature, $K$, is bounded from above by a negative constant $C$:

$$K(p) \leq C < 0 \text{ for all } p \in \Sigma$$

**Then:**

No past-directed timelike curve from $\Sigma$ can have eigentime, i.e. proper length, larger than $\frac{3}{2}$.

All past-directed timelike geodesics are incomplete.
Pirsa: 09110120

Thus: Assuming \( K(p) \leq \kappa < 0 \) meant that spacetime has a finite minimum expansion rate everywhere on \( \Sigma \).

The strong energy condition?

Recall: The "weak energy condition":

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 Meaning? For an observer with unit tangent \( v \) the local energy density is: \( T_{\mu \nu} v^\mu v^\nu \geq 0 \)
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where \( v \) is any timelike vector and \( K_\mu := T^{\mu \nu} v^\nu \)

Meaning? The local energy-momentum flow vector \( K \) may not be conserved but has to be non-space-like: Flow should be into the future \( \Rightarrow \) need for causality.

Definition:

Matter is said to obey the strong energy condition iff:
Strong: $g + \sum_{i=1}^{3} p_i \geq 0$ and $g + p_i \geq 0$ for $i \in \{1, 2, 3\}$

Note: Could possibly be also negative.

A cosmological constant $\Lambda$ is to be viewed as a contribution to $T_{\mu\nu}$. Exercise: $\Lambda$ contributes a positive $g_{\mu\nu}$. Implications?

**Essence of point c):**

Given, in particular, the strong energy condition, one can show that geodesics meet a divergence of a quantity called expansion, $\Theta$, in finite proper time:

The "expansion", $\Theta$:

Consider a "congruence of timelike geodesics"
Essence of point $e$:

Given, in particular, the strong energy condition, one can show that geodesics meet a divergence of a quantity called expansion, $\Theta$, in finite proper time:

The "expansion", $\Theta$:

Consider a "congruence of timelike geodesics" e.g., freely falling dust.

through $\Sigma$, i.e., a smooth family of timelike geodesics, exactly one through each $p \in \Sigma$. If parametrized by proper time, their tangent vector field $\xi$, namely
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Given, in particular, the strong energy condition, one can show that geodesics meet a divergence of a quantity called expansion, \( \Theta \), in finite proper time:

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Consider a "congruence of timelike geodesics" through \( \Sigma \), i.e., a smooth family of timelike geodesics, exactly one through each \( p \in \Sigma \). If parametrized by proper time, their tangent vector field \( \xi \), namely
Essence of point (1):

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Important motion also e.g. in study of gravitational collapse of stars.

The "expansion", $\Theta$:

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Essence of point e:

Given, in particular, the strong energy condition, one can show that geodesics meet a divergence of a quantity called \( \Theta \), in finite proper time:

The "expansion" \( \Theta \):

Consider a "congruence of timelike geodesics" e.g., freely falling dust:

through \( \Sigma \), i.e., a smooth family of timelike geodesics, exactly one through each \( p \in \Sigma \). If parametrized by proper time, their tangent vector field \( \xi \), namely
The "expansion", $\Theta$:

Consider a "congruence of timelike geodesics" through $\Sigma$, i.e., a smooth family of timelike geodesics, exactly one through each $p \in \Sigma$. If parametrized by proper time, their tangent vector field $\xi$, namely

$$\xi := \frac{d}{dt}$$

will obey: $g(\xi, \xi) = -1$.
Consider a congruence of timelike geodesics through $\Sigma$, i.e., a smooth family of timelike geodesics, exactly one through each $p \in \Sigma$. If parametrized by proper time, their tangent vector field $\xi$, namely

$$\xi := \frac{d}{d\tau}$$

will obey: $g(\xi, \xi) = -1$.

Consider now a one-parametric sub family of these geodesics:

$$g(\xi, \xi)$$
through $\Sigma$, i.e., a smooth family of timelike geodesics, exactly one through each $p \in \Sigma$. If parametrized by proper time, their tangent vector field $\xi$, namely

$$\xi = \frac{d}{d\tau}$$

will obey: $g(\xi, \xi) = -1 \quad \text{A} \ p$.

Consider now a one-parametric sub family of these geodesics:

$y(\tau, s)$

$a$ "connecting vector field"
through $\Sigma$, i.e., a smooth family of timelike geodesics, exactly one through each $p \in \Sigma$. If parametrized by proper time, their tangent vector field $\xi$, namely

\[ \xi = \frac{\partial}{\partial \tau} \text{ proper time} \]

will obey: $g(\xi, \xi) = 1$ \( \forall p \).

Consider now a one-parameter subfamily of these geodesics: $\gamma(t, s)$, parameter of family of neighboring geodesics. Then, we define the deviation vector:
\[ \zeta := \frac{d}{d\tau} \text{ proper time} \]

will obey: \( g(\zeta, \zeta) = -1 \) \( \forall \rho \).

Consider now a one-parameter subfamily of these geodesics:

\[ y(\tau, s) \]

parametric of family of neighboring geodesics.

\( \text{a "connecting vector field" } \)

Then, we define the deviation vector:

\[ \eta := \frac{d}{ds} \]

\( \text{a line of constant \( \zeta \) value } \)

\( \text{a geodesic, i.e., a line of constant } \zeta \text{ value } \)
Consider now a one-parameter subfamily of these geodesics:

\[ y(t, s) \]

parametric of family of neighboring geodesics.

Then, we define the deviation vector:

\[ \eta = \frac{d}{ds} \]

a line of constant \( \xi \) value

\[ \xi \]

g a geodesic, i.e., a line of constant \( s \) value

How does \( \eta \) change along a geodesic?
Consider now a one-parametric subfamily of these geodesics:

\[ y(t, s) \]

parametric family of neighboring geodesics.

Then, we define the deviation vector:

\[ \eta := \frac{d}{ds} \]

a line of constant \( t \) value

\[ \xi \]

a geodesic, i.e., a line of constant \( s \) value

How does \( \eta \) change along a geodesic?

We have \( \frac{d}{ds} \frac{d}{dt} = \frac{d}{dt} \frac{d}{ds} \), i.e., in any coordinate system.
How does $\gamma$ change along a geodesic?

We have $\frac{d}{ds} \frac{d}{dt} = \frac{d}{d\tau} \frac{d}{d\tau}$, i.e., in an arbitrary coordinate system:

$$\nabla_\xi \gamma = \nabla_\xi \gamma$$

$$\Rightarrow \quad \xi^\alpha \nabla_\alpha \eta^\mu = \eta^\mu \nabla_\alpha \xi^\alpha$$

$$\Rightarrow \quad \xi^\alpha \nabla_\alpha \eta^\mu = \eta^\mu \xi^\alpha \nabla_\alpha$$

$$\Rightarrow \quad \xi^\alpha \nabla_\alpha \eta^\mu = \eta^\mu \xi^\alpha \nabla_\alpha$$

$$\Rightarrow \quad B^\alpha_{\mu \nu} := \xi^\alpha \nabla_\alpha \eta^\mu$$

Along the geodesic, $\xi$, the deviation vector $\eta^\mu$ changes its direction and length by $B^\alpha_{\mu \nu}$.

The tensor $B^\alpha_{\mu \nu}$ can be decomposed covariantly.
How does $\gamma$ change along a geodesic?

We have $\frac{d}{ds} \frac{d}{dc} = \frac{d}{dc} \frac{d}{ds}$, i.e., in any coordinate system:

\[ \frac{\partial}{\partial s} \gamma^a = \frac{\partial}{\partial c} \gamma^a \]

\[ \Rightarrow \quad \nabla^c \gamma^a \eta^c e_0 = \gamma^a \nabla^c \eta^c e_0 \]

\[ \Rightarrow \quad \nabla^c \gamma^a \eta^b e_b \]

\[ \Rightarrow \quad \nabla^c \gamma^a \eta^b e_b = \nabla^c \gamma^a \eta^b e_b \]

\[ \Rightarrow \quad \gamma^a \nabla^c \eta^b e_b = \nabla^c \gamma^a \eta^b e_b \]

\[ \Rightarrow \quad \gamma^a \nabla^c \eta^b e_b = \gamma^a \nabla^c \eta^b e_b \]

Along the geodesic, $\gamma$, the deviation vector $\gamma^a$ changes its direction and length by $B^a_\mu \gamma^\mu$.

