Abstract:
The generalized notion of noncontextuality
Problems with the traditional definition of noncontextuality:
- applies only to projective measurements
- applies only to deterministic hidden variable models
- applies only to models of quantum theory
Problems with the traditional definition of noncontextuality:
- applies only to projective measurements
- applies only to deterministic hidden variable models
- applies only to models of quantum theory

A better notion of noncontextuality would determine
- whether any given theory admits a noncontextual model
- whether any given experimental data can be explained by a noncontextual model
A realist model of an operational theory

\[ \int \mu_P(\lambda) d\lambda = 1 \]

\[ \mu_P(\lambda) \]
A realist model of an operational theory

\[ \int \mu_P(\lambda) d\lambda = 1 \]

\[ 0 \leq \xi_{M,k} \leq 1 \]

\[ \sum_k \xi_{M,k}(\lambda) = 1 \text{ for all } \lambda \]
A realist model of an operational theory

\[ \int \mu_P(\lambda) d\lambda = 1 \]

\[ \mu_P(\lambda) \]

\[ 0 \leq \xi_{M,k} \leq 1 \]

\[ \sum_k \xi_{M,k}(\lambda) = 1 \text{ for all } \lambda \]

\[ \xi_{M,1}(\lambda) \]

\[ \xi_{M,2}(\lambda) \]

\[ \xi_{M,3}(\lambda) \]
Generalized definition of noncontextuality:

A realist model of an operational theory is noncontextual if

Operational equivalence of two experimental procedures \[ ightarrow \] Equivalent representations in the realist model
Operational equivalence classes
Operational equivalence classes
Operational equivalence classes

P is equivalent to P' if
\[ \forall M \forall k : p(k|P, M) = p(k|P', M) \]
Difference of Equivalence class
Difference of context
Example from quantum theory

Different density op's

\[ \rho \quad \rho' \]
Example from quantum theory

\[ I = \frac{1}{2} |0\rangle \langle 0| + \frac{1}{2} |1\rangle \langle 1| \]

\[ \frac{1}{2} I = \frac{1}{2} |+\rangle \langle +| + \frac{1}{2} |-_\rangle \langle -| \]

Example from quantum theory

\[ I = \text{Tr}_B \left[ \frac{1}{\sqrt{2}} (|0\rangle \langle 0| + |1\rangle \langle 1|) \right] \]

\[ \frac{1}{2} I = \text{Tr}_B \left[ \frac{1}{\sqrt{2}} (|0\rangle \langle +| + |1\rangle \langle -|) \right] \]
Preparation noncontextual model

\[ \mu(\lambda) \]
Example from quantum theory

\[ I = \text{Tr}_B\left[ \frac{1}{\sqrt{2}} (|0\rangle |0\rangle + |1\rangle |1\rangle) \right] \]

\[ \frac{1}{2} I = \text{Tr}_B\left[ \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle - |1\rangle + |0\rangle) \right] \]
Example from quantum theory

\[ I = \frac{1}{2} |0\rangle \langle 0| + \frac{1}{2} |1\rangle \langle 1| \]

\[ \frac{1}{2} I = \frac{1}{2} |+\rangle \langle +| + \frac{1}{2} |-\rangle \langle -| \]
Example from quantum theory

\[ I = \text{Tr}_B[\frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)] \]

\[ \frac{1}{2}I = \text{Tr}_B[\frac{1}{\sqrt{2}}(|0\rangle|+\rangle + |1\rangle|-\rangle)] \]
Example from quantum theory

\[ I = \frac{1}{2} |0\rangle \langle 0| + \frac{1}{2} |1\rangle \langle 1| \]

\[ \frac{1}{2} I = \frac{1}{2} |+\rangle \langle +| + \frac{1}{2} |-\rangle \langle -| \]
Example from quantum theory

\[ I = \text{Tr}_B \left[ \frac{1}{\sqrt{2}} (\langle 0 | 0 \rangle + |1 \rangle |1 \rangle) \right] \]

\[ \frac{1}{2} I = \text{Tr}_B \left[ \frac{1}{\sqrt{2}} (\langle 0 | + \rangle + |1 \rangle | - \rangle) \right] \]
Example from quantum theory

