Contextuality
Problems with the traditional definition of noncontextuality:
- applies only to sharp measurements
- applies only to deterministic hidden variable models
- applies only to models of quantum theory
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A better notion of noncontextuality would determine
- whether any given theory admits a noncontextual model
- whether any given experimental data can be explained by a noncontextual model
Operational Quantum Mechanics

Preparation $P$

Measurement $M$

Density operator $\rho$

Positive operator-valued measure (POVM) $\{E_k\}$

$$Pr(k|P, M) = \text{Tr}[E_k \rho]$$
General Operational Theories

Preparation $P$

Element of a convex set $\vec{p}$

Measurement $M$

Set of elements of a positive cone $\{\vec{r}_k\}$

$$Pr(k|P, M) = \vec{r}_k \cdot \vec{p}$$

See e.g. L. Hardy, quant-ph/0101012 and J. Barrett, quant-ph/0508211
A realist model of an operational theory

\[
\int \mu_P(\lambda) d\lambda = 1
\]

\[
\mu_P(\lambda)
\]
A realist model of an operational theory

\[ \int \mu_P(\lambda) d\lambda = 1 \]

\[ 0 \leq \xi_{M,k} \leq 1 \]

\[ \sum_k \xi_{M,k}(\lambda) = 1 \text{ for all } \lambda \]

\[ \xi_{M,1}(\lambda) \]

\[ \xi_{M,2}(\lambda) \]

\[ \xi_{M,3}(\lambda) \]
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\[ 0 \leq \xi_{M,k} \leq 1 \]

\[ \sum_k \xi_{M,k}(\lambda) = 1 \text{ for all } \lambda \]

\[ p(k|P, M) = \int d\lambda \xi_{M,k}(\lambda) \mu_P(\lambda) \]
Generalized definition of noncontextuality:

A realist model of an operational theory is noncontextual if

Operational equivalence of two experimental procedures $\rightarrow$ Equivalent representations in the HV model
Generalized definition of noncontextuality:

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Generalized definition of noncontextuality:

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Operational equivalence of two experimental procedures \(\rightarrow\) Equivalent representations in the HV model
Operational equivalence classes
Operational equivalence classes
Operational equivalence classes

$P$ is equivalent to $P'$ if

$\forall M \forall k : p(k | P, M) = p(k | P', M)$
Difference of context
Example from quantum theory

Different density op’s
Example from quantum theory

\[ I = \frac{1}{2} |0\rangle \langle 0| + \frac{1}{2} |1\rangle \langle 1| \]

\[ \frac{1}{2} I = \frac{1}{2} |+\rangle \langle +| + \frac{1}{2} |-\rangle \langle -| \]
Example from quantum theory

\[ I = \frac{1}{2} |0\rangle \langle 0| + \frac{1}{2} |1\rangle \langle 1| \]

\[ \frac{1}{2} I = \frac{1}{2} |+\rangle \langle +| + \frac{1}{2} |-\rangle \langle -| \]
Example from quantum theory

\[ I = \text{Tr}_B\left[ \frac{1}{\sqrt{2}} (|0\rangle |0\rangle + |1\rangle |1\rangle) \right] \]

\[ \frac{1}{2} I = \text{Tr}_B\left[ \frac{1}{\sqrt{2}} (|0\rangle |+\rangle + |1\rangle |-\rangle) \right] \]
Preparation noncontextual model

$\mu(\lambda)$

$\lambda$

$P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9$

$M_1, M_2, M_3, M_4, M_5, M_6, M_7, M_8, M_9, M_{10}$
Preparation noncontextual model

\[ \mu(\lambda) \quad \lambda \]
Preparation noncontextual model

\[ \mu(\lambda) \]
Definition of preparation noncontextual model:

\[ \forall M : p(k|P, M) = p(k|P', M) \]

\[ \implies p(\lambda|P) = p(\lambda|P') \]
(a) Some states of a qubit

(b) A preparation noncontextual model of these
(RWS, PRA 75, 032110, 2007)

(c) A preparation contextual model of these
(Kochen-Specker, 1967)

\[
\begin{align*}
\mu_{|0\rangle}(\lambda) &= \frac{1}{2} \mu_{|0\rangle}(\lambda) + \frac{1}{2} \mu_{|1\rangle}(\lambda) \\
\mu_{|+\rangle}(\lambda) &= \frac{1}{2} \mu_{|+\rangle}(\lambda) + \frac{1}{2} \mu_{|-\rangle}(\lambda) \\
\mu_{|-\rangle}(\lambda) &= \frac{1}{2} \mu_{|\rangle}(\lambda) + \frac{1}{2} \mu_{|-\rangle}(\lambda)
\end{align*}
\]
\[
\begin{align*}
&\frac{1}{2} \left( \frac{1}{2}, \frac{1}{2}, 0, 0 \right) \\
&+ \frac{1}{2} \left( 0, 0, \frac{1}{2}, \frac{1}{2} \right) = \left( \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \right) \\
&\left( \frac{1}{2}, 0, \frac{1}{2}, 0 \right) \\
&\left( 0, \frac{1}{2}, 0, \frac{1}{2} \right)
\end{align*}
\]
(a) Some states of a qubit

