Evidence in favour of $\psi$-epistemic models
The analogy to Liouville mechanics
Example 1: The impossibility of discriminating non-orthogonal states

Consider

\[ |\psi_1\rangle \]

\[ |\psi_2\rangle \]
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Consider $|\psi_1\rangle$ and $|\psi_2\rangle$.

Newtonian analogy: Mysterious. No analogue of non-orthogonality.
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\[ |\psi_1\rangle \quad |\psi_2\rangle \]

Newtonian analogy: Mysterious. No analogue of non-orthogonality.

Liouville analogy: Natural

\[ \mu_1(x, p) \quad \mu_2(x, p) \]
Example 2: Lack of exponential divergence of states under chaotic evolution

In Newtonian mechanics, exponential divergence of ontic states is the signature of chaos.
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In Newtonian mechanics, exponential divergence of ontic states is the signature of chaos.

In quantum theory,

$$\langle \psi_1(t) | \psi_2(t) \rangle = \langle \psi_1(0) | U^* U | \psi_2(0) \rangle$$

$$= \langle \psi_1(0) | \psi_2(0) \rangle$$

No divergence!
Example 2: Lack of exponential divergence of states under chaotic evolution

In Newtonian mechanics, exponential divergence of ontic states is the signature of chaos.

In quantum theory,
\[
\langle \psi_1(t) | \psi_2(t) \rangle = \langle \psi_1(0) | U^\dagger U | \psi_2(0) \rangle \\
= \langle \psi_1(0) | \psi_2(0) \rangle
\]

No divergence!

Newtonian analogy: This is puzzling

Liouville analogy: This is natural, due to Liouville’s theorem
\[
\int dx dp \sqrt{\mu_1(x, p, t)} \sqrt{\mu_2(x, p, t)} = \int dx dp \sqrt{\mu_1(x, p, 0)} \sqrt{\mu_2(x, p, 0)}
\]
Example 3: The impossibility of cloning non-orthogonal states

(C. Fuchs, 1996)

Cloning the set \( \{ |\psi_1\rangle, |\psi_2\rangle \} \) implies \( |\psi_s\rangle \chi \rightarrow |\psi_s\rangle |\psi_s\rangle \) for \( s = 1, 2 \)
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Cloning the set $\{|\psi_1\rangle, |\psi_2\rangle\}$ implies $|\psi_s\rangle |\chi\rangle \rightarrow |\psi_s\rangle |\psi_s\rangle$ for $s = 1, 2$

By unitarity, the inner product must be constant
But $|\langle \psi_1 | \langle \chi | (|\psi_2\rangle |\chi\rangle)|^2 = |\langle \psi_1 | \psi_2\rangle|^2$

while $|\langle \psi_1 | \langle \psi_1 | (|\psi_2\rangle |\psi_2\rangle)|^2 = |\langle \psi_1 | \psi_2\rangle|^2$
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These are equal iff \( \langle \psi_1 | \psi_2 \rangle = 0 \) or 1 i.e. orthogonal or identical
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Liouville analogy:
Cloning the set \( \{ \mu_1(z), \mu_2(z) \} \) implies \( \mu_s(z) \nu(y) \rightarrow \mu_s(z) \mu_s(y) \) for \( s = 1, 2 \)
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By Liouville’s theorem, the classical fidelity must be constant
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But $\int dz dy \sqrt{\mu_1(z)\nu(y)} \sqrt{\mu_2(z)\nu(y)} = \int dz \sqrt{\mu_1(z)} \sqrt{\mu_2(z)}$

while $\int dz dy \sqrt{\mu_1(z)\mu_1(y)} \sqrt{\mu_2(z)\mu_2(y)} = \left( \int dz \sqrt{\mu_1(z)} \sqrt{\mu_2(z)} \right)^2$
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Cloning the set $\{ | \psi_1 \rangle, | \psi_2 \rangle \}$ implies $| \psi_s \rangle \langle \chi | \rightarrow | \psi_s \rangle | \psi_s \rangle$ for $s = 1, 2$