\[ I = \frac{1}{2} |0\rangle \langle 0| + \frac{1}{2} |1\rangle \langle 1| \]

\[ \frac{1}{2} I = \frac{1}{2} |+\rangle \langle +| + \frac{1}{2} |-\rangle \langle -| \]
Example from quantum theory

\[ I = \text{Tr}_B[\frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)] \]

\[ \frac{1}{2}I = \text{Tr}_B[\frac{1}{\sqrt{2}}(|0\rangle|+\rangle + |1\rangle|-\rangle)] \]
Preparation noncontextual model

\[ \mu(\lambda) \]

\[ \lambda \]
Example from quantum theory

\[ I = \frac{1}{2} |0\rangle \langle 0| + \frac{1}{2} |1\rangle \langle 1| \]

\[ \frac{1}{2} I = \frac{1}{2} |+\rangle \langle +| + \frac{1}{2} |-\rangle \langle -| \]
Example from quantum theory

\[ I = \frac{1}{2} |0\rangle \langle 0| + \frac{1}{2} |1\rangle \langle 1| \]

\[ \frac{1}{2} I = \frac{1}{2} |+\rangle \langle +| + \frac{1}{2} |-\rangle \langle -| \]
Preparation contextual model

$P_4(\lambda)$

$\mu_{P_1}(\lambda)$
Preparation noncontextual model

$\mu(\lambda)$

$\lambda$
Example from quantum theory

\[ I = \text{Tr}_B\left[ \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle) \right] \]

\[ \frac{1}{2}I = \text{Tr}_B\left[ \frac{1}{\sqrt{2}}(|0\rangle|+\rangle + |1\rangle|-\rangle) \right] \]
Example from quantum theory

\[ I = \frac{1}{2} |0\rangle \langle 0| + \frac{1}{2} |1\rangle \langle 1| \]

\[ \frac{1}{2} I = \frac{1}{2} |+\rangle \langle +| + \frac{1}{2} |-\rangle \langle -| \]
Preparation contextual model

$P_4(\lambda)$

$\mu_{P_1}(\lambda)$
Definition of preparation noncontextual model:

\[ \forall M : p(k|P, M) = p(k|P', M) \]

\[ p(\lambda|P) = p(\lambda|P') \]
(a) Some states of a qubit

(b) A preparation noncontextual model of these

(c) A preparation contextual model of these
   (Kochen-Specker, 1967)

\[
\mu_{1/2}(\lambda) = \frac{1}{2} \mu_{|0\rangle}(\lambda) + \frac{1}{2} \mu_{|1\rangle}(\lambda) \\
\mu_{1/2, p_{b_1}}(\lambda) = \frac{1}{2} \mu_{|0\rangle}(\lambda) + \frac{1}{2} \mu_{|1\rangle}(\lambda)
\]
\[
\{ |\psi_1\rangle\langle\psi_1|, I - |\psi_1\rangle\langle\psi_1| \} \quad \{ |\psi_1\rangle\langle\psi_1|, I - |\psi_1\rangle\langle\psi_1| \} \\
I - |\psi_1\rangle\langle\psi_1| = |\psi_2\rangle\langle\psi_2| + |\psi_3\rangle\langle\psi_3| \quad I - |\psi_1\rangle\langle\psi_1| = |\psi_2^\prime\rangle\langle\psi_2^\prime| + |\psi_3^\prime\rangle\langle\psi_3^\prime| 
\]
\{E, I - E\}
\begin{align*}
E &= q \left\langle \frac{n}{4} \right| + (1 - q) \frac{1}{2} I \\
E &= \frac{1}{2} \left\langle 0 \right| + \frac{1}{2} \left\langle + \right|
\end{align*}
\begin{align*}
\{E, I - E\} \\
E &= q|\frac{n}{4}\rangle\langle\frac{n}{4}| + (1 - q)\frac{1}{2}I
\end{align*}

\begin{align*}
\{E, I - E\} \\
E &= \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|+\rangle\langle +|
\end{align*}
\begin{align*}
\{E, I - E\} \\
E &= q|\frac{n}{4}\rangle\langle\frac{n}{4}| + (1 - q)\frac{1}{2}I \\
\{E, I - E\} \\
E &= \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|+\rangle\langle +| 
\end{align*}
\begin{align*}
\{E, I - E\} = q\frac{n}{4}\left\langle \frac{n}{4} \right\rangle + (1 - q)\frac{1}{2} I \\
E = \frac{1}{2}\left| 0 \right\rangle \left\langle 0 \right| + \frac{1}{2}\left| + \right\rangle \left\langle + \right|.
\end{align*}
universal noncontextuality
= noncontextuality for preparations and measurements
Generalized noncontextuality in quantum theory
Defining noncontextuality in quantum theory