(b) A preparation noncontextual model of these
(RWS, PRA 75, 032110, 2007)

(c) A preparation contextual model of these
(Kochen-Specker, 1967)
Difference of context
\begin{align*}
\{ |\psi_1\rangle\langle\psi_1|, I - |\psi_1\rangle\langle\psi_1| \} & \quad \{ |\psi_1\rangle\langle\psi_1|, I - |\psi_1\rangle\langle\psi_1| \} \\
I - |\psi_1\rangle\langle\psi_1| & = |\psi_2\rangle\langle\psi_2| + |\psi_3\rangle\langle\psi_3| \\
& = |\psi_2\rangle\langle\psi_2| + |\psi_3\rangle\langle\psi_3| \\
I - |\psi_1\rangle\langle\psi_1| & = |\psi_2\rangle\langle\psi_2| + |\psi_3\rangle\langle\psi_3| \\
\end{align*}
\[ \{E, I - E\} \]
\[ E = q\frac{n}{4}\langle\frac{n}{4}\rangle + (1 - q)\frac{1}{2}I \]
\[ \{E, I - E\} \]
\[ E = \frac{1}{2}\ket{0}ra{0} + \frac{1}{2}\ket{+}ra{+} \]
\[
\{E, I - E\} \\
E = q \left| \frac{n}{4} \right\rangle \langle \frac{n}{4} \right| + (1 - q) \frac{1}{2} I
\]
universal noncontextuality
= noncontextuality for preparations and measurements
Preparation-based proof of contextuality

(i.e. of the impossibility of a noncontextual realist model of quantum theory)
Important features of realist models

Let $P \leftrightarrow \mu(\lambda)$

$P' \leftrightarrow \mu'(\lambda)$

Representing one-shot distinguishability:

If $P$ and $P'$ are distinguishable with certainty

then $\mu(\lambda) \mu'(\lambda) = 0$
(a) Some states of a qubit

(b) A preparation noncontextual model of these
(RWS, PRA 75, 032110, 2007)

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Important features of realist models

Let \( P \leftrightarrow \mu(\lambda) \)
\( P' \leftrightarrow \mu'(\lambda) \)

Representing one-shot distinguishability:
If \( P \) and \( P' \) are distinguishable with certainty
then \( \mu(\lambda) \mu'(\lambda) = 0 \)

Representing convex combination:
If \( P'' = P \) with prob. \( p \) and \( P' \) with prob. \( 1 - p \)
Then \( \mu''(\lambda) = p \mu(\lambda) + (1 - p) \mu'(\lambda) \)
\[ P(\neg P') = P(\neg P) P(P) + P(\neg P') P(P') = \frac{1}{2} \left( \frac{1}{2}, \frac{1}{2}, 0, 0 \right) + \frac{1}{2} \left( 0, 0, \frac{1}{2}, \frac{1}{2} \right) = 1 - P \left( \begin{array}{c}
\frac{1}{2}, 0, \frac{1}{2}, 0 \\
0, \frac{1}{2}, 0, \frac{1}{2} 
\end{array} \right) \]
Proof based on finite construction in 2d

\[
\begin{align*}
P_a & \leftrightarrow \psi_a = (1, 0) \\
P_A & \leftrightarrow \psi_A = (0, 1) \\
P_b & \leftrightarrow \psi_b = (1/2, \sqrt{3}/2) \\
P_B & \leftrightarrow \psi_B = (\sqrt{3}/2, -1/2) \\
P_c & \leftrightarrow \psi_c = (1/2, -\sqrt{3}/2) \\
P_C & \leftrightarrow \psi_C = (\sqrt{3}/2, 1/2)
\end{align*}
\]
\[ P(NP) p(P) + P(NP') p(P') \cdot \frac{1}{2} \left( \frac{1}{2}, \frac{1}{2}, 0, 0 \right) + \frac{1}{2} \left( 0, 0, \frac{1}{2}, \frac{1}{2} \right) = \left( \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \right) \]

\[ 1 - P \left( \frac{1}{2}, 0, \frac{1}{2}, 0 \right) = \left( 0, \frac{1}{2}, 0, \frac{1}{2} \right) \]

\[ 14 \times 4 / 14^4 = 0 \]
Proof based on finite construction in 2d

\[ P_a \leftrightarrow \sigma_a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \]
\[ P_A \leftrightarrow \sigma_A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \]
\[ P_b \leftrightarrow \sigma_b = \begin{pmatrix} \frac{3}{4} & \frac{1}{4}\sqrt{3} \\ -\frac{1}{4}\sqrt{3} & \frac{1}{4} \end{pmatrix} \]
\[ P_B \leftrightarrow \sigma_B = \begin{pmatrix} \frac{1}{4}\sqrt{3} & \frac{1}{4} \\ -\frac{1}{4}\sqrt{3} & \frac{1}{4} \end{pmatrix} \]
\[ P_c \leftrightarrow \sigma_c = \begin{pmatrix} \frac{3}{4} & -\frac{1}{4}\sqrt{3} \\ \frac{3}{4} & -\frac{1}{4}\sqrt{3} \end{pmatrix} \]
\[ P_C \leftrightarrow \sigma_C = \begin{pmatrix} \frac{3}{4} & \frac{1}{4}\sqrt{3} \\ \frac{3}{4} & \frac{1}{4}\sqrt{3} \end{pmatrix} \]

\[ \sigma_a \sigma_A = 0 \]
\[ \sigma_b \sigma_B = 0 \]
\[ \sigma_c \sigma_C = 0 \]