The tensor $B^a_\mu$ can be decomposed covariantly and uniquely into:
How does $\gamma$ change along a geodesic? 

We have \( \frac{d}{ds} \frac{d}{dt} = \frac{d}{dt} \frac{d}{ds} \), i.e., in any coordinate system:

\[
\nabla_t \gamma^i = \partial_t \gamma^i
\]

\[
\Rightarrow \quad \xi^\mu \nabla_\mu \eta^i e_\nu = \eta^i \nabla_\nu \xi^\mu e_\rho
\]

\[
\Rightarrow \quad \xi^\mu \gamma_\mu e_\nu = \eta^i \nabla_\nu \xi^\mu e_\rho
\]

\[
\Rightarrow \quad \xi^\mu \gamma_\mu = \eta^i \xi^\rho = \xi^\mu B^\rho_{\ \mu}
\]

Along the geodesic, $\gamma$, the deviation vector $\gamma^i$ changes its direction and length by $B^\rho_{\ \mu} \gamma^\mu$.

The tensor $B^\rho_{\ \mu}$ can be decomposed covariantly and uniquely into:
Then, we define the deviation vector:

$$\eta := \frac{d}{ds}$$

a line of constant $\tau$ value

a geodesic, i.e., a line of constant $s$ value

How does $\eta$ change along a geodesic?

We have

$$\frac{d}{ds} \frac{d}{dt} = \frac{d}{dt} \frac{d}{ds}$$

i.e., in an arbitrary coordinate system.
How does $\xi$ change along a geodesic?

We have $\frac{d}{ds} \frac{d}{dt} = \frac{d}{ds} \frac{d}{d\tau}$, i.e., in arb. coordinate system:

\[ \nabla_{\xi} \eta = \eta \nabla^\eta \]

\[ \Rightarrow \xi^\mu \nabla_{\xi} \eta^\nu = \eta^\nu \nabla^\xi \xi^\mu \xi^\nu \]

\[ \Rightarrow \xi^\mu \xi^i \eta^\nu = \eta^\nu \xi^i \xi^\mu \]

\[ \Rightarrow \xi^\mu \xi^i \xi^j = \eta^\nu \xi^i \xi^j = \xi^\nu B^\mu_{\nu i} \]

\[ B^\mu_{\nu i} : = \xi^\nu \xi^i \mu \]

Along the geodesic, $\xi$, the deviation vector $\eta^\nu$ changes its direction and length by $B^\mu_{\nu i}$. 
\[ \eta := \frac{d}{ds} \]

- a line of constant \( \tau \) value

- a geodesic, i.e., a line of constant \( s \) value

**How does \( \eta \) change along a geodesic?**

We have \[ \frac{d}{ds} \frac{d}{d\tau} = \frac{d}{d\tau} \frac{d}{ds}, \text{ i.e., in any coordinate system:} \]

\[ \nabla_{\tau} \eta = \nabla_{\tau} \eta \]

\[ \Rightarrow \quad \xi^\tau \nabla_{\tau} \eta \xi^e \omega = \eta^e \xi^b \xi^\tau \xi^b e_\tau \]

\[ \Rightarrow \quad \xi^\tau \xi^b \xi^b \xi^\tau \xi^b e_\tau = \eta^e \xi^b \xi^b e_\tau \]
a line of constant $\tau$ value

a geodesic, i.e., a line of constant $s$ value

How does $\eta$ change along a geodesic?

We have $\frac{d}{ds} \frac{d}{d\tau} = \frac{d}{d\tau} \frac{d}{ds}$, i.e., in orb. coordinate system.

to see this, have them act on a scalar function

$\tau \equiv \frac{\partial}{\partial \tau} = \frac{\partial}{\partial \eta} \eta$

$\Rightarrow \quad \tau^a \partial_a \eta \equiv \eta = \eta^a \partial_a$ 

$\Rightarrow \quad \tau^a \partial_a \eta \equiv \eta = \partial_a \eta \Rightarrow \eta^b e_b$

$\Rightarrow \quad \tau^a \partial_a \eta \equiv \eta = \partial_b \eta^b e_b$ 

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$\Rightarrow \quad \tau^a \partial_a \eta \equiv \eta = \partial_b \eta^b e_b$
How does $\gamma$ change along a geodesic?

We have \( \frac{d}{ds} \frac{d}{dt} = \frac{d}{dt} \frac{d}{ds} \), i.e., in arcl. coordinate system:

\[
\mathbb{D}_s \gamma = \mathbb{D}_t \gamma
\]

\[
\Rightarrow \quad \xi^a \nabla_x \eta \omega = \xi^a \nabla_x \xi^b \epsilon_b
\]

\[
\Rightarrow \quad \xi^a \partial_i \eta \omega = \xi^a \xi^b \partial_i \epsilon_b
\]

\[
\Rightarrow \quad \xi^a \partial_{ij} \omega = \xi^a \xi^b \partial_{ij} \epsilon_b
\]

\[
\Rightarrow \quad B^\omega_{\mu} : = \xi^a \gamma_{ij}^b
\]

Along the geodesic, $\gamma$, the deviation vector $\xi^a$ changes its direction and length by $B^\omega_{\mu} \xi^a$.
How does $\eta$ change along a geodesic?

We have \( \frac{d}{ds} \frac{\partial}{\partial s} = \frac{\partial}{\partial t} \), i.e., in arbitrary coordinate system:

\[ \nabla_s \eta = \nabla_t \eta \]

\[ \Rightarrow \eta^\nu \nabla_\nu \eta^\alpha = \eta^\nu \nabla_\nu \eta^\alpha \]

\[ \Rightarrow \eta^\nu \nabla_\nu \eta = \eta^\nu \nabla_\nu \eta \]

\[ \Rightarrow \eta^\nu \nabla_\nu \eta = \eta^\nu \eta^\alpha \eta^\alpha \]

\[ \Rightarrow B^\nu_\mu := \eta^\nu \nabla_\mu \eta \]

Along the geodesic, $\eta$, the deviation vector $\eta^\nu$ changes its direction and length by $B^\nu_\mu \eta^\mu$.

The tensor $B^\nu_\mu$ can be decomposed covariantly and uniquely into:
\[ \nabla_\nu \eta^\sigma = \eta^\sigma \nabla_\nu \eta^\tau = \eta^\tau \nabla_\nu \eta^\sigma = \eta^\sigma \\nabla_\nu \eta^\tau = \eta^\tau \\nabla_\nu \eta^\sigma \]

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\[ \Rightarrow \quad \nabla_\nu \eta^\sigma = \eta^\tau \nabla_\nu \eta^\tau = \eta^\tau \nabla_\nu \eta^\tau = \eta^\tau \nabla_\nu \eta^\tau \]

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\[ \Rightarrow \quad \nabla_\nu \eta^\sigma = \eta^\tau \nabla_\nu \eta^\tau = \eta^\tau \nabla_\nu \eta^\tau = \eta^\tau \nabla_\nu \eta^\tau \]

Along the geodesic, $\gamma$, the deviation vector $\eta^\tau$ changes its direction and length by $B^\mu_\nu \eta^\tau$.