Preparation Noncontextuality in QT

if $P, P' \rightarrow \rho$ then $\mu_P(\lambda) = \mu_{P'}(\lambda) = \mu_\rho(\lambda)$
Defining noncontextuality in quantum theory

Measurement Noncontextuality in QT

if $M, M' \rightarrow \{E_k\}$ then $\xi_{M,k}(\lambda) = \xi_{M',k}(\lambda) = \xi_{E_k}(\lambda)$
Preparation-based proof of contextuality

(i.e. of the impossibility of a noncontextual realist model of quantum theory)
Important features of realist models

Let \( P \leftrightarrow \mu(\lambda) \)

\( P' \leftrightarrow \mu'(\lambda) \)

Representing one-shot distinguishability:

If \( P \) and \( P' \) are distinguishable with certainty

then \( \mu(\lambda) \mu'(\lambda) = 0 \)
Important features of realist models

Let $P \leftrightarrow \mu(\lambda)$

$P' \leftrightarrow \mu'(\lambda)$

Representing one-shot distinguishability:

If $P$ and $P'$ are distinguishable with certainty then $\mu(\lambda) \mu'(\lambda) = 0$

Representing convex combination:

If $P'' = P$ with prob. $p$ and $P'$ with prob. $1 - p$

Then $\mu''(\lambda) = p \mu(\lambda) + (1 - p) \mu'(\lambda)$
Proof based on finite construction in 2d

\[ P_a \leftrightarrow \psi_a = (1, 0) \]
\[ P_A \leftrightarrow \psi_A = (0, 1) \]
\[ P_b \leftrightarrow \psi_b = (1/2, \sqrt{3}/2) \]
\[ P_B \leftrightarrow \psi_B = (\sqrt{3}/2, -1/2) \]
\[ P_c \leftrightarrow \psi_c = (1/2, -\sqrt{3}/2) \]
\[ P_C \leftrightarrow \psi_C = (\sqrt{3}/2, 1/2) \]
Proof based on finite construction in 2d

\[ \sigma_a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \]
\[ \sigma_A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \]
\[ \sigma_b = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \sqrt{3} \\ \frac{1}{4} \sqrt{3} & \frac{3}{4} \end{pmatrix} \]
\[ \sigma_B = \begin{pmatrix} \frac{3}{4} & -\frac{1}{4} \sqrt{3} \\ -\frac{1}{4} \sqrt{3} & \frac{1}{4} \end{pmatrix} \]
\[ \sigma_c = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \sqrt{3} \\ -\frac{1}{4} \sqrt{3} & \frac{3}{4} \end{pmatrix} \]
\[ \sigma_C = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \sqrt{3} \\ \frac{1}{4} \sqrt{3} & \frac{1}{4} \end{pmatrix} \]

\[ \sigma_a \sigma_A = 0 \]
\[ \sigma_b \sigma_B = 0 \]
\[ \sigma_c \sigma_C = 0 \]
Proof based on finite construction in 2d

\[ P_a \leftrightarrow \sigma_a = \begin{pmatrix} 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]
\[ \sigma_a \sigma_A = 0 \]
\[ P_A \leftrightarrow \sigma_A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \]
\[ \sigma_b \sigma_B = 0 \]
\[ P_b \leftrightarrow \sigma_b = \begin{pmatrix} 1 & \frac{1}{4} \sqrt{3} \\ \frac{1}{4} \sqrt{3} & \frac{3}{4} \end{pmatrix} \]
\[ \sigma_c \sigma_C = 0 \]
\[ P_B \leftrightarrow \sigma_B = \begin{pmatrix} \frac{3}{4} & -\frac{1}{4} \sqrt{3} \\ -\frac{1}{4} \sqrt{3} & \frac{3}{4} \end{pmatrix} \]
\[ P_c \leftrightarrow \sigma_c = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \sqrt{3} \\ -\frac{1}{4} \sqrt{3} & \frac{3}{4} \end{pmatrix} \]
\[ P_C \leftrightarrow \sigma_c = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \sqrt{3} \\ \frac{1}{4} \sqrt{3} & \frac{3}{4} \end{pmatrix} \]

\[ \sigma_a, \sigma_b, \sigma_c, \sigma_A, \sigma_B, \sigma_C \]

