Abstract: In this talk (based on arXiv:1001.0354) we give a quantum statistical interpretation for the Kauffman bracket polynomial state sum $\langle K \rangle$ for the Jones polynomial. We use this quantum mechanical interpretation to give a new quantum algorithm for computing the Jones polynomial. This algorithm is useful for its conceptual simplicity, and it applies to all values of the polynomial variable that lie on the unit circle in the complex plane. Letting $C(K)$ denote the Hilbert space for this model, there is a natural unitary transformation $U$ from $C(K)$ to itself such that $\langle K \rangle = \langle F | U | F \rangle$; where $|F\rangle$ is a sum over basis states for $C(K)$. The quantum algorithm arises directly from this formula via the Hadamard Test. We then show that the framework for our quantum model for the bracket polynomial is a natural setting for Khovanov homology. The Hilbert space $C(K)$ of our model has basis in one-to-one correspondence with the enhanced states of the bracket state summation and is isomorphic with the chain complex for Khovanov homology with coefficients in the complex numbers. We show that for the Khovanov boundary operator $d$ defined on $C(K)$ we have the relationship $dU + Ud = 0$. Consequently, the unitary operator $U$ acts on the Khovanov homology, and we therefore obtain a direct relationship between Khovanov homology and this quantum algorithm for the Jones polynomial. The formula for the Jones polynomial as a graded Euler characteristic is now expressed in terms of the eigenvalues of $U$ and the Euler characteristics of the eigenspaces of $U$ in the homology. The quantum algorithm given here is inefficient, and so it remains an open problem to determine better quantum algorithms that involve both the Jones polynomial and the Khovanov homology.
Topological Quantum Information, Khovanov Homology and the Jones Polynomial

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Quantum Mechanics in a Nutshell

A state of a physical system corresponds to a unit vector $|S\rangle$ in a complex vector space.

(measurement free) Physical processes are modeled by unitary transformations applied to the state vector: $|S\rangle \rightarrow U|S\rangle$

If $|S\rangle = z_1|1\rangle + z_2|2\rangle + ... + z_n|n\rangle$

in a measurement basis $\{|1\rangle,|2\rangle,...,|n\rangle\}$, then measurement of $|S\rangle$ yields $|i\rangle$ with probability $|z|^2$. 
\[ H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} / \text{Sqrt}(2) \]
\[ M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \]
\[ \text{HMH} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \]
Mach-Zender Interferometer

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\[ H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \]

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Hadamard Test

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\[ |0\rangle \text{ occurs with probability } 1/2 + \text{Re}[<\phi|U|\phi>] / 2. \]
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Figure 2 - The Reidemeister Moves.

Reidemeister Moves reformulate knot theory in terms of graph combinatorics.
Bracket Polynomial Model for the Jones Polynomial

\[ \langle \times \rangle = A \langle \otimes \rangle + A^{-1} \langle \rangle \langle \rangle \]

\[ \langle K \circ \rangle = (-A^2 - A^{-2}) \langle K \rangle \]

\[ \langle \vdash \rangle = (-A^3) \langle \circ \rangle \]

\[ \langle \backslash \rangle = (-A^{-3}) \langle \circ \rangle \]
Reformulating the Bracket

Let \( c(K) \) = number of crossings on link \( K \).
Form \( A^{-c(K)} \langle K \rangle \) and replace \( A \) by \( -q^{-1} \).

Then the skein relation for \( \langle K \rangle \) will be replaced by:

\[
\langle X \rangle = \langle \bigodot \rangle - q \langle \rangle \langle \rangle
\]

\[
\langle \bigodot \rangle = (q + q^{-1})
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Bracket Polynomial Model for the Jones Polynomial

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\[ \langle K \bigcirc \rangle = (-A^2 - A^{-2}) \langle K \rangle \]

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Use enhanced states by labeling each loop with $+1$ or $-1$. 

\[ q + q^{-1} \]
Use enhanced states by labeling each loop with +1 or -1.

\[ \begin{align*}
\text{=} & \quad +1 & \quad + & \quad -1 \\
\text{q} & \quad + & \quad \text{q}^{-1}
\end{align*} \]
Enhanced States

\[ q^{-1} \iff -1 \iff X \]

\[ q^{+1} \iff +1 \iff 1 \]

For reasons that will soon become apparent, we let \(-1\) be denoted by \(X\) and \(+1\) be denoted by \(1\).
An enhanced state that contributes

\[\frac{1}{\sqrt{|q|}} \frac{1}{\sqrt{|q|}} \frac{1}{\sqrt{|q|}}\] \[\frac{|q|}{\sqrt{|q|}} \frac{|q|}{\sqrt{|q|}} \frac{|q|}{\sqrt{|q|}}\]

- to the revised bracket state sum.
Enhanced State Sum Formula for the Bracket

\[ \langle K \rangle = \sum_{s} q^{j(s)} (-1)^{i(s)} \]
A Quantum Statistical Model for the Bracket Polynomial.