The tensor $B^\nu_\mu$ can be decomposed covariantly and uniquely into:

$B^\nu_\mu = \omega^\nu_\mu + \sigma^\nu_\mu + \xi^\nu_\mu$

(all 3 terms are tensors because the split is covariant)
\( \Rightarrow \quad \xi^{\nu} \nabla_{\nu} \eta^{\mu} e_{\sigma} = \eta^{\mu} \nabla_{\nu} \xi^{\nu} e_{\sigma} \)

\( \Rightarrow \quad \xi^{\nu} \zeta_{\nu} e_{\sigma} = \eta^{\mu} \xi^{\mu} e_{\sigma} \)

\( \Rightarrow \quad \xi^{\nu} \zeta_{\nu} = \eta^{\mu} \xi^{\mu} = \eta^{\mu} \mathbf{B}_{\mu} \) for

\[ \eta^\mu : \xi^\nu \]

Along the geodesic, \( \xi \), the deviation vector \( \eta^\mu \) changes its direction and length by \( \mathbf{B}_{\mu} \eta^\nu \).

The tensor \( \mathbf{B}_{\mu} \) can be decomposed covariantly and uniquely into:

\[ \mathbf{B}_{\mu} = \omega_{\mu \nu} + \xi_{\mu \nu} + \xi_{\nu \mu} \]

(\text{all 3 terms are tensors because the split is covariant})

\[ \text{symmetric and trace = 0} \]

\[ \text{antisymmetric} \]
\[ \Rightarrow \quad \xi^b \nabla_b \eta^a e^c = \eta^a \nabla_b \xi^b e^c \]
\[ \Rightarrow \quad \xi^b \nabla_b \eta^a = \eta^a \nabla_b \xi^b \]
\[ \Rightarrow \quad \xi^b \nabla_b \eta^a = \eta^a \nabla_b \xi^b = \eta^a B^a_{\mu} \]

Along the geodesic, \( \xi \), the deviation vector \( \eta^a \) changes its direction and length by \( B^a_{\mu} \eta^a \).

The tensor \( B^a_{\mu} \) can be decomposed covariantly and uniquely into:

\[ B_{\mu \nu} = \omega_{\mu \nu} + g_{\mu \nu} + \delta_{\mu \nu} \] (all 3 terms are tensors because the split is covariant)

We have: \( \omega_{\mu \nu} = \frac{1}{2} (B_{\mu \nu} - B_{\nu \mu}) \), clearly.
\[ \Rightarrow \quad \overset{\gamma}{\nabla} \overset{\gamma}{\nabla} = \overset{\gamma}{\nabla} \overset{\gamma}{\nabla} = D_{\nu} B_{\mu} \]

\[ B_{\mu} = \varepsilon_{\mu} \]

Along the geodesic, \( \gamma \), the deviation vector \( \gamma^{\mu} \) changes its direction and length by \( B_{\mu} \gamma^{\mu} \).

The tensor \( B_{\mu} \) can be decomposed covariantly and uniquely into:

\[ B_{\mu} = \omega_{\mu} + \Theta_{\mu} + \varepsilon_{\mu} \]

where symmetric and trace = 0

We have: \( \omega_{\mu} = \frac{1}{2} \left( B_{\mu} - B_{\nu} \right) \), clearly.

But \( \Theta_{\mu}, \varepsilon_{\mu} = ? \)
But $g_{\mu \nu}, t_{\mu \nu} = ?$

In preparation: define the projector $h_{\mu \nu}$ onto $(\mathbb{R}^4)^\perp$ i.e. onto the spatial components:

$$h_{\mu \nu} := g_{\mu \nu} + \xi^\alpha \xi_\alpha$$

Check: is $h_{\mu \nu}$ really always $\perp$ to $g$?

Indeed: $g^{\alpha \beta} h_{\mu \nu} w^\alpha w^\beta = (\xi, w) + (\xi, \xi)(\xi, w) = 0$

Define: The "expansion", $\Theta$, is defined as the magnitude of the spatial part of $B$:
and uniquely into:

$$B_{\mu\nu} = \omega_{\mu\nu} + \omega_{\nu\mu} + t_{\mu\nu}$$

symmetric and trace = 0

We have: $$\omega_{\mu\nu} = \frac{1}{2}(B_{\mu\nu} - B_{\nu\mu}),$$ clearly.

But $$\omega_{\mu\nu}, t_{\mu\nu} = ?$$

In preparation: define the projector $$h_{\mu\nu}$$ onto $$(\mathbb{R}^3)_{\perp}$$ i.e. onto the spatial components:

$$h_{\mu\nu} = g_{\mu\nu} + 5_{\mu\nu}$$

Check: is $$h_{\mu\nu}$$ really always $$\perp$$ to $$g$$?
But $G_{\mu \nu}$, $\mathcal{T}_{\mu \nu}$ = ?

In preparation: define the projector $h_{\mu \nu}$ onto $(\mathbb{R} g)^{-1}$, i.e. onto the spatial components:

$$h_{\mu \nu} := g_{\mu \nu} + \delta_{\mu \nu} \delta_{\nu \lambda}$$

Check: is $h_{\mu \nu}$ really always $\perp$ to $g$?

Indeed: $g^{\nu \lambda} h_{\mu \nu} w^\lambda = (g, w) + (\delta, \delta)(g, w) = 0$

Define: The "expansion", $\Theta$, is defined as the magnitude of the spatial part of $B$.
\[ h_{\mu\nu} = g_{\mu\nu} + \delta_{\mu\nu} \]

Check: is \( h_{\mu\nu} \) really always \( 1 \) to \( 5 \)?

Indeed: \( \delta^{\mu} h_{\mu\nu} = (\delta, \omega) + (\delta, \delta)(\delta, \omega) = 0 \)

Define: The "expansion", \( \Theta \), is defined as the magnitude of the spatial part of \( B \):

\[ \Theta := B^{\mu\nu} h_{\mu\nu} \]

Claim: \( Tr(B) = \Theta \)

Indeed: \( \Theta = B^{\mu\nu} h_{\mu\nu} = B^{\mu\nu} g_{\mu\nu} + \delta^{\mu} \delta_{\nu} B_{\mu\nu} = Tr(B) + \delta^{\mu} \delta_{\nu} \Theta \)
Define: The "expansion", $\Theta$, is defined as the magnitude of the spatial part of $B$:

$$\Theta := B^{\mu\nu} h_{\mu\nu}$$

Claim: $Tr(B) = \Theta$

Indeed: $\Theta = B^{\mu\nu} h_{\mu\nu} = B^{\mu\nu} g_{\mu\nu} + \delta^{\mu}_{\nu} \delta^{\nu}_{\mu} B_{\mu}^{\nu} = Tr(B) + \delta^{\mu}_{\nu} \delta^{\nu}_{\mu} \partial_{\mu} \partial_{\nu}$

Therefore: $\phi_{\mu\nu} = \frac{1}{2} (B_{\mu\nu} + B_{\nu\mu}) - \frac{1}{3} \Theta h_{\mu\nu}$

and: $\tau_{\mu\nu} = \frac{1}{2} \Theta h_{\mu\nu}$ - the rest term.
vector $\gamma'$ changes its direction and length by $B^\nu{}_{\mu} \gamma'$. The tensor $B^\nu{}_{\mu}$ can be decomposed covariantly and uniquely into:

$B^\nu{}_{\mu} = \omega^\nu{}_{\mu} + \delta^\nu{}_{\mu} + \epsilon^\nu{}_{\mu}$

(all 3 terms are tensors because the split is covariant)

We have: $\omega^\nu{}_{\mu} = \frac{1}{2}(B^\nu{}_{\mu} - B^\mu{}_{\nu})$, clearly.