\( P_a \) and \( P_A \) are distinguishable with certainty
\( P_b \) and \( P_B \) are distinguishable with certainty
\( P_c \) and \( P_C \) are distinguishable with certainty

\[ \mu_a(\lambda) \mu_A(\lambda) = 0 \]
\[ \rightarrow \mu_b(\lambda) \mu_B(\lambda) = 0 \]
\[ P_{aA} \equiv P_a \text{ and } P_A \text{ with prob. } 1/2 \text{ each} \]
\[ P_{bB} \equiv P_b \text{ and } P_B \text{ with prob. } 1/2 \text{ each} \]
\[ P_{cC} \equiv P_c \text{ and } P_C \text{ with prob. } 1/2 \text{ each} \]
\[ P_{abc} \equiv P_a, P_b \text{ and } P_c \text{ with prob. } 1/3 \text{ each} \]
\[ P_{ABC} \equiv P_A, P_B \text{ and } P_C \text{ with prob. } 1/3 \text{ each} \]
\( P_{aA} \equiv P_a \text{ and } P_A \text{ with prob. } 1/2 \text{ each} \)
\( P_{bB} \equiv P_b \text{ and } P_B \text{ with prob. } 1/2 \text{ each} \)
\( P_{cC} \equiv P_c \text{ and } P_C \text{ with prob. } 1/2 \text{ each} \)
\( P_{abc} \equiv P_a, P_b \text{ and } P_c \text{ with prob. } 1/3 \text{ each} \)
\( P_{ABC} \equiv P_A, P_B \text{ and } P_C \text{ with prob. } 1/3 \text{ each} \)

\[
\begin{align*}
\mu_{aA}(\lambda) &= \frac{1}{2} \mu_a(\lambda) + \frac{1}{2} \mu_A(\lambda) \\
\mu_{bB}(\lambda) &= \frac{1}{2} \mu_b(\lambda) + \frac{1}{2} \mu_B(\lambda) \\
\mu_{cC}(\lambda) &= \frac{1}{2} \mu_c(\lambda) + \frac{1}{2} \mu_C(\lambda) \\
\mu_{abc}(\lambda) &= \frac{1}{3} \mu_a(\lambda) + \frac{1}{3} \mu_b(\lambda) + \frac{1}{3} \mu_c(\lambda) \\
\mu_{ABC}(\lambda) &= \frac{1}{3} \mu_A(\lambda) + \frac{1}{3} \mu_B(\lambda) + \frac{1}{3} \mu_C(\lambda)
\end{align*}
\]
\[ I/2 = \frac{1}{2} \sigma_a + \frac{1}{2} \sigma_A \]
\[ = \frac{1}{2} \sigma_b + \frac{1}{2} \sigma_B \]
\[ = \frac{1}{2} \sigma_c + \frac{1}{2} \sigma_C \]
\[ = \frac{1}{3} \sigma_a + \frac{1}{3} \sigma_b + \frac{1}{3} \sigma_c \]
\[ = \frac{1}{3} \sigma_A + \frac{1}{3} \sigma_B + \frac{1}{3} \sigma_C. \]
\[ I/2 = \frac{1}{2} \sigma_a + \frac{1}{2} \sigma_A \]
\[ = \frac{1}{2} \sigma_b + \frac{1}{2} \sigma_B \]
\[ = \frac{1}{2} \sigma_c + \frac{1}{2} \sigma_C \]
\[ = \frac{1}{3} \sigma_a + \frac{1}{3} \sigma_b + \frac{1}{3} \sigma_c \]
\[ = \frac{1}{3} \sigma_A + \frac{1}{3} \sigma_B + \frac{1}{3} \sigma_C. \]

\[ P_{aA} \simeq P_{bB} \simeq P_{cC} \]
\[ \simeq P_{abc} \simeq P_{ABC} \]

