Abstract: In an ontological model of quantum theory that is Bell-local, one can assume without loss of generality that the outcomes of measurements are determined deterministically by the ontic states (i.e. the values of the local hidden variables). The question I address in this talk is whether such determinism can always be assumed in a noncontextual ontological model of quantum theory, in particular whether it can be assumed for nonprojective measurements. While it is true that one can always represent a measurement by a deterministic response function by incorporating ancillary degrees of freedom into one's description (for instance those of the apparatus), I show that in moving to such a representation, one typically loses the warrant to apply the assumption of measurement noncontextuality. The implications for experimental tests of measurement noncontextuality will be discussed.
The status of determinism in noncontextual models of quantum theory

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My view on the correct interpretation of quantum mechanics and how contextuality comes out as the key phenomenon to tackle in this research program

One understands something best when one can apply it to useful purposes. Therefore, we need to understand the technological applications of contextuality
The traditional notion of a noncontextual hidden variable model of quantum theory
Traditional notion of a noncontextual hidden variable model:
For every hidden state $\lambda$, every basis of vectors receives a 0-1 valuation, wherein exactly one element is assigned the value 1 (corresponding to the outcome that would occur for $\lambda$), and every vector is assigned the same value regardless of the basis it is considered a part (i.e. the context).
Traditional notion of a noncontextual hidden variable model: For every hidden state $\lambda$, every basis of vectors receives a 0-1 valuation, wherein exactly one element is assigned the value 1 (corresponding to the outcome that would occur for $\lambda$), and every vector is assigned the same value regardless of the basis it is considered a part (i.e. the context).
Traditional notion of a noncontextual hidden variable model:
For every $\lambda$, every projector $II$ is assigned a value 0 or 1 regardless of how it is measured (i.e. the context)

$$v(\Pi) = 0 \text{ or } 1 \quad \text{for all } II$$

$\{\Pi_1, \Pi_2, \Pi_3\}$ \hspace{1cm} $v(\Pi_1) = 1$

$\{\Pi_1, \Pi'_2, \Pi'_3\}$ \hspace{1cm} $v(\Pi_1) = 1$
Bell-Kochen-Specker theorem: A traditional noncontextual hidden variable model of quantum theory for Hilbert spaces of dimension 3 or greater is impossible.
The most general sort of measurement in quantum theory

<table>
<thead>
<tr>
<th>Standard Measurements</th>
<th>Generalized Measurements</th>
</tr>
</thead>
<tbody>
<tr>
<td>{\Pi_i}</td>
<td>{E_d}</td>
</tr>
<tr>
<td>\langle \psi</td>
<td>\Pi_i</td>
</tr>
<tr>
<td>\sum_i \Pi_i = I</td>
<td>\sum_d E_d = I</td>
</tr>
<tr>
<td>\text{P}(i) = \text{tr}(\rho \Pi_i)</td>
<td>\text{P}(d) = \text{tr}(\rho E_d)</td>
</tr>
<tr>
<td>\Pi_i \Pi_j = \delta_{ij} \Pi_i</td>
<td>\text{---}</td>
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Measurement by coupling to an ancilla

\[ p(k) = \text{Tr}_{s a}[\prod_k^{(s a)} (\rho_s \otimes \tau_a)] \]
\[ = \text{Tr}_s[\text{Tr}_a(\prod_k^{(s a)} \tau_a) \rho_s] \]

**Naimark's theorem**: Every POVM can be implemented by coupling to an ancilla and implementing a projective measurement.
Example

\[ |\theta\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle \]

\[ \{ |\Phi_1\rangle, |\Phi_2\rangle, |\Phi_3\rangle, |\Phi_4\rangle \} = \{ \sqrt{2}^{-1}(|0\rangle|0\rangle + |1\rangle|1\rangle), \]
\[ \sqrt{2}^{-1}(|0\rangle|0\rangle - |1\rangle|1\rangle), \]
\[ \sqrt{2}^{-1}(|0\rangle|1\rangle + |1\rangle|0\rangle), \]
\[ \sqrt{2}^{-1}(|0\rangle|1\rangle - |1\rangle|0\rangle) \}\]