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But $\langle \psi_1 | \langle \chi | (| \psi_2 \rangle | \chi \rangle) = | \langle \psi_1 | \psi_2 \rangle |

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These are equal iff $\int dz \sqrt{\mu_1 (z)} \sqrt{\mu_2 (z)} = 0 \text{ or } 1$ i.e. disjoint or identical
Where the Liouville analogy fails

**Pure states:** In Liouville mechanics, they are Dirac-delta functions on phase space

Thus, they have strictly disjoint support
(hence distinguishable, clonable)

State of complete knowledge = Newtonian state

In other words: Quantum states are analogous to states of incomplete knowledge

Consider: **Liouville mechanics with an epistemic restriction**
Where the Liouville analogy fails

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In other words: Quantum states are analogous to states of incomplete knowledge

**Consider:** Liouville mechanics with an epistemic restriction
The analogy to Liouville mechanics with an epistemic restriction

Based primarily on unpublished work with Stephen Bartlett and Terry Rudolph
Liouville mechanics

\[ \mu(x, p) \]

What is a good epistemic restriction to apply? -- look to quantum mechanics
\[ \Delta x \Delta p \geq \frac{\hbar}{2} \]

\[ C_{x,p} = \frac{\langle \hat{x} \hat{p} + \hat{p} \hat{x} \rangle - \langle \hat{x} \rangle \langle \hat{p} \rangle}{\hbar} \]
Liouville mechanics

$\mu(x, p)$

What is a good epistemic restriction to apply?
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Quantum mechanics

Uncertainty principle:

$$\Delta^2 x \Delta^2 p - C^2_{x,p} \geq (\hbar/2)^2$$

where

$$\Delta^2 x \equiv \langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2$$

$$C_{x,p} \equiv \frac{1}{2} \langle \hat{x} \hat{p} + \hat{p} \hat{x} \rangle - \langle \hat{x} \rangle \langle \hat{p} \rangle$$

$$\langle \hat{A} \rangle \equiv \text{Tr}(\hat{A} \hat{\rho})$$
**Liouville mechanics**

\[ \mu(x, p) \]

What is a good epistemic restriction to apply? -- look to quantum mechanics

**Quantum mechanics**

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\[ \langle A \rangle \equiv \text{Tr}(\hat{A} \hat{\rho}) \]

**Liouville mechanics with an epistemic restriction**

Uncertainty principle:

\[ \Delta^2 x \Delta^2 p - C_{x,p}^2 \geq (\hbar/2)^2 \]

where

\[ \Delta^2 x \equiv \langle x^2 \rangle - \langle x \rangle^2 \]

\[ C_{x,p} \equiv \langle xp \rangle - \langle x \rangle \langle p \rangle \]

\[ \langle f(x, p) \rangle \equiv \int dx dp f(x, p) \mu(x, p) \]
Liouville mechanics with an epistemic restriction

Assume:

The classical uncertainty principle (for a single particle in 1D):

The only Liouville distributions that can be prepared are those that satisfy

$$\Delta^2_x \Delta^2_p - C_{x,p}^2 \geq (\hbar/2)^2$$

and that have maximal entropy for a given set of second-order moments.
Liouville mechanics with an epistemic restriction

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and that have maximal entropy for a given set of second-order moments.

Among $\mu(x,p)$ with a given set of second-order moments, Gaussian distributions maximize the entropy
Valid epistemic states for one canonical system

\[ \mu(x, p) \geq 0 \]

\[ \int \mu(x, p) dx dp = 1 \]
Pure epistemic states

Mixed epistemic states
Multiplicity of convex decompositions of a mixed epistemic state into pure epistemic states
Quantum mechanics