But $\delta^\nu{}_{\mu}$, $\epsilon^\nu{}_{\mu}$, $\omega^\nu{}_{\mu}$? In preparation: define the projector $h_{\mu\nu}$ onto $(1, \overline{\epsilon})$. \[\text{Page 99/147}\]
\[ -1 \quad \Rightarrow \quad \xi \cdot (\mathbf{v} - \mathbf{v}_1) = \mathbf{v}_2 - \mathbf{v}_1 \] 
\[ \Rightarrow \quad \xi = \mathbf{v}_2 \] 
\[ \Rightarrow \quad \xi \cdot (\mathbf{v} - \mathbf{v}_1) = \xi \cdot \mathbf{v}_2 \] 
\[ \Rightarrow \quad \xi \cdot (\mathbf{v} - \mathbf{v}_1) = \xi \cdot \mathbf{v}_2 \] 
\[ \Rightarrow \quad B^\nu_{\mu} := \xi \cdot \mathbf{v}_2 \] 

\[ \Rightarrow \quad \text{Along the geodesic, } \xi, \text{ the deviation vector } \xi \cdot \mathbf{v} \text{ changes its direction and length by } B^\nu_{\mu} \xi. \]

- The tensor \( B^\nu_{\mu} \) can be decomposed covariantly and uniquely into:

\[ B_{\mu \nu} = \omega_{\mu \nu} + \xi_{\mu} \xi_{\nu} + \tau_{\mu \nu} \]  
\[ \text{symmetric and trace = 0} \] 
\[ \text{antisymmetric} \]

\[ \text{We have: } \omega_{\mu \nu} = \frac{1}{2} (B_{\mu \nu} - B_{\nu \mu}), \text{ clearly.} \]
\[ \Rightarrow \quad \xi^\alpha \dot{\xi}_\alpha = \eta^\nu \eta_\nu \]

\[ \Rightarrow \quad \xi^\alpha \dot{\xi}_\alpha = \eta^\nu \eta_\nu = \xi^\nu \mathcal{B}^\nu_\mu \eta_\mu \]

Along the geodesic, \( \xi \), the deviation vector \( \eta^\nu \) changes its direction and length by \( \mathcal{B}^\nu_\mu \eta^\mu \).

- The tensor \( \mathcal{B}^\nu_\mu \) can be decomposed covariantly and uniquely into:

\[ \mathcal{B}^\nu_\mu = \omega^\nu_\mu + d^\nu_\mu + \xi^\nu \eta_\mu \]

\( \text{symmetric and traceless} \)

\( \text{antisymmetric} \)

\( \text{rest} \)

(all 3 terms are tensors (because the split is covariant))

We have: \( \omega^\nu_\mu = \frac{1}{2} (\mathcal{B}^\nu_\mu - \mathcal{B}^\mu_\nu) \), clearly.

But \( \xi^\nu \eta_\mu = ? \)
Check: is $h_{\mu\nu} \omega^\nu$ really always 1 to 0?

Indeed: $\omega^\nu h_{\mu\nu} = (\omega,\omega) + (\omega,\tilde{\omega})(\omega,\omega) = 0$

Define: The "expansion", $\theta$, is defined as the magnitude of the spatial part of $B$:

$\theta := B^\mu{}^\nu h_{\mu\nu}$

Claim: $Tr(B) = \theta$

Indeed: $\theta = B^\mu{}^\nu h_{\mu\nu} = B^\mu{}^\nu g_{\mu\nu} + 5^\mu\tilde{\omega} B_{\mu\nu} = Tr(B) + 5^\mu\tilde{\omega} \alpha \delta^\mu_\nu$

($= 0$ because $\tilde{\omega} \delta^\mu_\nu = 0$ for $\mu = \nu$)
Indeed: $\tilde{g}^{-1} h_{\mu \nu} w = (w, w) + (w, j)(w, w) = 0$

Define: The "expansion" $\theta$, is defined as the magnitude of the spatial part of $B$:

$$\theta := B^{\mu \nu} h_{\mu \nu}$$

Claim: $T_r(B) = \theta$

Indeed: $\theta = B^{\mu \nu} h_{\mu \nu} = B^{\mu \nu} g_{\mu \nu} + \tilde{g}^{\mu \nu} \tilde{g}_{\rho \sigma} B^{\rho \sigma}$

$= T_r(B) + \tilde{g}^{\mu \nu} \tilde{g}_{\rho \sigma} \partial_{[\rho} \partial_{\sigma]} w$

Therefore: $\phi_{\mu \nu} = \frac{1}{2} (B_{\mu \nu} + B_{\nu \mu}) - \frac{1}{3} \theta h_{\mu \nu}$
Define: The "expansion", θ, is defined as the magnitude of the spatial part of \( B \):

\[
\theta := B^{\mu\nu} h_{\mu\nu}
\]

Claim: \( T_\nu(B) = \theta \)

Indeed:

\[
\theta = B^{\mu\nu} h_{\mu\nu} = B^{\mu\nu} g_{\mu\nu} + \nabla^\nu \nabla_\nu B^{\mu\nu} = T_\nu(B) + \nabla^\nu \nabla_\nu \theta
\]

Therefore:

\[
\sigma_{\mu\nu} = \frac{1}{2} (B_{\mu\nu} + B_{\nu\mu}) - \frac{1}{3} \Theta h_{\mu\nu}
\]

and:

\[
t_{\mu\nu} = \frac{1}{3} \Theta h_{\mu\nu} \quad \text{is the "rest term"}
\]
Define: The "expansion", \( \Theta \), is defined as the magnitude of the spatial part of \( B \):

\[
\Theta = B^{\mu \lambda} h_{\mu \lambda}
\]

Claim: \( T_{\tau} (B) = \Theta \)

Indeed: \[
\Theta = B^{\mu \lambda} h_{\mu \lambda} = B^{\mu \lambda} g_{\mu \lambda} + \mathbf{\varepsilon}^{\mu \lambda \rho \sigma} B_{\rho \sigma}
\]

\[
= T_{\tau} (B) + \mathbf{\varepsilon}^{\mu \lambda \rho \sigma} \partial_{\mu} \mathbf{\varepsilon}_{\rho \sigma}
\]

because \( \varepsilon^{\mu \lambda \rho \sigma} = 0 \) for g-endcs.

Therefore: \[
\rho_{\mu \lambda} = \frac{1}{2} (B_{\mu \lambda} + B_{\lambda \mu}) - \frac{1}{3} \Theta h_{\mu \lambda}
\]

the part of \( B_{\mu \lambda} \) which is symmetric and traceless.

and:

\[
t_{\mu \lambda} = \frac{1}{3} \Theta h_{\mu \lambda}
\]

the "rest term".
Indeed: \( \theta = B^{\mu \nu} h_{\mu \nu} = B^{\mu \nu} g_{\mu \nu} + 3 g^{\mu \nu} B_{\mu \nu} \)

\[ = T^\nu (B) + 3 g^{\mu \nu} \partial_\mu \partial_\nu \theta \]

\( = 0 \) because \( \partial_\mu \partial_\nu \theta = 0 \)

for vanishing.

Therefore:

\[ d_{\mu \nu} = \frac{1}{2} (B_{\mu \nu} + B_{\nu \mu}) - \frac{1}{3} \theta h_{\mu \nu} \]

because:

\[ T^\nu (B) = \frac{1}{2} g^{\mu \nu} B_{\mu \nu} \]

\[ = \frac{1}{4} (g_{\mu \nu} + g_{\nu \mu}) \]

\[ = \frac{1}{4} \theta \]

the part of \( B_{\mu \nu} \) which is symmetric and traceless.

and:

\[ t_{\mu \nu} = \frac{1}{3} \theta h_{\mu \nu} \]

\( \approx \) the "rest term".

\[ \square \] Interpretation:

a) \( \omega_{\mu \nu} \) is antisymmetric: \( \omega_{\mu \nu} = -\omega_{\nu \mu} \)

it generates Lorentz transformation for \( \eta \).
Therefore:
\[ \phi_{\mu \nu} = \frac{1}{4} (B_{\mu \nu} + B_{\nu \mu}) - \frac{1}{3} \Theta h_{\mu \nu} \]

and:
\[ t_{\mu \nu} = \frac{1}{3} \Theta h_{\mu \nu} \]

the part of \( B_{\mu \nu} \) which is symmetric and traceless.