\[ E^{(s)}_k = \text{Tr}_a(\Pi^{(sa)}_k \tau_a) \]
\[ = \langle \theta | a | \Phi_k \rangle_{sa} \langle \Phi_k | sa | \theta \rangle_a \]

\[ \langle \theta | a | \Phi_{1(2)} \rangle_{sa} = \sqrt{2}^{-1} [ \cos(\theta/2)|0\rangle_s + \sin(\theta/2)|1\rangle_s ] = \]
\[ \langle \theta | a | \Phi_{3(4)} \rangle_{sa} = \sqrt{2}^{-1} [ \sin(\theta/2)|0\rangle_s + \cos(\theta/2)|1\rangle_s ] = \]

\[ \{ E_k \} = \{ \frac{1}{2} |\theta\rangle\langle \theta |, \frac{1}{2} |\theta\rangle\langle -\theta |, \frac{1}{2} |\pi - \theta\rangle\langle \pi - \theta |, \frac{1}{2} |\pi + \theta\rangle\langle \pi + \theta | \} \]

\[ \theta = \pi/4 \]
Generalizing the notion of noncontextuality

From projective measurements to POVMs
So that we can model real experiments, where inevitable decoherence (coupling to the environment) implies that no measurement is truly projective
A popular proposal for how to generalize the notion of noncontextuality to POVMs
Quantum States and Generalized Observables: A Simple Proof of Gleason’s Theorem

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(Received 29 May 2003; published 19 September 2003)

“An interpretation of valuations as truth value assignments would require the numbers \( v(E) \) to be either 1 or 0, indicating the occurrence or nonoccurrence of an outcome associated with \( E \). Valuations with this property are referred to as dispersion-free. The above theorem entails immediately that dispersion-free effect valuations [...] do not exist. It follows that noncontextual hidden variables, understood as dispersion-free, globally defined, valuations, are excluded in quantum mechanics.”
Kochen-Specker Theorem for a Single Qubit using Positive Operator-Valued Measures

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(Received 2 October 2002; published 12 May 2003)

“Each equation contains eight positive-semidefinite operators whose sum is the identity. Therefore, a noncontextual hidden-variable theory must assign the answer yes to one and only one of these eight operators.”
An alternative proposal for how to generalize the notion of noncontextuality to POVMs

And for how to generalize it from quantum theory to any operational theory

So that for any given experimental data, we can say whether it can be explained by a noncontextual model regardless of the empirical status of quantum theory
Operational theories

These are defined as lists of instructions

An operational theory specifies

\[ p(k|P, M) \equiv \text{The probability of outcome } k \text{ of } M \text{ given } P \]
Operational Quantum Mechanics

Preparation $P$

Measurement $M$

Density operator $\rho$

Positive operator-valued measure (POVM) $\{E_k\}$

$$p(k|P, M) = \text{Tr}[E_k \rho]$$
A hidden variable model of an operational theory

\[ p(k|P, M) = \int d\lambda \xi_{M,k}(\lambda) \mu_P(\lambda) \]
Generalized definition of noncontextuality:

A hidden variable model of an operational theory is **noncontextual** if

Operational equivalence of two experimental procedures $\rightarrow$ Equivalent representations in the realist model
Example from quantum theory

Different POVMs

\{E_1, E_2\} \quad \{E_1', E_2'\}
Example from quantum theory

\[
\langle \psi_1 | \langle \psi_1 | , I - |\psi_1 \rangle \langle \psi_1 | \rangle \rangle
\]

\[
I - |\psi_1 \rangle \langle \psi_1 |
\]

\[
= |\psi_2 \rangle \langle \psi_2 | + |\psi_3 \rangle \langle \psi_3 |
\]

\[
I - |\psi_1 \rangle \langle \psi_1 |
\]