Uncertainty principle:

\[ \gamma(\hat{\rho}) + i\hbar \Sigma \geq 0 \]

\[ \gamma(\hat{\rho}) = 2 \begin{pmatrix} \Delta^2 x_1 & C_{x_1, p_1} & C_{x_1, x_2} & C_{x_1, p_2} & \ldots \\ C_{p_1, x_1} & \Delta^2 p_1 & C_{p_1, x_2} & C_{p_1, p_2} & \ldots \\ C_{x_2, x_1} & C_{x_2, p_1} & \Delta^2 x_2 & C_{x_2, p_2} & \ldots \\ C_{p_2, x_1} & C_{p_2, p_1} & C_{p_2, x_2} & \Delta^2 p_2 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \]

\[ \Sigma = \begin{pmatrix} 0 & -1 & \ldots \\ 1 & 0 & \ldots \\ \vdots & \vdots & \ddots \end{pmatrix} \]
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\[ \Sigma = \begin{pmatrix} 0 & -1 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & -1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \]

Single particle in 1d:

\[ 2 \begin{pmatrix} \Delta^2 x & C_{x,p} \\ C_{p,x} & \Delta^2 p \end{pmatrix} + i\hbar \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \geq 0 \]

\[ 2 \begin{pmatrix} \Delta^2 x & C_{x,p} - \frac{1}{2}i\hbar \\ C_{p,x} + \frac{1}{2}i\hbar & \Delta^2 p \end{pmatrix} \geq 0 \]

\[ \Delta^2 x \Delta^2 p - C_{x,p}^2 \geq (\hbar/2)^2 \]
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\[ \Sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 0 & -1 \\ 1 & 0 \\ \vdots & \vdots \end{pmatrix} \]

Liouville mechanics with an epistemic restriction

Uncertainty principle:

\[ \gamma(\mu) + i\hbar \Sigma \geq 0 \]

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\[ \Sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 0 & -1 \\ 1 & 0 \\ \vdots & \vdots \end{pmatrix} \]
Quantum mechanics

Uncertainty principle:
\[ \gamma(\hat{\rho}) + i\hbar \Sigma \geq 0 \]

Single particle in 1d:
\[
2 \left( \begin{array}{cc}
\Delta^2 x & C_{x,p} \\
C_{p,x} & \Delta^2 p
\end{array} \right) + i\hbar \left( \begin{array}{cc}
0 & -1 \\
1 & 0
\end{array} \right) \geq 0
\]
\[
2 \left( \begin{array}{cc}
\Delta^2 x & C_{x,p} - \frac{1}{2}i\hbar \\
C_{p,x} + \frac{1}{2}i\hbar & \Delta^2 p
\end{array} \right) \geq 0
\]
\[
\Delta^2 x \Delta^2 p - C_{x,p}^2 \geq (\hbar/2)^2
\]
\[ A \geq 0 \]
\[ \langle \psi | A | \psi \rangle = 0 \quad \forall | \psi \rangle \]
\[ A = \sum_{n} a_{n} | \phi_{n} \rangle \langle \phi_{n} | \]
\[ a_{n} \geq 0 \]
\[ \Delta x \Delta p \geq \frac{\hbar}{2} \]
\[ C_{x, \rho} = \frac{\langle x | \hat{\rho} + \hat{\rho} x \rangle - \langle x \rangle \langle \hat{\rho} \rangle}{\rho} \]

\( \rho \)
Liouville mechanics with an epistemic restriction

Assume:

The classical uncertainty principle:

The only Liouville distributions that can be prepared are those that satisfy

$$\gamma(\mu) + i\hbar \Sigma \geq 0$$

and that have maximal entropy for a given set of second-order moments.
**Quantum mechanics**

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\gamma(\hat{\rho}) = 2 \begin{pmatrix}
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C_{x_2, x_1} & C_{x_2, p_1} & \Delta^2 x_2 & C_{x_2, p_2} \\
C_{p_2, x_1} & C_{p_2, p_1} & C_{p_2, x_2} & \Delta^2 p_2 \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

\[ \Sigma = \begin{pmatrix}
0 & -1 & & \\
1 & 0 & & \\
& & \ddots & \\
\vdots & \vdots & \ddots & \ddots
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C_{x_2, x_1} & C_{x_2, p_1} & \Delta^2 x_2 & C_{x_2, p_2} & \cdots \\
C_{p_2, x_1} & C_{p_2, p_1} & C_{p_2, x_2} & \Delta^2 p_2 & \cdots \\
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\end{pmatrix}
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Assume:

The classical uncertainty principle:

The only Liouville distributions that can be prepared are those that satisfy

\[ \gamma(\mu) + i\hbar \Sigma \geq 0 \]

and that have maximal entropy for a given set of second-order moments.