\( P_a \) and \( P_A \) are distinguishable with certainty
\( P_b \) and \( P_B \) are distinguishable with certainty
\( P_c \) and \( P_C \) are distinguishable with certainty

\[ \mu_a(\lambda) \mu_A(\lambda) = 0 \]
\[ \rightarrow \mu_b(\lambda) \mu_B(\lambda) = 0 \]
\[ P_{aA} \equiv P_a \text{ and } P_A \text{ with prob. } 1/2 \text{ each} \]
\[ P_{bB} \equiv P_b \text{ and } P_B \text{ with prob. } 1/2 \text{ each} \]
\[ P_{cC} \equiv P_c \text{ and } P_C \text{ with prob. } 1/2 \text{ each} \]
\[ P_{abc} \equiv P_a, P_b \text{ and } P_c \text{ with prob. } 1/3 \text{ each} \]
\[ P_{ABC} \equiv P_A, P_B \text{ and } P_C \text{ with prob. } 1/3 \text{ each} \]
\[ P_{aA} \equiv P_a \text{ and } P_A \text{ with prob. } 1/2 \text{ each} \]
\[ P_{bB} \equiv P_b \text{ and } P_B \text{ with prob. } 1/2 \text{ each} \]
\[ P_{cC} \equiv P_c \text{ and } P_C \text{ with prob. } 1/2 \text{ each} \]
\[ P_{abc} \equiv P_a, P_b \text{ and } P_c \text{ with prob. } 1/3 \text{ each} \]
\[ P_{ABC} \equiv P_A, P_B \text{ and } P_C \text{ with prob. } 1/3 \text{ each} \]

\[ \mu_{aA}(\lambda) = \frac{1}{2} \mu_a(\lambda) + \frac{1}{2} \mu_A(\lambda) \]
\[ \mu_{bB}(\lambda) = \frac{1}{2} \mu_b(\lambda) + \frac{1}{2} \mu_B(\lambda) \]
\[ \mu_{cC}(\lambda) = \frac{1}{2} \mu_c(\lambda) + \frac{1}{2} \mu_C(\lambda) \]
\[ \mu_{abc}(\lambda) = \frac{1}{3} \mu_a(\lambda) + \frac{1}{3} \mu_b(\lambda) + \frac{1}{3} \mu_c(\lambda) \]
\[ \mu_{ABC}(\lambda) = \frac{1}{3} \mu_A(\lambda) + \frac{1}{3} \mu_B(\lambda) + \frac{1}{3} \mu_C(\lambda) \]
\[
I/2 = \frac{1}{2} \sigma_a + \frac{1}{2} \sigma_A \\
= \frac{1}{2} \sigma_b + \frac{1}{2} \sigma_B \\
= \frac{1}{2} \sigma_c + \frac{1}{2} \sigma_C \\
= \frac{1}{3} \sigma_a + \frac{1}{3} \sigma_b + \frac{1}{3} \sigma_c \\
= \frac{1}{3} \sigma_A + \frac{1}{3} \sigma_B + \frac{1}{3} \sigma_C.
\]
\begin{align*}
I/2 &= \frac{1}{2} \sigma_a + \frac{1}{2} \sigma_A \\
&= \frac{1}{2} \sigma_b + \frac{1}{2} \sigma_B \\
&= \frac{1}{2} \sigma_c + \frac{1}{2} \sigma_C \\
&= \frac{1}{3} \sigma_a + \frac{1}{3} \sigma_b + \frac{1}{3} \sigma_c \\
&= \frac{1}{3} \sigma_A + \frac{1}{3} \sigma_B + \frac{1}{3} \sigma_C.
\end{align*}

P_{aA} \approx P_{bB} \approx P_{cC} \\
\approx P_{abc} \approx P_{ABC}
\[ I/2 = \frac{1}{2} \sigma_a + \frac{1}{2} \sigma_A \]
\[ = \frac{1}{2} \sigma_b + \frac{1}{2} \sigma_B \]
\[ = \frac{1}{2} \sigma_c + \frac{1}{2} \sigma_C \]
\[ = \frac{1}{3} \sigma_a + \frac{1}{3} \sigma_b + \frac{1}{3} \sigma_c \]
\[ = \frac{1}{3} \sigma_A + \frac{1}{3} \sigma_B + \frac{1}{3} \sigma_C. \]

\[ P_{aA} \sim P_{bB} \sim P_{cC} \]
\[ \sim P_{abc} \sim P_{ABC} \]