Interpretation:

a) \( \omega_{\mu \nu} \) is antisymmetric: \( \omega_{\mu \nu} = -\omega_{\nu \mu} \)

\[ \Rightarrow \text{it generates Lorentz transformation for } \gamma. \]

but all \( \gamma \) are \( \perp \) to the time direction

\[ \Rightarrow \omega_{\mu \nu} \text{ generates spatial rotations of neighboring} \]
Interpretation:

a) \( \omega_{\mu \nu} \) is antisymmetric: \( \omega_{\mu \nu} = -\omega_{\nu \mu} \)

\( \Rightarrow \) it generates Lorentz transformation for \( \eta \).

but all \( \eta \) are \( \pm 1 \) to the time direction

\( \Rightarrow \) \( \omega_{\mu \nu} \) generates spatial rotations of neighboring geodesics around another. So, \( \omega_{\mu \nu} \) is called

\[ \omega = "Twiists tensor" \]

One can prove: (nontrivial)

If one chooses the congruence \( \alpha \)
**Interpretation:**

a) \( \omega_{\nu} \) is antisymmetric: \( \omega_{\nu} = -\omega_{\nu} \)

\[ \Rightarrow \text{it generates Lorentz transformation for } \eta. \]

but all \( \eta \) are \( \perp \) to the time direction

\[ \Rightarrow \omega_{\nu} \text{ generates spatial rotations of neighboring geodesics around another. So, } \omega_{\nu} \text{ is called} \]

\[ \omega = "\text{Torsion tensor"} \]

One can prove: (non-trivial)

If one chooses the congruence of geodesics \( \perp \) to \( \Sigma \) then \( \omega_{\nu} = 0. \)
Interpretation:

a) $\omega_{\rho \sigma} \text{ is antisymmetric: } \omega_{\rho \sigma} = -\omega_{\sigma \rho}$

$\Rightarrow$ it generates Lorentz transformation for $\eta$.

but all $\eta$ are $\perp$ to the time direction

$\Rightarrow$ $\omega_{\rho \sigma}$ generates spatial rotations of neighboring geodesics around another. So, $\omega_{\rho \sigma}$ is called

$\omega = \text{"Twists tensor"}$

One can prove: (non-trivial)

If one chooses the congruence of geodesics $I$ to $\Sigma$ then $\omega_{\rho \sigma} = 0$. 
a) \( \omega_{\mu} \) is antisymmetric: \( \omega_{\mu} = -\omega_{\mu} \)

\[ \Rightarrow \text{it generates Lorentz transformation for } \gamma. \]

but all \( \gamma \) are \( \perp \) to the time direction

\[ \Rightarrow \omega_{\mu} \text{ generates spatial rotations of neighboring geodesics around another. So, } \omega_{\mu} \text{ is called } \omega = "Twists tensor" \]

One can prove: (non-trivial)

If one chooses the congruence of geodesics \( \perp \) to \( \Sigma \) then \( \omega_{\mu} = 0. \)
\( \omega \) is a one-form symmetric. \( \omega_{\mu} = \omega_{\nu} \)

\( \Rightarrow \) it generates Lorentz transformation for \( \gamma \).

but all \( \gamma \) are \( \perp \) to the time direction

\( \Rightarrow \) \( \omega_{\mu} \) generates spatial rotations of neighboring geodesics around another. So, \( \omega_{\mu} \) is called

\[ \omega = \text{"Twists tensor"} \]

One can prove: (non-trivial)

If one chooses the congruence of geodesics \( \perp \) to \( \Sigma \) then \( \omega_{\mu} = 0 \).

b.) \( \omega_{\mu} \) is symmetric, \( \omega_{\mu} = \omega_{\nu} \) (i.e. hermitian)
\[ \omega = \text{Twist's tensor} \]

One can prove: (nontrivial)

If one chooses the congruence of geodesics \( T \) to \( \Sigma \) then \( \omega_{\mu\nu} = 0 \).

b.) \( g_{\mu\nu} \) is symmetric, \( g_{\mu\nu} = g_{\nu\mu} \). (i.e. hermitian)

Consider "diagonalized", by suitable choice of cd basis. ✋

\( \Rightarrow \) \( g_{\mu\nu} \) changes the relative lengths of the basis vectors, by multiplying them with its eigenvalues.

i.e. points on a sphere will under geodesic flow become points on an ellipsoid.
6.) \( \sigma_{\mu} \) is symmetric, \( \sigma_{\mu} = \sigma_{\nu} \) (i.e. hermitean)

Consider "diagonalized", by suitable choice of \( \sigma \) basis.

\( \Rightarrow \) \( \sigma_{\mu} \) changes the relative lengths of the basis vectors, by multiplying them with its eigenvalues.

i.e. points on a sphere will under geodesic flow become points on an ellipsoid.

Note: Since \( Tr(\sigma) = 0 \) we have \( det(e^{\sigma}) = 1 \)

\( \Rightarrow \) The volume spanned by basis vectors stays the same under the action of \( \sigma \).
b.) \( \sigma \) is symmetric, \( \sigma_{ij} = \sigma_{ji} \). (i.e. hermitian)

Consider "diagonalized", by suitable choice of \( \mathbf{e} \) basis.

\[ \Rightarrow \sigma \text{ changes the relative lengths of the basis vectors, by multiplying them with its eigenvalues.} \]

i.e. points on a sphere will under geodesic flow become points on an ellipsoid.

\[ \Rightarrow \text{The volume spanned by basis vectors stays the same under the action of } \sigma. \]

Note: Since \( \text{Tr} \ (\sigma) = 0 \) we have \( \det(\mathbf{e}^{\sigma}) = 1 \)

\[ \Rightarrow \text{infinitesimal transport along geodesics} \]
b.) $\bar{g}_{\mu\nu}$ is symmetric, $\bar{g}_{\mu\nu} = \bar{g}_{\nu\mu}$. (i.e. hermitean)

Consider "diagonalized", by suitable choice of cd basis.

$\Rightarrow \bar{g}_{\mu\nu}$ changes the relative lengths of the basis vectors, by multiplying them with its eigenvalues.

i.e. points on a sphere will under geodesic flow become points on an ellipsoid.

Note: Since $Tr(\bar{g}) = 0$ we have $\det(e^{\bar{g}}) = 1$

$\Rightarrow$ The volume spanned by basis vectors stays the same under the action of $\bar{g}$.

$\Rightarrow$ Definition: $\bar{g}_{\mu\nu} = \text{"Shear tensor"}$
$\Rightarrow$ $\Gamma$ changes the relative lengths of the basis vectors, by multiplying them with its eigenvalues.

i.e. points on a sphere will under geodesic flow become points on an ellipsoid.

Note: Since $T^\dagger (\sigma) = 0$ we have $\det(e^{\sigma \sigma}) = 1$.

$\Rightarrow$ The volume spanned by basis vectors stays the same under the action of $\sigma$.

$\Longrightarrow$ Definition: $\Gamma_{\mu\nu} =$ "Shear tensor"
c.) While the twist and shear tensors are both traceless and therefore volume-preserving, we see that the trace part, $\Theta$, i.e., more precisely

$$\varepsilon_{\mu} = \frac{1}{3} \Theta h_{\mu} =: "Expansion\ tensor"$$

is indeed generating the spatial expansion or contraction of nearby geodesics!

Evolution of $\Theta$ along a geodesic?
c.) While the twist and shear tensors are both traceless and therefore volume-preserving, we see that the trace part, $\Theta$, i.e., more precisely

$$t_{\mu\nu} = \frac{1}{3} \Theta h_{\mu\nu} = \text{"Expansion tensor"}$$

is indeed generating the spatial expansion or contraction of nearby geodesics!