Among valid \( \mu \) with a given \( \gamma \), multi-variate Gaussians maximize the entropy

\[ \mu(z) = \frac{1}{(2\pi)^{n/2}|\gamma|^{1/2}} \exp \left( -\frac{1}{2} (z - \langle z \rangle)^T \gamma^{-1} (z - \langle z \rangle) \right) \]
Quantum mechanics

Uncertainty principle:

\[ \gamma(\hat{\rho}) + i\hbar \Sigma \geq 0 \]

\[ \gamma(\hat{\rho}) = 2 \begin{pmatrix} \Delta^2 x_1 & C_{x_1,p_1} & C_{x_1,x_2} & C_{x_1,p_2} & \cdots \\ C_{p_1,x_1} & \Delta^2 p_1 & C_{p_1,x_2} & C_{p_1,p_2} \\ C_{x_2,x_1} & C_{x_2,p_1} & \Delta^2 x_2 & C_{x_2,p_2} \\ C_{p_2,x_1} & C_{p_2,p_1} & C_{p_2,x_2} & \Delta^2 p_2 \\ \vdots & & & & \end{pmatrix} \]

\[ \Sigma = \begin{pmatrix} 0 & -1 & & \cdots \\ 1 & 0 & & \\ & 0 & -1 & \cdots \\ & 1 & 0 & \cdots \\ & & \vdots & \end{pmatrix} \]

\[ \Delta^2 x = \langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2 \]
Quantum mechanics

Uncertainty principle:
\[ \gamma(\hat{\rho}) + i\hbar \Sigma \geq 0 \]

\[ \hat{R} = (\hat{x}_1, \hat{p}_1, \hat{x}_2, \hat{p}_2, \ldots) \]
\[ \gamma_{ij} = 2(\frac{1}{2}\langle\{R_i, R_j\}\rangle - \langle R_i \rangle \langle R_j \rangle) \]
\[ [R_i, R_j] = i\hbar \Sigma_{ij} \]

\[ \gamma(\hat{\rho}) = 2\begin{pmatrix} \Delta^2 x_1 & C_{x_1,p_1} & C_{x_1,x_2} & C_{x_1,p_2} & \cdots \\ C_{p_1,x_1} & \Delta^2 p_1 & C_{p_1,x_2} & C_{p_1,p_2} & \cdots \\ C_{x_2,x_1} & C_{x_2,p_1} & \Delta^2 x_2 & C_{x_2,p_2} & \cdots \\ C_{p_2,x_1} & C_{p_2,p_1} & C_{p_2,x_2} & \Delta^2 p_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \]

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\[ \Delta^2 x = \langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2 \]
\[ C_{xx} = \frac{1}{2} (\langle \hat{x} \hat{x} \rangle + \langle \hat{x} \rangle^2) - (\langle \hat{x} \rangle \langle \hat{x} \rangle) \]
Quantum mechanics

Uncertainty principle:

$$\gamma(\hat{\rho}) + i\hbar \Sigma \geq 0$$

$$\gamma(\hat{\rho}) = 2 \left( \begin{array}{cccc} \Delta^2 x_1 & C_{x_1,p_1} & C_{x_1,x_2} & C_{x_1,p_2} & \cdots \\ C_{p_1,x_1} & \Delta^2 p_1 & C_{p_1,x_2} & C_{p_1,p_2} & \cdots \\ C_{x_2,x_1} & C_{x_2,p_1} & \Delta^2 x_2 & C_{x_2,p_2} & \cdots \\ C_{p_2,x_1} & C_{p_2,p_1} & C_{p_2,x_2} & \Delta^2 p_2 & \cdots \\ \vdots & & & & \ddots \end{array} \right)$$

$$\Sigma = \left( \begin{array}{ccc} 0 & -1 & \cdots \\ 1 & 0 & \cdots \\ & & \ddots \end{array} \right)$$