**By preparation noncontextuality**

\[ \mu_{aA}(\lambda) = \mu_{bB}(\lambda) = \mu_{cC}(\lambda) \]
\[ = \mu_{abc}(\lambda) = \mu_{ABC}(\lambda) \]
\[ \equiv \nu(\lambda) \]
\[ I/2 = \frac{1}{2} \sigma_a + \frac{1}{2} \sigma_A \]
\[ = \frac{1}{2} \sigma_b + \frac{1}{2} \sigma_B \]
\[ = \frac{1}{2} \sigma_c + \frac{1}{2} \sigma_C \]
\[ = \frac{1}{3} \sigma_a + \frac{1}{3} \sigma_b + \frac{1}{3} \sigma_c \]
\[ = \frac{1}{3} \sigma_A + \frac{1}{3} \sigma_B + \frac{1}{3} \sigma_C. \]

\[ P_{aA} \simeq P_{bB} \simeq P_{cC} \]
\[ \simeq P_{abc} \simeq P_{ABC} \]

By preparation noncontextuality

\[ \mu_{aA}(\lambda) = \mu_{bB}(\lambda) = \mu_{cC}(\lambda) \]
\[ \equiv \mu_{abc}(\lambda) = \mu_{ABC}(\lambda) \implies \nu(\lambda) \]

\[ \nu(\lambda) = \frac{1}{2} \mu_a(\lambda) + \frac{1}{2} \mu_A(\lambda) \]
\[ = \frac{1}{2} \mu_b(\lambda) + \frac{1}{2} \mu_B(\lambda) \]
\[ = \frac{1}{2} \mu_c(\lambda) + \frac{1}{2} \mu_C(\lambda) \]
\[ = \frac{1}{3} \mu_a(\lambda) + \frac{1}{3} \mu_b(\lambda) + \frac{1}{3} \mu_c(\lambda) \]
\[ = \frac{1}{3} \mu(\lambda) + \frac{1}{3} \mu(\lambda) + \frac{1}{3} \mu(\lambda) \]
Our task is to find
\[ \mu_a(\lambda), \mu_A(\lambda), \mu_b(\lambda), \]
\[ \mu_B(\lambda), \mu_c(\lambda), \mu_C(\lambda), \]
and \( \nu(\lambda) \) such that

\[ \mu_a(\lambda) \mu_A(\lambda) = 0 \]
\[ \mu_b(\lambda) \mu_B(\lambda) = 0 \]
\[ \mu_c(\lambda) \mu_C(\lambda) = 0 \]

\[ \nu(\lambda) = \frac{1}{2} \mu_a(\lambda) + \frac{1}{2} \mu_A(\lambda) \]
\[ = \frac{1}{2} \mu_b(\lambda) + \frac{1}{2} \mu_B(\lambda) \]
\[ = \frac{1}{2} \mu_c(\lambda) + \frac{1}{2} \mu_C(\lambda) \]
\[ = \frac{1}{3} \mu_a(\lambda) + \frac{1}{3} \mu_b(\lambda) + \frac{1}{3} \mu_c(\lambda) \]
\[ = \frac{1}{3} \mu_A(\lambda) + \frac{1}{3} \mu_B(\lambda) + \frac{1}{3} \mu_C(\lambda). \]
Our task is to find
\[ \mu_a(\lambda), \mu_A(\lambda), \mu_b(\lambda), \mu_B(\lambda), \mu_c(\lambda), \mu_C(\lambda), \]
and \( \nu(\lambda) \) such that
\[ \mu_a(\lambda) \mu_A(\lambda) = 0 \]
\[ \mu_b(\lambda) \mu_B(\lambda) = 0 \]
\[ \mu_c(\lambda) \mu_C(\lambda) = 0 \]
i.e., paralleling the quantum structure:
\[ \sigma_a \sigma_A = 0 \]
\[ \sigma_b \sigma_B = 0 \]
\[ \sigma_c \sigma_C = 0 \]
\[ \frac{I}{2} = \frac{1}{2} \sigma_a + \frac{1}{2} \sigma_A \]
\[ = \frac{1}{2} \sigma_b + \frac{1}{2} \sigma_B \]
\[ = \frac{1}{2} \sigma_c + \frac{1}{2} \sigma_C \]
\[ = \frac{1}{3} \sigma_a + \frac{1}{3} \sigma_b + \frac{1}{3} \sigma_c \]
\[ = \frac{1}{3} \sigma_A + \frac{1}{3} \sigma_B + \frac{1}{3} \sigma_C. \]