\[
= |\psi_2' \rangle \langle \psi_2' | + |\psi_3' \rangle \langle \psi_3' |
\]
Example from quantum theory
Example from quantum theory
New notion versus traditional notion for representation of projective measurements
How to formulate the traditional notion of noncontextuality:
This is equivalent to assuming:

\[ \{ \vert \psi_1 \rangle \langle \psi_1 \vert, I - \vert \psi_1 \rangle \langle \psi_1 \vert \} \]

\[ \{ \vert \psi_1 \rangle \langle \psi_1 \vert, I - \vert \psi_1 \rangle \langle \psi_1 \vert \} \]
But recall that the most general representation was

$$\{ P_k \} \xrightarrow{M} \xi_{P_1}(\lambda) \xrightarrow{\lambda} \xi_{P_2}(\lambda) \xrightarrow{\lambda} \xi_{P_3}(\lambda) \xrightarrow{\lambda}$$

Therefore:

traditional notion of noncontextuality for projective mmts = revised notion of noncontextuality for projective mmts and outcome determinism for projective mmts
So, the new definition of noncontextuality is not simply a *generalization* of the traditional notion

For projective measurements, it is a *revision* of the traditional notion
Local determinism:
We ask: Does the outcome depend on space-like separated events (in addition to local settings and \( \lambda \))?

Local causality:
We ask: Does the probability of the outcome depend on space-like separated events (in addition to local settings and \( \lambda \))?

Traditional notion of measurement noncontextuality:
We ask: Does the outcome depend on the measurement context (in addition to the observable and \( \lambda \))?

The revised notion of measurement noncontextuality:
We ask: Does the probability of the outcome depend on the measurement context (in addition to the observable and \( \lambda \))?

Noncontextuality and determinism are separate issues
traditional notion of noncontextuality for projective mmts = revised notion of noncontextuality for projective mmts and outcome determinism for projective mmts

No-go theorems for previous notion of noncontextuality are not necessarily no-go theorems for the new notion!

In face of contradiction, could give up determinism!
Can we justify an assumption of outcome determinism in some way?

Many people have a strong intuition that allowing outcome indeterminism does not add any generality and that consequently we may as well assume outcome determinism. Recall Fine’s theorem for instance.
Outcome-deterministic ontic extension

For any given model, we can always build one that is outcome-deterministic on a larger system

Example: Replace

\[ \xi_1(\lambda) \rightarrow \lambda \]
\[ \xi_2(\lambda) \rightarrow \lambda \]
\[ \xi_3(\lambda) \rightarrow \lambda \]

with

\[ \mu(\lambda_a) \rightarrow \lambda_a \]
\[ \chi_1(\lambda_s, \lambda_a) \rightarrow \lambda_s \]
\[ \chi_2(\lambda_s, \lambda_a) \rightarrow \lambda_s \]
\[ \chi_3(\lambda_s, \lambda_a) \rightarrow \lambda_s \]
The argument in favor of the popular proposal

Premiss: If two measurements have the same statistics for all preparations, then they should be represented by the same response function in the hidden variable model
Operational equivalence → Ontic equivalence
The assumption of measurement noncontextuality

Premiss: Every measurement can be represented by an outcome-deterministic response function on a larger system
Operational unsharpness is consistent with ontic sharpness
Outcome-deterministic ontic extensions of measurements

Purported conclusion: If two measurements have the same statistics for all preparations, then they should be represented by the same outcome-deterministic response functions
A reason to be suspicious

Premiss: If two measurements have the same statistics for all preparations, then they can be represented in the quantum formalism by the same POVM

An assumption of quantum theory

Premiss: Every measurement can be represented by a projective-valued measure on a larger system

Naimark’s theorem

Purported conclusion: If two measurements have the same statistics for all preparations, then they can be represented by the same projector-valued measure