$$\Delta^2 x = \langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2$$

$$C_{x_1,x_2} = \frac{1}{2} (\langle \hat{x}_1 \hat{x}_2 \rangle + \langle \hat{x}_2 \hat{x}_1 \rangle) - \langle \hat{x}_1 \rangle \langle \hat{x}_2 \rangle$$

$$\hat{\mathbf{R}} = (\hat{x}_1, \hat{p}_1, \hat{x}_2, \hat{p}_2, \ldots)$$

$$\gamma_{ij} = 2 \left( \frac{1}{2} \langle \{ R_i, R_j \} \rangle - \langle R_i \rangle \langle R_j \rangle \right)$$

$$[R_i, R_j] = i\hbar \Sigma_{ij}$$

$$2\langle (R_i - \langle R_i \rangle)(R_j - \langle R_j \rangle) \rangle$$

$$= 2\langle (R_i R_j) - \langle R_i \rangle \langle R_j \rangle \rangle$$

$$= \langle \{ R_i, R_j \} \rangle + \langle [R_i, R_j] \rangle - 2\langle R_i \rangle \langle R_j \rangle$$

$$= \gamma_{ij} + i\hbar \Sigma_{ij}$$

$$(Y, (\gamma(\hat{\rho}) + i\hbar \Sigma)Y)$$

$$= \sum_{i,j} Y_i^* (\gamma_{ij} + i\hbar \Sigma_{ij}) Y_j$$

$$= 2\sum_i Y_i^* (R_i - \langle R_i \rangle) \sum_j Y_j (R_j - \langle R_j \rangle)$$
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\gamma(\hat{\rho}) = 2 \begin{pmatrix}
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C_{p_1,x_1} & \Delta^2 p_1 & C_{p_1,x_2} & C_{p_1,p_2} & \cdots \\
C_{x_2,x_1} & C_{x_2,p_1} & \Delta^2 x_2 & C_{x_2,p_2} & \cdots \\
C_{p_2,x_1} & C_{p_2,p_1} & C_{p_2,x_2} & \Delta^2 p_2 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

\[
\Sigma = \begin{pmatrix}
0 & -1 & & & \\
1 & 0 & & & \\
0 & -1 & & & \\
1 & 0 & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots
\end{pmatrix}
\]

\[ \Delta^2 x = \langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2 \]

\[ C = \frac{1}{2} (\hat{\sigma} \cdot \hat{\sigma} + \hat{\sigma} \cdot \hat{\sigma}) - (\langle \hat{\sigma} \rangle \langle \hat{\sigma} \rangle) \]

\[ \hat{R} = (\hat{x}_1, \hat{p}_1, \hat{x}_2, \hat{p}_2, \ldots) \]

\[ \gamma_{ij} = 2\left( \frac{1}{2} \langle \{R_i, R_j\} \rangle - \langle R_i \rangle \langle R_j \rangle \right) \]

\[ [R_i, R_j] = i\hbar \Sigma_{ij} \]

\[ 2\langle (R_i - \langle R_i \rangle) (R_j - \langle R_j \rangle) \rangle = 2\langle (R_i R_j) - \langle R_i \rangle \langle R_j \rangle \rangle \]

\[ = \langle \{R_i, R_j\} \rangle + \langle [R_i, R_j] \rangle - 2\langle R_i \rangle \langle R_j \rangle \]

\[ = \gamma_{ij} + i\hbar \Sigma_{ij} \]

\[ (Y, (\gamma(\hat{\rho}) + i\hbar \Sigma) Y) \]

\[ = \sum_{i,j} Y_i^* (\gamma_{ij} + i\hbar \Sigma_{ij}) Y_j \]

\[ = 2\langle \sum_i Y_i^* (R_i - \langle R_i \rangle) \sum_j Y_j (R_j - \langle R_j \rangle) \rangle \]

\[ = 2\langle A_Y^\dagger A_Y \rangle \geq 0 \quad \forall Y \]
Valid epistemic states for a pair of canonical systems

Uncorrelated distributions
\[ \mu(x_1, p_1, x_2, p_2) = \mu(x_1, p_1) \mu(x_2, p_2) \]