By preparation noncontextuality

\[ \mu_{aA}(\lambda) = \mu_{bB}(\lambda) = \mu_{cC}(\lambda) \]
\[ = \mu_{abc}(\lambda) = \mu_{ABC}(\lambda) \]
\[ \equiv \nu(\lambda) \]
\[ P_{aA} \equiv P_a \text{ and } P_A \text{ with prob. } \frac{1}{2} \text{ each} \]

\[ P_{bB} \equiv P_b \text{ and } P_B \text{ with prob. } \frac{1}{2} \text{ each} \]

\[ P_{cC} \equiv P_c \text{ and } P_C \text{ with prob. } \frac{1}{2} \text{ each} \]

\[ P_{abc} \equiv P_a, P_b \text{ and } P_c \text{ with prob. } \frac{1}{3} \text{ each} \]

\[ P_{ABC} \equiv P_A, P_B \text{ and } P_C \text{ with prob. } \frac{1}{3} \text{ each} \]

\[ \mu_{aA}(\lambda) = \frac{1}{2} \mu_a(\lambda) + \frac{1}{2} \mu_A(\lambda) \]

\[ \mu_{bB}(\lambda) = \frac{1}{2} \mu_b(\lambda) + \frac{1}{2} \mu_B(\lambda) \]

\[ \mu_{cC}(\lambda) = \frac{1}{2} \mu_c(\lambda) + \frac{1}{2} \mu_C(\lambda) \]

\[ \mu_{abc}(\lambda) = \frac{1}{3} \mu_a(\lambda) + \frac{1}{3} \mu_b(\lambda) + \frac{1}{3} \mu_c(\lambda) \]

\[ \mu_{ABC}(\lambda) = \frac{1}{3} \mu_A(\lambda) + \frac{1}{3} \mu_B(\lambda) + \frac{1}{3} \mu_C(\lambda) \]
Our task is to find
\[ \mu_a(\lambda), \mu_A(\lambda), \mu_b(\lambda), \]
\[ \mu_B(\lambda), \mu_c(\lambda), \mu_C(\lambda), \]
and \( \nu(\lambda) \) such that

\[ \mu_a(\lambda) \mu_A(\lambda) = 0 \]
\[ \mu_b(\lambda) \mu_B(\lambda) = 0 \]
\[ \mu_c(\lambda) \mu_C(\lambda) = 0 \]

\[ \nu(\lambda) = \frac{1}{2} \mu_a(\lambda) + \frac{1}{2} \mu_A(\lambda) \]
\[ = \frac{1}{2} \mu_b(\lambda) + \frac{1}{2} \mu_B(\lambda) \]
\[ = \frac{1}{2} \mu_c(\lambda) + \frac{1}{2} \mu_C(\lambda) \]
\[ = \frac{1}{3} \mu_a(\lambda) + \frac{1}{3} \mu_b(\lambda) + \frac{1}{3} \mu_c(\lambda) \]
\[ = \frac{1}{3} \mu_A(\lambda) + \frac{1}{3} \mu_B(\lambda) + \frac{1}{3} \mu_C(\lambda). \]
Our task is to find
\[ \mu_a(\lambda), \mu_A(\lambda), \mu_b(\lambda), \mu_B(\lambda), \mu_c(\lambda), \mu_C(\lambda), \]
and \( \nu(\lambda) \) such that
\[ \mu_a(\lambda) \mu_A(\lambda) = 0 \]
\[ \mu_b(\lambda) \mu_B(\lambda) = 0 \]
\[ \mu_c(\lambda) \mu_C(\lambda) = 0 \]

\[ \nu(\lambda) = \frac{1}{2} \mu_a(\lambda) + \frac{1}{2} \mu_A(\lambda) \]
\[ = \frac{1}{2} \mu_b(\lambda) + \frac{1}{2} \mu_B(\lambda) \]
\[ = \frac{1}{2} \mu_c(\lambda) + \frac{1}{2} \mu_C(\lambda) \]
\[ = \frac{1}{3} \mu_a(\lambda) + \frac{1}{3} \mu_b(\lambda) + \frac{1}{3} \mu_c(\lambda) \]
\[ = \frac{1}{3} \mu_A(\lambda) + \frac{1}{3} \mu_B(\lambda) + \frac{1}{3} \mu_C(\lambda). \]

i.e., paralleling the quantum structure:
\[ \sigma_a \sigma_A = 0 \]
\[ \sigma_b \sigma_B = 0 \]
\[ \sigma_c \sigma_C = 0 \]

\[ I/2 = \frac{1}{2} \sigma_a + \frac{1}{2} \sigma_A \]
\[ = \frac{1}{2} \sigma_b + \frac{1}{2} \sigma_B \]
\[ = \frac{1}{2} \sigma_c + \frac{1}{2} \sigma_C \]
\[ = \frac{1}{3} \sigma_a + \frac{1}{3} \sigma_b + \frac{1}{3} \sigma_c \]
\[ = \frac{1}{3} \sigma_A + \frac{1}{3} \sigma_B + \frac{1}{3} \sigma_C. \]
Our task is to find 
\[ \mu_a(\lambda), \mu_A(\lambda), \mu_b(\lambda), \]
\[ \mu_B(\lambda), \mu_c(\lambda), \mu_C(\lambda), \]
and \( \nu(\lambda) \) such that

\[ \mu_a(\lambda) \mu_A(\lambda) = 0 \]
\[ \mu_b(\lambda) \mu_B(\lambda) = 0 \]
\[ \mu_c(\lambda) \mu_C(\lambda) = 0 \]