Let $C(K)$ denote a Hilbert space with basis $|s\rangle$ where $s$ runs over the enhanced states of a knot or link diagram $K$.

We define a unitary transformation.

$$U : C(K) \longrightarrow C(K)$$

$$U|s\rangle = (-1)^{i(s)} q^{j(s)} |s\rangle$$

$q$ is chosen on the unit circle in the complex plane.
\[ |\psi\rangle = \sum_s |s\rangle \]

The evaluation of the bracket polynomial is given by the following formula:

\[ \langle K \rangle = \langle \psi | U | \psi \rangle. \]

This gives a new quantum algorithm for the Jones polynomial (via Hadamard Test).
\[ q^i(s)(-1)^i(s) \]

Khovanov Homology - Jones Polynomial as an Euler Characteristic

Two key motivating ideas are involved in finding the Khovanov invariant. Of all, one would like to categorify a link polynomial such as \( \langle K \rangle \). There are numerous meanings to the term categorify, but here the quest is to find a way to express the polynomial as a graded Euler characteristic \( \langle K \rangle = \chi_q(H(K)) \) for some homology theory associated with \( \langle K \rangle \).

We will formulate Khovanov homology in the context of our quantum statistical model for the bracket polynomial.
Khovanov Homology - Jones Polynomial as an Euler Characteristic

Two key motivating ideas are involved in finding the Khovanov invariant. First, one would like to categorify a link polynomial such as $\langle K \rangle$. There are many ways to the term categorify, but here the quest is to find a way to express the link polynomial as a graded Euler characteristic $\langle K \rangle = \chi_q \langle H(K) \rangle$ for some homology associated with $\langle K \rangle$.

We will formulate Khovanov Homology in the context of our quantum statistical model for the bracket polynomial.
The Khovanov Complex
\[ \partial(s) = \sum_{\tau} \partial_{\tau}(s) \]

The boundary is a sum of partial differentials corresponding to resmoothings on the states.

Each state loop is a module.

A collection of state loops corresponds to a tensor product of these modules.
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The boundary is a sum of partial differentials corresponding to resmoothings on the states.

Each state loop is a module.

A collection of state loops corresponds to a tensor product of these modules.
\[ \sqrt{\{1, x^2\}} \quad \text{basic} \quad \sqrt{\text{module/field}} \quad \text{e.g., } k = \mathbb{C} \]

\[ x^2 = 0, \quad \text{1} \cdot x = x = x \cdot 1 \]

\[ \Delta(x) = x \otimes x \]

\[ \Delta(1) = 1 \otimes x + x \otimes x \]
\[\sqrt{\{1, x^2\}} \quad \text{basis} \quad \text{V mod len/k field} \quad \text{e.g., } k = \mathbb{C}\]

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\[1 \cdot 1 = 1\]

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Enhanced State Sum Formula for the Bracket

\[ \langle K \rangle = \sum_s q^j(s) (-1)^i(s) \]

\[ j(s) = n_B(s) + \lambda(s) \]

\[ i(s) = n_B(s) = \text{number of B-smoothings in the state } s. \]

\[ \lambda(s) = \text{number of +1 loops minus number of -1 loops.} \]

\[ \langle K \rangle = \sum_{i,j} (-1)^i q^j \dim(C^{ij}) \]

\[ C^{ij} = \text{module generated by enhanced states with } i = n_B \text{ and } j \text{ as above.} \]
Enhanced State Sum Formula for the Bracket

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Hovanov constructs differential acting in the form

\[ \partial : C^{ij} \rightarrow C^{i+1,j} \]

For \( j \) to be constant as \( i \) increases by 1, we need 

\[ \lambda(s) \] to decrease by 1.
\[ \sqrt{3}, \sqrt{5} \] are basic.

\[ x^2 = \sqrt{3}, x = \sqrt{3}, 1 \]

\[ 1^2 = 1 \]

\[ \Delta(n) = n \bigtriangledown n \]

\[ \Delta(1) = 1 \bigtriangledown 1 \]
\[ \langle K \rangle = \sum_{i,j} (-1)^i q^j \dim(C^{ij}) \]

Hovanov constructs differential acting in the form

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For \( j \) to be constant as \( i \) increases by 1, we need \( \lambda(s) \) to decrease by 1.
The differential increases the homological grading $i$ by 1 and leaves fixed the quantum grading $j$.