Evolution of $\Theta$ along a geodesic?
c.) While the twist and shear tensors are both traceless and therefore volume-preserving, we see that the trace part, \( \Theta \), i.e., more precisely

\[ \tau_{\nu} = \frac{1}{3} \Theta h_{\nu} = \text{"Expansion tensor"} \]

recall: is projector on spatial part.

is indeed generating the spatial expansion or contraction of nearby geodesics!

**Evolution of \( \Theta \) along a geodesic?**
is indeed generating the spatial expansion or contraction of nearby geodesics!

Evolution of $\theta$ along a geodesic?

The Raychaudhuri equation:

Consider:

$\xi^c \nabla_c \xi^a = \xi^c \nabla_a \xi^a + R_{abac} \xi^b \xi^d$

by definition of the curvature tensor

$\xi^c \nabla_c \xi^{b,c} = \nabla_c (\xi^c \xi^d) - (\nabla^c \xi^d)(\nabla_c \xi^a) + R_{cdac} \xi^b$
is indeed generating the spatial expansion or contraction of nearby geodesics!

Evolution of $\Theta$ along a geodesic?

The Raychaudhuri equation:

Consider:

$\xi^c \nabla_c B_{ab} \quad$ by def.

$\xi^c \nabla_c \xi^a = \xi^c \nabla_b \xi^a + R_{cba} \xi^c \xi^d$

by definition of the curvature tensor

$\xi^c \nabla_c \xi^a = \nabla_c (\xi^c \nabla_c \xi^a) - (\nabla_b \xi^c)(\nabla_c \xi^a) + R_{cba} \xi^c \xi^d$

$\xi^c \nabla_c B_{ab} = \xi^c \nabla_c \xi^a$
The Raychaudhuri equation:

Consider:

\[ \xi^c \nabla_c B_{ab} = \xi^c \nabla_c \xi^a = \xi^c \nabla_c \xi^a + R_{cbda} \xi^c \xi^d \]

or

\[ \xi^c B_{babc} = \nabla_b (\xi^c \nabla_c \xi^a) - \left( \nabla_b \xi^c \right) \left( \nabla_c \xi^a \right) + R_{cbda} \xi^c \xi^d \]

by definition of the curvature tensor

by definition of the curvature tensor

by definition of the curvature tensor

\[ = -B^c_{\;\;b} B_{ac} + R_{cbda} \xi^c \xi^d \]

use next: \( B_{\mu\nu} = \frac{1}{3} \Theta_{\mu\nu} + \xi_{\mu\nu} + \omega_{\mu\nu} \)

The Raychaudhuri equation is the trace of this equation:

\[ \frac{d \theta}{d t} = \Theta \theta = - \frac{1}{3} \theta^2 - G_{ab} c^{ab} + \omega_{ab} \omega^{ab} - R_{cd} \xi^c \xi^d \]

recall: Ricci tensor is

\[ \left( R_{cd} \right) = Reda_a \]
The Raychaudhuri equation:

Consider:

\[ \xi^c \nabla_c B_{ab} = \xi^c \nabla_c \xi^a = \xi^c \nabla_b \xi^a + R_{cbad} \xi^c \xi^d \]

\[ \xi^c B_{abc} = \nabla_\xi (\xi^c \nabla_c \xi^a) = \xi^c \nabla_c \xi^a - (\nabla_b \xi^c)(\nabla_c \xi^a) + R_{cbad} \xi^c \xi^d \]

\[ \frac{d}{dt} B_{ab} = -B_{ab} + R_{cbad} \xi^c \xi^d \]

Use next: \( B_{\mu\nu} = \frac{1}{3} \Theta h_{\mu\nu} + \omega_{\mu\nu} + \omega_{\mu\nu} \)

The Raychaudhuri equation is the trace of this equation:

Recall: Ricci tensor is \( R_{ab} = R_{ba} \)

\[ \frac{d}{dt} \Theta = \nabla_\xi \Theta = -\frac{1}{3} \Theta^2 - G_{ab} \xi^a \xi^b + \omega_{ab} \omega^{ab} - R_{cd} \xi^c \xi^d \]

\[ \Theta = \frac{\partial}{\partial t} \text{ and } \text{Tr}(B) = \Theta \]
The Raychaudhuri equation is the trace of this equation:

\[
\frac{d \Theta}{dt} = \nabla^c \Theta = - \frac{1}{3} \Theta^2 - G_{ab} G^{ab} + \omega_{ab} \omega^{ab} - R_{cd} \delta^c \delta^d
\]

\[
\frac{d}{dt} = \Theta \quad \text{and} \quad \text{Tr} (B) = \Theta
\]

Dynamics?

Assume that
Consider:

\[ \xi^c \nabla_c B_{ab} = \xi^c \nabla_c \xi_a = \xi^c \nabla_b \xi_c \xi_a + R_{cba}^\phantom{c} \xi^d \xi^e \xi_d \]

by definition of the curvature tensor

\[ \xi^c B_{ab,c} = \]

Labute rule

\[ = \nabla_b (\xi^c \nabla_c \xi_a) - (\nabla_b \xi^c) (\nabla_c \xi_a) + R_{cba}^\phantom{c} \xi^d \xi^e \xi_d \]

is because geodesic

\[ \frac{\partial}{\partial b} = -B^c_b B_{ac} + R_{cba}^\phantom{c} \xi^d \xi^e \xi_d \]

use next: \( B_{\mu\nu} = \frac{1}{3} \Theta h_{\mu\nu} + \epsilon_{\mu\nu} + \epsilon_{\mu\nu} \)

The Raychaudhuri equation is the trace of this equation:

\[ \frac{d}{d\tau} \Theta = \mathcal{V} \Theta = -\frac{1}{3} \Theta^2 - G_{ab} \xi^{ab} + \omega_{\mu\nu} \omega^{\mu\nu} - R_{\mu\nu} \xi^\mu \xi^\nu \]

recall: \( \xi \) Ricci tensor is

\[ \mathcal{R}_{\mu\nu} = \mathcal{R}_{\mu\nu}^a \xi^a \]

\[ \frac{d}{d\tau} \Theta = \mathcal{V} \Theta = \frac{1}{3} \Theta^2 - G_{ab} \xi^{ab} + \omega_{\mu\nu} \omega^{\mu\nu} - R_{\mu\nu} \xi^\mu \xi^\nu \]

always positive number if

choose 

\[ \Sigma \]

pos. or neg.?
\[ \frac{\partial}{\partial t} \mathbf{B}_{\text{obj}} = \nabla \times (\mathbf{E} \times \mathbf{B}_{\text{obj}}) - (\nabla \times \mathbf{E}) \times \mathbf{B}_{\text{obj}} + \mathbf{R}_{\text{c6a} d} \mathbf{E}^{c d} \]

- because gradient

\[
\frac{\partial}{\partial t} \mathbf{B}_{\text{obj}} = - \mathbf{E} \times \mathbf{B}_{\text{obj}} + \mathbf{R}_{\text{c6a} d} \mathbf{E}^{c d}
\]

use next: \( B_{\mu \nu} = \frac{1}{3} \theta \mathbf{h}_{\mu \nu} + \mathbf{h}_{\mu \nu} + \mathbf{w}_{\mu \nu} \)

The Raychaudhuri equation is the trace of this equation:

\[
\frac{d \theta}{d t} = \frac{\partial}{\partial t} \theta = - \frac{1}{3} \theta^2 - C_{ab} E^{ab} + \omega_{ab} \omega^{ab} - R \mathbf{E} \mathbf{E} - \text{pos. or neg?}
\]

\[
\frac{d}{d t} = \theta \text{ and } \text{Tr}(B) = \theta
\]

Dynamics?

Assume that

\[ R \mathbf{E} \mathbf{E} \geq 0 \text{ for all times} \]
Dynamics?