From decompositions (1)-(3), for \( \lambda = \lambda' \)

\[ \mu_a(\lambda') = 0 \text{ or } 2\nu(\lambda') \]
\[ \mu_b(\lambda') = 0 \text{ or } 2\nu(\lambda') \]
\[ \mu_c(\lambda') = 0 \text{ or } 2\nu(\lambda') \]

\[ \nu(\lambda) = \frac{1}{2} \mu_a(\lambda) + \frac{1}{2} \mu_A(\lambda) \]
\[ = \frac{1}{2} \mu_b(\lambda) + \frac{1}{2} \mu_B(\lambda) \]
\[ = \frac{1}{2} \mu_c(\lambda) + \frac{1}{2} \mu_C(\lambda) \]
\[ = \frac{1}{3} \mu_a(\lambda) + \frac{1}{3} \mu_b(\lambda) + \frac{1}{3} \mu_c(\lambda) \]
\[ = \frac{1}{3} \mu_A(\lambda) + \frac{1}{3} \mu_B(\lambda) + \frac{1}{3} \mu_C(\lambda) \]
Our task is to find 
\( \mu_a(\lambda), \mu_A(\lambda), \mu_b(\lambda), \mu_B(\lambda), \mu_c(\lambda), \mu_C(\lambda), \) 
and \( \nu(\lambda) \) such that 

\[
\begin{align*}
\mu_a(\lambda) \mu_A(\lambda) &= 0 \\
\mu_b(\lambda) \mu_B(\lambda) &= 0 \\
\mu_c(\lambda) \mu_C(\lambda) &= 0
\end{align*}
\]

From decompositions (1)-(3), for \( \lambda = \lambda' \)

\[
\begin{align*}
\mu_a(\lambda') &= 0 \text{ or } 2\nu(\lambda') \\
\mu_b(\lambda') &= 0 \text{ or } 2\nu(\lambda') \\
\mu_c(\lambda') &= 0 \text{ or } 2\nu(\lambda')
\end{align*}
\]

But then the RHS of decomposition (4) is 

\[
0, \frac{2}{3}\nu(\lambda'), \frac{4}{3}\nu(\lambda'), 2\nu(\lambda') \\
\neq \nu(\lambda')
\]

for \( \lambda' \) such that \( \nu(\lambda') \neq 0 \)

**CONTRADICTION**
Measurement contextuality

New definition versus traditional definition
How to formulate the traditional notion of noncontextuality:

\[ |\psi_1\rangle \quad |\psi_2\rangle \quad |\psi_3\rangle \quad \chi_1(\lambda) \quad \chi_2(\lambda) \quad \chi_3(\lambda) \]

\[ |\psi'_1\rangle \quad |\psi'_2\rangle \quad |\psi'_3\rangle \quad \chi'_1(\lambda) \quad \chi'_2(\lambda) \quad \chi'_3(\lambda) \]
This is equivalent to assuming:

\[
\{ |\psi_1\rangle\langle\psi_1|, \ I \ - \ |\psi_1\rangle\langle\psi_1| \}
\]
How to formulate the traditional notion of noncontextuality:

\[ |\psi_1\rangle \quad |\psi_2\rangle \quad |\psi_3\rangle \quad \Leftrightarrow \quad \chi_1(\lambda) \quad \Box \quad \lambda \\
\chi_2(\lambda) \\
\chi_3(\lambda) \]

\[ |\psi_1\rangle \quad |\psi'_2\rangle \quad |\psi'_3\rangle \quad \Leftrightarrow \quad \chi_1(\lambda) \quad \Box \quad \lambda \\
\chi'_2(\lambda) \\
\chi'_3(\lambda) \]
This is equivalent to assuming:

\[
\{ |\psi_1\rangle\langle \psi_1|, I - |\psi_1\rangle\langle \psi_1| \}
\]
But recall that the most general representation was

\[\{P_k\} \xrightarrow{M} \xi_{P_1}(\lambda) \xrightarrow{\lambda} \xi_{P_2}(\lambda) \xrightarrow{\lambda} \xi_{P_3}(\lambda) \xrightarrow{\lambda}\]

Therefore:

traditional notion of noncontextuality  =  revised notion of noncontextuality for sharp measurements
and
outcome determinism for sharp measurements
This is equivalent to assuming:

\[ \{ |\psi_1\rangle \langle \psi_1|, I - |\psi_1\rangle \langle \psi_1| \} \]

measure \[ |\psi_2\rangle \text{ and } |\psi_3\rangle \]

coarse-grain \[ \chi_1(\lambda) \] and \[ \chi_{-1}(\lambda) \]

\[ \{ |\psi_1\rangle \langle \psi_1|, I - |\psi_1\rangle \langle \psi_1| \} \]

measure \[ |\psi_2\rangle \text{ and } |\psi_3\rangle \]

coarse-grain \[ \chi_1(\lambda) \] and \[ \chi_{-1}(\lambda) \]
But recall that the most general representation was

\[ \{ P_k \} \rightarrow M \leftrightarrow \xi_{P_1}(\lambda) \rightarrow \lambda \]

\[ \xi_{P_2}(\lambda) \rightarrow \lambda \]

\[ \xi_{P_3}(\lambda) \rightarrow \lambda \]

Therefore:

traditional notion of noncontextuality = revised notion of noncontextuality for sharp measurements

and

outcome determinism for sharp measurements
So, the new definition of noncontextuality is not simply a generalization of the traditional notion.