Then

$$K\rangle = \sum_j q^j \sum_i (-1)^i \dim(C^{ij}) = \sum_j q^j \chi(C^{\cdot j})$$

$$\chi(H(C^{\cdot j})) = \chi(C^{\cdot j})$$

$$\langle K \rangle = \sum_j q^j \chi(H(C^{\cdot j}))$$
\[ \partial(s) = \sum_{\tau} \partial_{\tau}(s) \]

The boundary is a sum of partial differentials corresponding to resmoothings on the states.

\[ \Delta(X) = X \otimes X \text{ and } \Delta(1) = 1 \otimes X + X \otimes 1. \]

\[ x^2 = 0 \]
Partial Boundaries

\[ \partial \]
The partial differentials $\partial_{\tau}(s)$ are uniquely determined by the condition $j'(s) = j(s)$ for all $s'$ involved in the action of the partial differential on the en-state $s$. This unique form of the partial differential can be described by the fol-structures of multiplication and comultiplication on the algebra $A = k[X]/(X^2)$, $k = \mathbb{Z}/2\mathbb{Z}$ for mod-2 coefficients, or $k = \mathbb{Z}$ for integral coefficients.

The element 1 is a multiplicative unit and $X^2 = 0$.

$m(1) = 1 \otimes X + X \otimes 1$ and $\Delta(X) = X \otimes X$. 
Quantum Statistical Model for Khovanov Homolgy and the Bracket Polynomial.

Let $C(K)$ denote a Hilbert space with basis $|s\rangle$ where $s$ runs over the enhanced states of a knot or link diagram $K$.

We define a unitary transformation.

$$U : C(K) \longrightarrow C(K)$$

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This gives a new quantum algorithm for the Jones polynomial (via Hadamard Test).
With \[ U |s\rangle = (-1)^{i(s)} q^{j(s)} |s\rangle, \]

\[
\partial : C^{ij} \longrightarrow C^{i+1,j}
\]

\[ U \partial + \partial U = 0. \]

This means that the unitary transformation \( U \) acts on the homology so that \[ U : H(C(K)) \quad \longrightarrow \quad H(C(K)) \]
With \( U |s\rangle = (-1)^{i(s)} q^{j(s)} |s\rangle, \)

\[ \partial : C^{ij} \longrightarrow C^{i+1 j} \]

\[ U \partial + \partial U = 0. \]

This means that the unitary transformation \( U \) acts on the homology so that

\[ U : H(C(K)) \longrightarrow H(C(K)) \]
\[ C_{\bullet,j} = \bigoplus_i C^{i,j} \]

\[ \langle \rangle = \sum_s q^j(s) (-1)^{i(s)} = \sum_j q^j \sum_i (-1)^i \text{dim}(C^{ij}) \]

\[ = \sum_j q^j \chi(C_{\bullet,j}) = \sum_j q^j \chi(H(C_{\bullet,j})) . \]

This shows how \(<K>\) as a quantum amplitude contains information about the homology.
Eigenspace Picture

\[ C^0 = \bigoplus \lambda C^0_\lambda \]

\[ c^\bullet_\lambda : C^0_\lambda \rightarrow C^1_{-\lambda} \rightarrow C^2_{+\lambda} \rightarrow \cdots C^n_{(-1)^n \lambda} \]

\[ C = \bigoplus \lambda C^\bullet_\lambda \]

\[ \langle \psi | U | \psi \rangle = \sum_\lambda \lambda \chi(H(C^\bullet_\lambda)) \]
SUMMARY

We have interpreted the bracket polynomial as a quantum amplitude by making a Hilbert space $\mathbb{C}(K)$ whose basis is the collection of enhanced states of the bracket.

This space $\mathbb{C}(K)$ is naturally interpreted as the chain space for the Khovanov homology associated with the bracket polynomial.

$$\langle K \rangle = \langle \psi | U | \psi \rangle.$$  

The homology and the unitary transformation $U$ speak to one another via the formula

$$U \partial + \partial U = 0.$$
Questions

We have shown how Khovanov homology fits into the context of quantum information related to the Jones polynomial and how the polynomial is placed in this context by a unitary transformation $U$ the Hilbert space of the model. This transformation $U$ acts on the homology, and its eigenspaces give a natural decomposition of the homology that is related to the partition amplitude corresponding to the Jones polynomial.

The states of the model are intensely combinatorial, related to the representation of the knot or link.

How can this formulation be used in quantum information theory and in statistical mechanics?
a: \epsilon^{ij} \rightarrow \epsilon^{i+1j}

Yes!

\langle K \rangle = \sum_{\sigma} X(H(C^{\sigma} \vartheta(K))}

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