Assume that

\[ R_{\mu\nu} S^\mu S^\nu \geq 0 \] for all timelike \( S \)

i.e., using the Einstein equation

\[ R_{\mu\nu} = 8\pi G (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T) \]

we are assuming that

\[ T_{\mu\nu} S^\mu S^\nu - \frac{1}{2} g_{\mu\nu} T \geq 0 \]

i.e. that

\[ T_{\mu\nu} S^\mu S^\nu \geq - \frac{1}{2} T \] whenever \( S^\mu S^\nu \leq 0 \)
Dynamics?

Assume that

\[ R_{\mu \nu} S^{\mu} S^{\nu} \geq 0 \quad \text{for all timelike } S \]

i.e., using the Einstein equation

\[ R_{\mu \nu} = 8\pi G (T_{\mu \nu} - \frac{1}{2} g_{\mu \nu} T) \]

we are assuming that

\[ T_{\mu \nu} S^{\mu} S^{\nu} - \frac{1}{2} S^{\mu} S_{\mu} T \geq 0 \]

i.e. that \( T_{\mu \nu} S^{\mu} S^{\nu} \geq -\frac{1}{2} T \) whenever \( S^{\mu} S_{\mu} < 0 \)
The Raychaudhuri equation is the trace of this equation:

\[ \frac{d\theta}{dt} = \Psi \theta = -\frac{1}{3} \theta^2 - \Gamma^a_{ab} \dot{\gamma}^a + \omega^a_{,b} \omega^b - \text{Red} \delta^a \]

recall Ricci tensor is \( \sqrt{\text{Red} = \text{Red}_{\text{ab}}} \)

Always positive \( \text{Volume of choose congruence} \) is pos. or neg?

Dynamics?

Assume that

\[ R_{\mu \nu} \delta^a \delta^b \geq 0 \quad \text{for all timelike} \ \delta \]

i.e., using the Einstein equation

\[ R_{\mu \nu} = 8\pi G (T_{\mu \nu} - \frac{1}{2} g_{\mu \nu} T^a) \]
recall: \[ \frac{\partial \gamma}{\partial p^2} = \Delta \theta = -\frac{1}{3} \theta - \frac{G_{a b} g^{a b}}{\omega_{a b} \omega^{a b}} - \frac{K c d g_{a b}}{\Sigma \text{pos. or neg?}} \]

\[ \sum \frac{d}{dt} = \dot{\theta} \text{ and } Tr(\dot{g}) = \theta \]

**Dynamics?**

Assume that

\[ R_{\mu \nu} g^{\mu \nu} \geq 0 \text{ for all timelike } \xi \]

i.e., using the Einstein equation

\[ R_{\mu \nu} = 8\pi G \left( T_{\mu \nu} - \frac{1}{2} g_{\mu \nu} T \right) \]

we are assuming that

\[ T_{\mu \nu} g^{\mu \nu} - \frac{1}{2} g_{\mu \nu} T \geq 0 \]
The Raychaudhuri equation is the trace of this equation:

\[ \frac{d\theta}{d\tau} = \frac{2}{3} \theta^2 - \frac{1}{3} \theta^3 - G_{ab} \theta^a \theta^b + \omega_{a b} \omega^a \theta^b - \text{Red} \delta \delta \delta \delta \]

Recall: Ricci tensor is \( \sqrt{\text{Red} = \text{Red}_a^a} \)

\[ \frac{d\theta}{d\tau} = \theta \text{ and } \text{Tr} (\mathbf{B}) = \theta \]

Dynamics?

Assume that

\[ R_{\mu\nu} \delta^\mu \delta^\nu \geq 0 \text{ for all timelike } \delta \]

i.e., using the Einstein equation

\[ R_{\mu\nu} = 8\pi G (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T_{\alpha\alpha}) \]
\[ R_{\mu \nu} \delta^{\mu}_0 \delta^{\nu}_0 \geq 0 \quad \text{for all timelike } \delta^{\mu}_0 \delta^{\nu}_0 \]

i.e., using the Einstein equation

\[ R_{\mu \nu} = 8\pi G \left( T_{\mu \nu} - \frac{1}{2} g_{\mu \nu} T \right) \]

we are assuming that

\[ T_{\mu \nu} \delta^{\mu}_0 \delta^{\nu}_0 - \frac{1}{2} \delta^{\mu}_0 \delta^{\nu}_0 T \geq 0 \]

i.e. that

\[ T_{\mu \nu} \delta^{\mu}_0 \delta^{\nu}_0 \geq -\frac{1}{2} T \quad \text{whenever } \delta^{\mu}_0 \delta^{\nu}_0 < 0 \]

i.e. the Strong Energy Condition.

\[ \boxed{\text{Then:} \quad \frac{d\theta}{\theta} + \frac{1}{2} \theta^2 \leq 0} \]
i.e. the Strong Energy Condition.

Then:

\[ \frac{d\theta}{dt} + \frac{1}{3} \theta^2 \leq 0 \]

\[ \Rightarrow \frac{d}{dt} \theta^{-1} \geq \frac{1}{3} \text{ Rewrite as } d\theta^{-1} \geq \frac{1}{3} dt \text{ and integrate:} \]

\[ \Rightarrow \theta^{-1} \geq \frac{1}{3} (\theta(2) - (\theta(2) - \theta^{-1}(2) - \frac{1}{3}) \text{ on } \Sigma \text{ at singularity } \Theta' \]

\[ \theta^{-1} \geq \frac{1}{3} (\theta(2) - (\theta^{-1}(2) + \theta^{-1}(2)) \text{ (**) } \]

Consider the cases when the geodesics are initially:

a.) diverging, i.e., \( \theta(2) > 0 \) (expanding universe)

b.) converging, i.e., \( \theta(2) < 0 \) (contracting universe)
The Raychaudhuri equation is the trace of this equation:

\[ \frac{d\theta}{d\tau} = \mathcal{V} \theta = -\frac{1}{3} \theta^2 - \mathcal{G}_{ab} \mathcal{G}^{ab} + \omega_a \omega^a - R_{cd} \delta^c_0 \delta^d_0 \]

Recall, Ricci tensor is
\[ \sqrt{\text{Red}} = \text{Ric} \]
always positive

\[ \theta \]
and \[ \text{Tr}(\mathcal{B}) = \theta \]

Dynamics?

Assume that

\[ R_{\mu
u} \delta^\nu \delta^\mu \geq 0 \]

for all timelike \( \delta \)

i.e., using the Einstein equation

\[ R_{\mu\nu} = 8\pi G (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^a) \]
i.e., using the Einstein equation

$$R_{\mu\nu} = 8\pi G (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T)$$

we are assuming that

$$T_{\mu\nu} > 0$$

e.g. that

$$T_{\mu\nu} > \frac{1}{2} T$$

where $$\phi, \phi > 0$$

i.e. the Strong Energy Condition.

Then:

$$\frac{d\theta}{dt} + \frac{1}{3} \theta^2 \leq 0$$

$$\Rightarrow \frac{d}{dt} \theta' \geq \frac{1}{3}$$

Rewrite as

$$d\theta' \geq \frac{1}{3} dt$$

Integrate on $$\Sigma$$ at constant $$t$$.
we are assuming that
\[ T_{\mu \nu} S_{\mu} S_{\nu} - \frac{1}{2} g_{\mu \nu} T > 0 \]
i.e. that \[ T_{\mu \nu} S_{\mu} S_{\nu} \geq -\frac{1}{2} T \] whenever \( S_{\mu} S_{\nu} < 0 \)
i.e. the Strong Energy Condition.

Then:
\[ \frac{d\theta}{dt} + \frac{1}{3} \theta^2 \leq 0 \]

\[ \Rightarrow \frac{d}{dt} \theta' \geq \frac{1}{3} \text{ Rewrite as } d\theta' \geq \frac{1}{3} dt \text{ and integrate:} \]
\[ \Rightarrow \theta''(t) \geq \frac{1}{3} (\frac{2}{3} - (2) (\frac{2}{3} - 2) - (2) (2)) + \theta''(t) \text{ (Case b)} \]
Dynamics?