Correlated distributions
\[ \mu(x_1, p_1, x_2, p_2) = \frac{1}{N} \delta(x_1 - x_2) \delta(p_1 + p_2) \]

This corresponds to the entangled state of Einstein, Podolsky and Rosen
\[ |\psi\rangle = \int dx_1 \, dx_2 \, \delta(x_1 - x_2) |x_1\rangle |x_2\rangle \]
\[ = \int dp_1 \, dp_2 \, \delta(p_1 + p_2) |p_1\rangle |p_2\rangle \]
Valid deterministic transformations

The group of canonical transformations with quadratic Hamiltonian

Only canonical transformations preserve the uncertainty principle
Only quadratic Hamiltonians preserve the gaussianity
Valid measurements

Sets of indicator functions \( \{ \xi_k(x, p) \} \)

\[ \xi_k(x, p) = \text{probability of } k \text{ given } (x,p) \]

\[ \xi_k(x_1, p_1) \]

\[ \mu(x_1, p_1, x_2, p_2) \]
Valid measurements

Sets of indicator functions \( \{ \xi_k(x, p) \} \)

\[ \xi_k(x, p) = \text{probability of } k \text{ given } (x, p) \]

\[ \xi_k(x_1, p_1) \]

\[ \mu(x_1, p_1, x_2, p_2) \]

\[ \mu(x_2, p_2) \propto \int dx_1 dp_1 \xi(x_1, p_1) \delta(x_1 - x_2) \delta(p_1 + p_2) \]

\[ = \xi(x_2, -p_2) \]

\[ \mu(x_2, p_2) \propto \int dx_1 dp_1 \xi(x_1, p_1) \mu(x_1, p_1, x_2, p_2) \]
Valid measurements for one canonical system

\[ \xi_k(x, p) \geq 0 \]

\[ \sum_k \xi_k(x, p) = 1 \quad \forall x \forall p \]
Measurement-induced transformations

Measure $x$ in a reproducible way
Measurement-induced transformations

Measure $x$ in a reproducible way

"Collapse" = Bayesian updating + uniformly random mixture of translations over $p$
Note: the evolution is deterministic if the apparatus is treated internally.

**Internal apparatus**

**External apparatus**

Measure x

Measure x

Unknown disturbance to p

Interact by: $H_{int} = x_{sys} \cdot p_{app}$

Final x of apparatus reflects initial x of system

Final p of system reflects initial p of apparatus

The position of the internal-external cut doesn’t matter
Note: the evolution is *deterministic* if the apparatus is treated internally

**Internal apparatus**

**External apparatus**

Measure $x$

Prepare $x$

Unknown disturbance to $p$

Interact by $H_{int} = x_{sys} p_{app}$

Final $x$ of apparatus reflects initial $x$ of system

Final $p$ of system reflects initial $p$ of apparatus

The position of the internal-external cut doesn’t matter
Non-commutativity of measurements

Prepare $|x\rangle$

Measure $X$ then $P$

Measure $P$ then $X$
Note: the evolution is **deterministic** if the apparatus is treated internally.

**Internal apparatus**

**External apparatus**

\[
|\psi\rangle = \int dx_1 \, dx_2 \, \delta(x_1 - x_2) |x_1\rangle |x_2\rangle \\
= \int dp_1 \, dp_2 \, \delta(p_1 + p_2) |p_1\rangle |p_2\rangle
\]

On particle 1, measure either X or P.
Outcomes for measurements of X or P on particle 2 become certain.
The EPR experiment

\[ |\psi\rangle = \int dx_1 \, dx_2 \, \delta(x_1 - x_2) |x_1\rangle |x_2\rangle \]

\[ = \int dp_1 \, dp_2 \, \delta(p_1 + p_2) |p_1\rangle |p_2\rangle \]

On particle 1, measure either X or P
Outcomes for measurements of X or P on particle 2 become certain
\[
\mu(x_1, p_1, x_2, p_2) = \frac{1}{N} \delta(x_1 - x_2) \delta(p_1 + p_2)
\]

\[
\mu(x_2, p_2) = \frac{1}{N}
\]