For sharp measurements, it is a revision of the traditional notion.
Local determinism:
We ask: Does the outcome depend on space-like separated events (in addition to local settings and $\lambda$)?

Bell’s local causality:
We ask: Does the probability of the outcome depend on space-like separated events (in addition to local settings and $\lambda$)?
Local determinism:
We ask: Does the outcome depend on space-like separated events (in addition to local settings and $\lambda$)?

Bell’s local causality:
We ask: Does the probability of the outcome depend on space-like separated events (in addition to local settings and $\lambda$)?

Traditional notion of measurement noncontextuality:
We ask: Does the outcome depend on the measurement context (in addition to the observable and $\lambda$)?

The revised notion of measurement noncontextuality:
We ask: Does the probability of the outcome depend on the measurement context (in addition to the observable and $\lambda$)?
Local determinism:
We ask: Does the outcome depend on space-like separated events (in addition to local settings and $\lambda$)?

Bell’s local causality:
We ask: Does the probability of the outcome depend on space-like separated events (in addition to local settings and $\lambda$)?

Traditional notion of measurement noncontextuality:
We ask: Does the outcome depend on the measurement context (in addition to the observable and $\lambda$)?

The revised notion of measurement noncontextuality:
We ask: Does the probability of the outcome depend on the measurement context (in addition to the observable and $\lambda$)?
traditional notion of noncontextuality = revised notion of noncontextuality for sharp measurements and outcome determinism for sharp measurements

No-go theorems for previous notion are not necessarily no-go theorems for the new notion!

In face of contradiction, could give up ODSM
However, one can prove that

- preparation noncontextuality → outcome determinism for sharp measurements

Therefore:

- measurement noncontextuality
- and
- preparation noncontextuality

and

- measurement noncontextuality
- and
- outcome determinism for sharp measurements
However, one can prove that

preparation noncontextuality $\rightarrow$ outcome determinism for sharp measurements

Therefore:

measurement noncontextuality and preparation noncontextuality $\rightarrow$ measurement noncontextuality and outcome determinism for sharp measurements
However, one can prove that

preparation noncontextuality → outcome determinism for sharp measurements

Therefore:

measurement noncontextuality and preparation noncontextuality → Traditional notion of noncontextuality
However, one can prove that

preparation noncontextuality → outcome determinism for sharp measurements

Therefore:

measurement noncontextuality and preparation noncontextuality → Traditional notion of noncontextuality

no-go theorems for the traditional notion of noncontextuality can be salvaged as no-go theorems for the generalized notion
Measurement-based proof of contextuality

(i.e. of the impossibility of a noncontextual realist model of quantum theory)
Proof of contextuality for unsharp measurements in 2d

\[ M_a \leftrightarrow \{ \Pi_a, \Pi_A \} \]
\[ M_b \leftrightarrow \{ \Pi_b, \Pi_B \} \]
\[ M_c \leftrightarrow \{ \Pi_c, \Pi_C \} \]

\( \Pi_x \) projects onto \( \psi_x \)

\[ \Pi_a + \Pi_A = I \]
\[ \Pi_b + \Pi_B = I \]
\[ \Pi_c + \Pi_C = I \]

\[ \Pi_a \Pi_A = 0 \]
\[ \Pi_b \Pi_B = 0 \]
\[ \Pi_c \Pi_C = 0 \]
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\[ \Pi_c \Pi_C = 0 \]

\[ M_a \leftrightarrow \{ \chi_a(\lambda), \chi_A(\lambda) \} \]
\[ M_b \leftrightarrow \{ \chi_b(\lambda), \chi_B(\lambda) \} \]
\[ M_c \leftrightarrow \{ \chi_c(\lambda), \chi_C(\lambda) \} \]

By definition

\[ \chi_a(\lambda) + \chi_A(\lambda) = 1 \]
\[ \chi_b(\lambda) + \chi_B(\lambda) = 1 \]
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\[ \Pi_c + \Pi_C = I \]

By definition

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\[ \chi_b(\lambda) + \chi_B(\lambda) = 1 \]
\[ \chi_c(\lambda) + \chi_C(\lambda) = 1 \]

By outcome determinism for sharp measurements

\[ \chi_a(\lambda)\chi_A(\lambda) = 0 \]
\[ \chi_b(\lambda)\chi_B(\lambda) = 0 \]
\[ \chi_c(\lambda)\chi_C(\lambda) = 0 \]