Assume that

\[ R_{\mu\nu} g^{\mu\nu} \geq 0 \quad \text{for all timelike } g \]

i.e., using the Einstein equation

\[ R_{\mu\nu} = 8\pi G \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) \]

we are assuming that

\[ T_{\mu\nu} g^{\mu\nu} - \frac{1}{2} g^{\mu\nu} T g_{\mu\nu} \geq 0 \]
The Raychaudhuri equation is the trace of this equation:

\[ \text{Recall: Ricci tensor is} \quad \nabla^a R_{ab} = R_{ab} \]

\[ \frac{d\theta}{dt} = \nabla_\theta \theta = -\frac{1}{3} \theta^2 - C_{abc} C^{abc} + \omega_{ab} \omega^{ab} - R_{cd} \gamma^c \gamma^d \]

\[ \text{Always positive} \quad \text{if} \quad \nabla (\theta) = \theta \]

Dynamics?

Assume that

\[ R_{\mu\nu} \delta^\mu \delta^\nu \geq 0 \quad \text{for all timelike} \ \delta \]

i.e., using the Einstein equation

\[ \phi \phi^*(\nabla^2 + m^2) \]

\[ \phi \phi^*(\nabla^2 + m^2) = 0 \]

\[ \nabla_\mu \phi \phi^* = 0 \]

\[ \nabla_\mu \phi \phi^* = 0 \]
i.e. that \( T_{\mu\nu} \geq -\frac{1}{2} T \) whenever \( T < 0 \).

i.e. the Strong Energy Condition.

Then:

\[
\frac{d\Theta}{dt} + \frac{1}{3} \Theta^2 \leq 0
\]

\[
\Rightarrow \frac{d}{d\tau} \Theta^{-1} \geq \frac{1}{3} \text{ Rewritten as } d\Theta^{-1} \geq \frac{1}{3} d\tau \text{ and integrate: }
\]

\[
\Theta^{-1} \geq \frac{1}{3} (\tau - \tau_0) - \Theta^{-1}(\tau_0) \]

\[
\Theta^{-1} \geq \frac{1}{3} (\tau - \tau_0) + \Theta^{-1}(\tau_0) \quad (\ast)
\]

Consider the cases when the geodesics are initially:

a.) diverging, i.e., \( \Theta(\tau_0) > 0 \) (expanding universe)

b.) converging, i.e., \( \Theta(\tau_0) < 0 \) (contracting universe)
Then:
\[ \frac{d\Theta}{d\tau} + \frac{1}{3} \Theta^2 \leq 0 \]
\[ \Rightarrow \frac{d}{d\tau} \Theta^- \geq \frac{1}{3} \] Rewrite as \[ d\Theta^- \geq \frac{1}{3} d\tau \] and integrate:
\[ \Theta^- \geq \frac{1}{3} (\tau_2 - \tau_0) + \Theta^- (\tau_0) \]

Consider the cases when the geodesics are initially:

a.) diverging, i.e., \( \Theta(\tau_0) > 0 \) (expanding universe)
b.) converging, i.e., \( \Theta(\tau_0) < 0 \) (contracting universe)

Now, follow the evolution in the two cases:

a.) backwards in time (which changes the sign of \( \frac{d\Theta}{d\tau} \))
b.) forward in time:
\[
\frac{d\theta}{d\tau} + \frac{1}{3}\theta^2 \leq 0
\]

\[\Rightarrow \frac{d}{d\tau} \theta \Rightarrow \frac{1}{3}\theta^2 \leq 0\]

\[\theta'(t) \Rightarrow \theta'(t) = \frac{1}{3}(\theta(t) - \theta_0(t)) \geq \frac{1}{3}(\theta(t) - \theta_0(t)) \]

\[\theta'(t) \Rightarrow \frac{1}{3}(\theta(t) - \theta_0(t)) \geq \frac{1}{3}(\theta(t) - \theta_0(t)) \] (\textbf{(*)})

Consider the cases when the geodesics are initially

\begin{itemize}
  \item [a.)] diverging, i.e., \(\theta(t) > 0\) (expanding universe)
  \item [b.)] converging, i.e., \(\theta(t) < 0\) (contracting universe)
\end{itemize}

Now, follow the evolution in the two cases

\begin{itemize}
  \item [a.)] backwards in time (which changes the sign of \(\frac{d\theta}{d\tau}\))
  \item [b.)] forward in time:
\end{itemize}
Consider the cases when the geodesics are initially

a.) diverging, i.e., $\Theta(\tau) > 0$ (expanding universe)

b.) converging, i.e., $\Theta(\tau) < 0$ (contracting universe)

Now, follow the evolution in the two cases

a.) backwards in time (which changes the sign of $d\Theta/d\tau$)

b.) forward in time:

Conclusion: Eq. (\star) implies that $\Theta(\tau)$ must go through 0, i.e.:

a.) for sufficiently early $\tau$, have $\Theta \to +\infty$, i.e.: Big Bang

b.) for sufficiently later $\tau$, have $\Theta \to -\infty$, i.e.: Big Crunch

Note: This type of reasoning leads also to further cosmological singularity theorems.
a.) diverging, i.e., $\theta(T) > 0$ (expanding universe)

b.) converging, i.e., $\theta(T) < 0$ (contracting universe)

Now, follow the evaluation in the two cases

a.) backwards in time (which changes the sign of $\frac{d\theta}{dt}$)

b.) forward in time:

Conclusion: Eq. (x) implies that $\theta(t)$ must go through 0, i.e.:

a.) for sufficiently early $t$, have $\theta \to +\infty$, i.e.: Big Bang

b.) for sufficiently later $t$, have $\theta \to -\infty$, i.e.: Big Crunch

Note: This type of reasoning leads also to further cosmological singularity theorems.

For another cosmological singularity theorem read
b. converging, i.e., $\Theta(\tau) < 0$ (contracting universe)

Assuming that $d\Theta/d\tau$ does not change before a time crossing of $\%$ occurs.

Now follow the evolution in the two cases

a. backwards in time (which changes the sign of $d\Theta/d\tau$)

b. forward in time:

**Conclusion:** Eq. (x) implies that $\Theta(\tau)$ must go through 0, i.e.,

a. for sufficiently early $\tau$, have $\Theta \to 0$, i.e.: Big Bang

b. for sufficiently late $\tau$, have $\Theta \to -\infty$, i.e.: Big Crunch

**Note:** This type of reasoning leads also to further cosmological singularity theorems.

E.g., another cosmological singularity theorem does not
Conclusion: Eq. (x) implies that $\Theta(t)$ must go through 0, i.e.,
a) for sufficiently early $t$, have $\Theta \to +\infty$, i.e.: Big Bang
b) for sufficiently late $t$, have $\Theta \to -\infty$, i.e.: Big Crunch

Note: This type of reasoning leads also to further cosmological singularity theorems.
E.g., another cosmological singularity theorem does not assume global hyperbolicity, and its conclusion is weak:
There is at least one incomplete timelike geodesic.

Remark: Singularity theorems suitable for black hole case also assume a trapped surface, i.e., a surface
Conclusion: Eq. (*) implies that $\Theta(t)$ must go through 0, i.e.:
a.) for sufficiently early $t$, have $\Theta \to +\infty$, i.e.: Big Bang
b.) for sufficiently late $t$, have $\Theta \to -\infty$, i.e.: Big Crunch

Note: This type of reasoning leads also to further cosmological singularity theorems.

E.g., another cosmological singularity theorem does not assume global hyperbolicity, and its conclusion is weaker:

There is at least one incomplete timelike geodesic.

Remark: Singularity theorems suitable for black hole case also assume a trapped surface, i.e., a surface on which both in- and outgoing null geodesics converge.