Initially A is completely ignorant of 2

If A measures x on 1, she infers x of 2

If A measures p on 1, she infers p of 2

A's decision does not affect the reality at 2, the x and p were already elements of reality.
The Wigner representation

Weyl operators \( \hat{w}(u, v) = \exp(-iv\hat{x} - iu\hat{p}) \)
The Wigner representation

Weyl operators \( \hat{w}(u, v) = \exp(-iv\hat{x} - iu\hat{p}) \)

Point operators \( \hat{A}(x, p) = \frac{1}{(2\pi)^2} \int \! du \! dv \exp(ivx + iup)\hat{w}(u, v) \)
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Wigner representation

\[ W_\rho(x, p) = \text{Tr}[\hat{\rho}\hat{A}(x, p)] \]

\[ W_E(x, p) = \text{Tr}[\hat{E}\hat{A}(x, p)] \]
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Wigner representation
\[
    W_{\hat{\rho}}(x,p) = \text{Tr}[\hat{\rho}\hat{A}(x,p)]
\]
\[
    W_{\hat{E}}(x,p) = \text{Tr}[\hat{E}\hat{A}(x,p)]
\]
\[
    \int dx dp W_{\hat{\rho}}(x,p) W_{\hat{E}}(x,p) = \text{Tr}[\hat{\rho}\hat{E}]
\]
The Wigner representation

Weyl operators \[ \hat{w}(u, v) = \exp(-iv\hat{x} - iu\hat{p}) \]

Point operators \[ \hat{A}(x, p) = \frac{1}{(2\pi)^2} \int dv du \exp(ivx + iup)\hat{w}(u, v) \]

Wigner representation \[ W_\hat{\rho}(x, p) = \text{Tr}[\hat{\rho}\hat{A}(x, p)] \]
\[ W_{\hat{E}}(x, p) = \text{Tr}[\hat{E}\hat{A}(x, p)] \]
\[ \int dx dp W_\hat{\rho}(x, p) W_{\hat{E}}(x, p) = \text{Tr}[\hat{\rho}\hat{E}] \]

This can be generalized \[ W_\hat{\rho}(x_1, p_1, x_2, p_2) = \text{Tr}[\hat{\rho}\hat{A}(x_1, p_1) \otimes \hat{A}(x_2, p_2)] \]
Gaussian quantum mechanics

**Gaussian state** $\rho$: one that has a Gaussian Wigner rep’n

$$W_\rho(z) = \frac{1}{(2\pi)^{n/2} |\gamma|^{1/2}} \exp \left( -\frac{1}{2} (z - \langle z \rangle)^T \gamma^{-1} (z - \langle z \rangle) \right)$$
Gaussian quantum mechanics

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Note: $\langle \hat{x}^k \hat{p}^l \rangle_\rho = \langle x^k p^l \rangle_{W_{\rho}}$ therefore $\gamma(\hat{\rho}) = \gamma(W_{\rho})$

Therefore, the Wigner rep’n satisfies the classical uncertainty principle
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Therefore, the Wigner rep’n satisfies the classical uncertainty principle

**Gaussian measurements and transformations:** preserve Gaussianity

One can prove

**Theorem:** Liouville mechanics with an epistemic restriction is empirically equivalent to Gaussian quantum mechanics
Categorizing quantum phenomena

Those arising in a restricted statistical classical theory

Those not arising in a restricted statistical classical theory
Categorizing quantum phenomena

Those arising in a restricted statistical classical theory

- Wave-particle duality
- Noncommutativity
- Entanglement
- Quantized spectra
- Key distribution
- Computational speed-up
- Improvements in metrology

Those not arising in a restricted statistical classical theory

- Collapse
- Teleportation
- No cloning
- Coherent superposition
- Bell inequality violations
- Quantum eraser
- Bell-Kochen-Specker theorem
- Pre and post-selection "paradoxes"
- Particle statistics
# Categorizing quantum phenomena

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Quantized spectra?  
Particle statistics?  
Others...
Categorizing quantum phenomena

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