Thus, \( \{ \chi(\lambda), \chi_V(\lambda) \} \)
M ≡ implement one of $M_a$, $M_b$ and $M_c$ with prob. 1/3 each, register only whether first or second outcome occurred
\( M \equiv \text{implement one of } M_a, M_b \text{ and } M_c \text{ with prob. } 1/3 \text{ each, register only whether first or second outcome occurred} \)

\[
M \leftrightarrow \left\{ \frac{1}{3} \Pi_a + \frac{1}{3} \Pi_b + \frac{1}{3} \Pi_c, \frac{1}{3} \Pi_A + \frac{1}{3} \Pi_B + \frac{1}{3} \Pi_C \right\}
\]
M ≡ implement one of $M_a$, $M_b$ and $M_c$ with prob. 1/3 each, register only whether first or second outcome occurred

\[ M \leftrightarrow \left\{ \frac{1}{3} \pi_a + \frac{1}{3} \pi_b + \frac{1}{3} \pi_c, \frac{1}{3} \pi_A + \frac{1}{3} \pi_B + \frac{1}{3} \pi_C \right\} \]

\[ M \leftrightarrow \left\{ \frac{1}{3} \chi_a(\lambda) + \frac{1}{3} \chi_b(\lambda) + \frac{1}{3} \chi_c(\lambda), \frac{1}{3} \chi_A(\lambda) + \frac{1}{3} \chi_B(\lambda) + \frac{1}{3} \chi_C(\lambda) \right\} \]
Proof of contextuality for unsharp measurements in 2d

\[ \text{M}_a \leftrightarrow \{\Pi_a, \Pi_A\} \]
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\[ \text{M}_c \leftrightarrow \{\chi_c(\lambda), \chi_C(\lambda)\} \]

By definition

\[ \chi_a(\lambda) + \chi_A(\lambda) = 1 \]
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M ≡ implement one of \( M_a, M_b, \) and \( M_c \) with prob. 1/3 each, register only whether first or second outcome occurred

\[
M \leftrightarrow \left\{ \frac{1}{3} \pi_a + \frac{1}{3} \pi_b + \frac{1}{3} \pi_c, \frac{1}{3} \pi_A + \frac{1}{3} \pi_B + \frac{1}{3} \pi_C \right\}
\]

\[
M \leftrightarrow \left\{ \frac{1}{3} \chi_a(\lambda) + \frac{1}{3} \chi_b(\lambda) + \frac{1}{3} \chi_c(\lambda), \frac{1}{3} \chi_A(\lambda) + \frac{1}{3} \chi_B(\lambda) + \frac{1}{3} \chi_C(\lambda) \right\}
\]
M ≡ implement one of $M_a$, $M_b$ and $M_c$ with prob. 1/3 each, register only whether first or second outcome occurred

\[
M \leftrightarrow \left\{ \frac{1}{3} \Pi_a + \frac{1}{3} \Pi_b + \frac{1}{3} \Pi_c, \frac{1}{3} \Pi_A + \frac{1}{3} \Pi_B + \frac{1}{3} \Pi_C \right\} = \left\{ \frac{1}{2} I, \frac{1}{2} I \right\}
\]

\[
M \leftrightarrow \left\{ \frac{1}{3} \chi_a(\lambda) + \frac{1}{3} \chi_b(\lambda) + \frac{1}{3} \chi_c(\lambda), \frac{1}{3} \chi_A(\lambda) + \frac{1}{3} \chi_B(\lambda) + \frac{1}{3} \chi_C(\lambda) \right\}
\]
M \equiv \text{implement one of } M_a, M_b \text{ and } M_c \text{ with prob. } 1/3 \text{ each, register only whether first or second outcome occurred}

\[ M \leftrightarrow \left\{ \frac{1}{3} \Pi_a + \frac{1}{3} \Pi_b + \frac{1}{3} \Pi_c, \frac{1}{3} \Pi_A + \frac{1}{3} \Pi_B + \frac{1}{3} \Pi_C \right\} = \left\{ \frac{1}{2} I, \frac{1}{2} I \right\} \]

\[ M \leftrightarrow \left\{ \frac{1}{3} \chi_a(\lambda) + \frac{1}{3} \chi_b(\lambda) + \frac{1}{3} \chi_c(\lambda), \frac{1}{3} \chi_A(\lambda) + \frac{1}{3} \chi_B(\lambda) + \frac{1}{3} \chi_C(\lambda) \right\} \]

\tilde{M} \equiv \text{ignore the system, flip a fair coin}

\[ \tilde{M} \leftrightarrow \left\{ \frac{1}{2} I, \frac{1}{2} I \right\} \]
M \equiv \text{implement one of } M_a, M_b \text{ and } M_c \text{ with prob. } 1/3 \text{ each, register only whether first or second outcome occurred}

\begin{align*}
M & \leftrightarrow \left\{ \frac{1}{3} \Pi_a + \frac{1}{3} \Pi_b + \frac{1}{3} \Pi_c, \frac{1}{3} \Pi_A + \frac{1}{3} \Pi_B + \frac{1}{3} \Pi_C \right\} = \left\{ \frac{1}{2} I, \frac{1}{2} I \right\} \\
\tilde{M} & \leftrightarrow \left\{ \frac{1}{3} \chi_a(\lambda) + \frac{1}{3} \chi_b(\lambda) + \frac{1}{3} \chi_c(\lambda), \frac{1}{3} \chi_A(\lambda) + \frac{1}{3} \chi_B(\lambda) + \frac{1}{3} \chi_C(\lambda) \right\}
\end{align*}