FALSE
Two partitions of an experiment

\[ P_a, \quad \rho_a \]
\[ P_s, \quad \rho_s \]

\[ M_{sa}, \quad \{ \prod_{k}^{(sa)} \} \]
\[ P_{sa} = (P_s, P_a) \]
\[ \rho_{sa} = \rho_s \otimes \rho_a \]
\[ p(k|P_{sa}, M_{sa}) = \text{Tr}_{sa}(\rho_{sa} \Pi_{k}^{(sa)}) \]
\[ M_{sa} = \text{(P}_a, M_{sa}) \]
\[ \{ E_k^{(s)} = \text{Tr}(\rho_a \Pi_{k}^{(sa)}) \} \]
\[ M_s = (P_a, M_{sa}) \]
\[ \{ E_k^{(s)} = \text{Tr}(\rho_a \Pi_{k}^{(sa)}) \} \]
\[ p(k|P_s, M_s) = \text{Tr}_s(\rho_s E_k^{(s)}) \]
Note:

It is not the case that a single measurement procedure can be represented either by a POVM or by a projector-valued measure.

There are many Neumark extensions of a given POVM.
$P_{sa} = (P_s, P_a)$

$\mu(\lambda_s, \lambda_a) = \mu(\lambda_s)\mu(\lambda_a)$

$p(k|P_{sa}, M_{sa}) = \int d\lambda_s d\lambda_a \mu(\lambda_s, \lambda_a) \xi_k(\lambda_s, \lambda_a)$

$P_s$

$\mu(\lambda_s)$

$M_{sa}$

$\{\xi_k(\lambda_s, \lambda_a)\}$

$M_s = (P_a, M_{sa})$

$\{\xi_k(\lambda_s) = \sum_{\lambda_a} \xi_k(\lambda_s, \lambda_a) \mu(\lambda_a)\}$

$p(k|P_s, M_s) = \int d\lambda_s \mu(\lambda_s) \xi_k(\lambda_s)$
Even if the response function is sharp on the composite space, it may not be sharp on the system space.

Do we always have a sharp ontic extension of a set of unsharp response functions?
A refinement of the argument in favor of the popular proposal

Premiss: If two measurements \textit{on} \textit{s} have the same statistics for all preparations \textit{on} \textit{s}, then they should be represented by the same response functions \textit{on} \textit{s}

Operational equivalence \textit{on} \textit{s} $\rightarrow$ Ontic equivalence \textit{on} \textit{s}

The assumption of measurement noncontextuality

Premiss: Every measurement \textit{on} \textit{s} can be represented by an outcome-deterministic response function \textit{on} \textit{sa} (and a distribution on \textit{a})

Operational unsharpness is consistent with ontic sharpness

Outcome-deterministic ontic extensions of measurements

Purported conclusion: If two measurements \textit{on} \textit{s} have the same statistics for all preparations \textit{on} \textit{s}, then they should be represented by the same outcome-deterministic response functions on some system (\textit{on} \textit{s} or \textit{on} \textit{sa}).
What needs to be shown to justify the popular proposal

Popular proposal on s: If two measurements on s have the same statistics for all preparations on s, then they should be represented by the same outcome-deterministic response functions on s.
What needs to be shown to justify the popular proposal

Or

Popular proposal on sa: If two measurements on s have the same statistics for all preparations on s, then they should be represented by the same outcome-deterministic response functions on sa.
The assumptions we can justify are...
P: Operational equivalence on $s$ implies equivalent response fns on $s$

\[ \xi_1(\lambda_s) \rightarrow \lambda_s \]
\[ \xi_2(\lambda_s) \rightarrow \lambda_s \]
\[ \xi_3(\lambda_s) \rightarrow \lambda_s \]
\[ \xi'_1(\lambda_s) \rightarrow \lambda_s \]
\[ \xi'_2(\lambda_s) \rightarrow \lambda_s \]
\[ \xi'_3(\lambda_s) \rightarrow \lambda_s \]

P: Every POVM on $s$ can be represented by a set of outcome-deterministic response functions on $sa$ (and a distribution on $a$)

\[ \mu(\lambda_a) \rightarrow \lambda_a \]
\[ \chi_1(\lambda_s, \lambda_a) \rightarrow \lambda_s \]
\[ \chi_2(\lambda_s, \lambda_a) \rightarrow \lambda_s \]
\[ \chi_3(\lambda_s, \lambda_a) \rightarrow \lambda_s \]
The assumptions we need to get the popular proposal on s are...
P: Operational equivalence on $s$ implies equivalent response fns on $s$

\[ \xi_1(\lambda_s) \quad \xi_2(\lambda_s) \quad \xi_3(\lambda_s) \quad \xi'_1(\lambda_s) \quad \xi'_2(\lambda_s) \quad \xi'_3(\lambda_s) \]

\[ \lambda_s \quad \lambda_s \quad \lambda_s \quad \lambda_s \quad \lambda_s \quad \lambda_s \]

P: Every POVMs on $s$ can be represented by a set of outcome-deterministic response functions on $s$

\[ \chi_1(\lambda_s) \quad \chi_2(\lambda_s) \quad \chi_3(\lambda_s) \]

\[ \lambda_s \quad \lambda_s \quad \lambda_s \]
Why a POVM on $s$ cannot be represented by an outcome-deterministic response function on $s$

POVM
\[ \{ \frac{I}{2}, \frac{I}{2} \} \]

Response functions
\[ \{ \frac{1}{2}, \frac{1}{2} \} \]

$\xi_1(\lambda_s)$

$\xi_2(\lambda_s)$
The assumptions we need to get the popular proposal on sa are...
P: Every POVMs on s can be represented by a set of outcome-deterministic response functions on sa (and a distribution on a)

P: Operational equivalence on s implies equivalent response fns on sa
Why operational equivalence on $s$ cannot imply equivalent response functions on $sa$
Can we justify an assumption of outcome determinism in some way?

Yes, from an assumption of noncontextuality for preparations but only for projective measurements
The notion of preparation noncontextuality
Difference of Equivalence class
Example from quantum theory

Different density op's
Example from quantum theory

\[ \frac{1}{2} I = \frac{1}{2} |0\rangle \langle 0| + \frac{1}{2} |1\rangle \langle 1| \]

\[ \frac{1}{2} I = \frac{1}{2} |+\rangle \langle +| + \frac{1}{2} |-\rangle \langle -| \]
Example from quantum theory

\[ \frac{1}{2}I = \text{Tr}_B\left[ \frac{1}{\sqrt{2}}(|0\rangle |0\rangle + |1\rangle |1\rangle) \right] \]

\[ \frac{1}{2}I = \text{Tr}_B\left[ \frac{1}{\sqrt{2}}(|0\rangle |+\rangle + |1\rangle |-\rangle) \right] \]
Preparation noncontextual model

$$\mu(\lambda)$$
One can prove that

Preparation noncontextuality → outcome determinism for projective measurements

Proof

\[
\mu_{I/3}(\lambda) = \frac{1}{3} \mu_{\psi_1}(\lambda) + \frac{1}{3} \mu_{\psi_2}(\lambda) + \frac{1}{3} \mu_{\psi_3}(\lambda)
\]

\[
\mu_{I/3}(\lambda) = p \mu_{\psi}(\lambda) + \ldots
\]
We’ve established that

preparation noncontextuality → outcome determinism for projective measurements

Therefore:

measurement noncontextuality
and
preparation noncontextuality → measurement noncontextuality
and
outcome determinism for projective measurements
We’ve established that

preparation noncontextuality → outcome determinism for projective measurements

Therefore:

measurement noncontextuality and preparation noncontextuality → Traditional notion of noncontextuality

no-go theorems for the traditional notion of noncontextuality can be salvaged as no-go theorems for the generalized notion

... and there are many new proofs
What needs to be done to obtain convincing experimental tests of universal noncontextuality (featuring measurements)

Theory side:
- Determine whether the implication from preparation noncontextuality to outcome determinism for sharp measurements holds for other operational theories
- Define robust notion of noncontextuality (operational closeness implies ontic closeness)

Experimental side:
Test operational equivalence of measurements and of preparations (the latter to justify outcome-determinism for projective measurements)