\tilde{M} \equiv \text{ignore the system, flip a fair coin}

\begin{align*}
\tilde{M} & \leftrightarrow \left\{ \frac{1}{2} I, \frac{1}{2} I \right\} \\
\tilde{\tilde{M}} & \leftrightarrow \left\{ \frac{1}{2}, \frac{1}{2} \right\}
\end{align*}

By the assumption of \textbf{measurement noncontextuality}

\begin{align*}
M \sim \tilde{\tilde{M}} & \rightarrow \left\{ \frac{1}{3} \chi_a + \frac{1}{3} \chi_b + \frac{1}{3} \chi_c, \frac{1}{3} \chi_A + \frac{1}{3} \chi_B + \frac{1}{3} \chi_C \right\} = \left\{ \frac{1}{2}, \frac{1}{2} \right\}
\end{align*}
M ≡ implement one of M_a, M_b and M_c with prob. 1/3 each, register only whether first or second outcome occurred

\[ M \leftrightarrow \{ \frac{1}{3} \eta_a + \frac{1}{3} \eta_b + \frac{1}{3} \eta_c, \frac{1}{3} \eta_A + \frac{1}{3} \eta_B + \frac{1}{3} \eta_C \} = \{ \frac{1}{2} I, \frac{1}{2} I \} \]

\[ M \leftrightarrow \{ \frac{1}{3} \chi_a(\lambda) + \frac{1}{3} \chi_b(\lambda) + \frac{1}{3} \chi_c(\lambda), \frac{1}{3} \chi_A(\lambda) + \frac{1}{3} \chi_B(\lambda) + \frac{1}{3} \chi_C(\lambda) \} \]

\( \tilde{M} \equiv \) ignore the system, flip a fair coin

\[ \tilde{M} \leftrightarrow \{ \frac{1}{2} I, \frac{1}{2} I \} \]

\[ \tilde{\tilde{M}} \leftrightarrow \{ \frac{1}{2}, \frac{1}{2} \} \]

By the assumption of measurement noncontextuality

\[ M \simeq \tilde{\tilde{M}} \rightarrow \{ \frac{1}{3} \chi_a + \frac{1}{3} \chi_b + \frac{1}{3} \chi_c, \frac{1}{3} \chi_A + \frac{1}{3} \chi_B + \frac{1}{3} \chi_C \} = \{ \frac{1}{2}, \frac{1}{2} \} \]

But \( \{0, 1\}, \{\frac{1}{3}, \frac{2}{3}\}, \{1, 0\}, \{\frac{2}{3}, \frac{1}{3}\} \neq \{ \frac{1}{2}, \frac{1}{2} \} \)

**CONTRADICTION**
Proof of contextuality for unsharp measurements in 2d

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\[ \chi_a(\lambda)\chi_A(\lambda) = 0 \]
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Thus, \( \{ \chi_\lambda, \chi_y(\lambda) \} \)
M \equiv \text{ implement one of } M_a, M_b \text{ and } M_c \text{ with prob. } 1/3 \text{ each, register only whether first or second outcome occurred}

M \leftrightarrow \{ \frac{1}{3} \eta_a + \frac{1}{3} \eta_b + \frac{1}{3} \eta_c, \frac{1}{3} \eta_a + \frac{1}{3} \eta_B + \frac{1}{3} \eta_C \} = \{ \frac{1}{2} I, \frac{1}{2} I \}

M \leftrightarrow \{ \frac{1}{3} \chi_a(\lambda) + \frac{1}{3} \chi_b(\lambda) + \frac{1}{3} \chi_c(\lambda), \frac{1}{3} \chi_a(\lambda) + \frac{1}{3} \chi_B(\lambda) + \frac{1}{3} \chi_C(\lambda) \}

\tilde{M} \equiv \text{ ignore the system, flip a fair coin}

\tilde{M} \leftrightarrow \{ \frac{1}{2} I, \frac{1}{2} I \}

\tilde{M} \leftrightarrow \{ \frac{1}{2}, \frac{1}{2} \}

By the assumption of measurement noncontextuality

M \simeq \tilde{M} \rightarrow \{ \frac{1}{3} \chi_a + \frac{1}{3} \chi_b + \frac{1}{3} \chi_c, \frac{1}{3} \chi_A + \frac{1}{3} \chi_B + \frac{1}{3} \chi_C \} = \{ \frac{1}{2}, \frac{1}{2} \}

But \{0, 1\}, \{\frac{1}{3}, \frac{2}{3}\}, \{1, 0\}, \{\frac{2}{3}, \frac{1}{3}\} \neq \{\frac{1}{2}, \frac{1}{2}\}

\text{CONTRADICTION}
The mystery of contextuality

There is a tension between

1) the dependence of representation on certain details of the experimental procedure

and

2) the independence of outcome statistics on those details of the experimental procedure