\[ \nu(\lambda) = \frac{1}{2} \mu_a(\lambda) + \frac{1}{2} \mu_A(\lambda) \]
\[ = \frac{1}{2} \mu_b(\lambda) + \frac{1}{2} \mu_B(\lambda) \]
\[ = \frac{1}{2} \mu_c(\lambda) + \frac{1}{2} \mu_C(\lambda) \]
\[ = \frac{1}{3} \mu_a(\lambda) + \frac{1}{3} \mu_b(\lambda) + \frac{1}{3} \mu_c(\lambda) \]
\[ = \frac{1}{3} \mu_A(\lambda) + \frac{1}{3} \mu_B(\lambda) + \frac{1}{3} \mu_C(\lambda). \]
Our task is to find
\[ \mu_a(\lambda), \mu_A(\lambda), \mu_b(\lambda), \]
\[ \mu_B(\lambda), \mu_c(\lambda), \mu_C(\lambda), \]
and \( \nu(\lambda) \) such that

\[ \mu_a(\lambda) \mu_A(\lambda) = 0 \]
\[ \mu_b(\lambda) \mu_B(\lambda) = 0 \]
\[ \mu_c(\lambda) \mu_C(\lambda) = 0 \]

\[ \nu(\lambda) = \frac{1}{2} \mu_a(\lambda) + \frac{1}{2} \mu_A(\lambda) \]
\[ = \frac{1}{2} \mu_b(\lambda) + \frac{1}{2} \mu_B(\lambda) \]
\[ = \frac{1}{2} \mu_c(\lambda) + \frac{1}{2} \mu_C(\lambda) \]
\[ = \frac{1}{3} \mu_a(\lambda) + \frac{1}{3} \mu_b(\lambda) + \frac{1}{3} \mu_c(\lambda) \]
\[ = \frac{1}{3} \mu_A(\lambda) + \frac{1}{3} \mu_B(\lambda) + \frac{1}{3} \mu_C(\lambda) \]
Our task is to find $\mu_a(\lambda), \mu_A(\lambda), \mu_b(\lambda), \mu_B(\lambda), \mu_c(\lambda), \mu_C(\lambda)$, and $\nu(\lambda)$ such that

\[
\begin{align*}
\mu_a(\lambda) \mu_A(\lambda) &= 0 \\
\mu_b(\lambda) \mu_B(\lambda) &= 0 \\
\mu_c(\lambda) \mu_C(\lambda) &= 0
\end{align*}
\]

Consider $\lambda'$ such that $\nu(\lambda') \neq 0$

From decompositions (1)-(3)

\[
\begin{align*}
\mu_a(\lambda') &= 0 \text{ or } 2\nu(\lambda') \\
\mu_b(\lambda') &= 0 \text{ or } 2\nu(\lambda') \\
\mu_c(\lambda') &= 0 \text{ or } 2\nu(\lambda')
\end{align*}
\]

\[
\nu(\lambda) = \frac{1}{2} \mu_a(\lambda) + \frac{1}{2} \mu_A(\lambda) \\
= \frac{1}{2} \mu_b(\lambda) + \frac{1}{2} \mu_B(\lambda) \\
= \frac{1}{2} \mu_c(\lambda) + \frac{1}{2} \mu_C(\lambda) \\
= \frac{1}{3} \mu_a(\lambda) + \frac{1}{3} \mu_b(\lambda) + \frac{1}{3} \mu_c(\lambda) \\
= \frac{1}{3} \mu_A(\lambda) + \frac{1}{3} \mu_B(\lambda) + \frac{1}{3} \mu_C(\lambda)
\]
Our task is to find 
\[ \mu_a(\lambda), \mu_A(\lambda), \mu_b(\lambda), \mu_B(\lambda), \mu_c(\lambda), \mu_C(\lambda), \]
and \( \nu(\lambda) \) such that
\[
\begin{align*}
\mu_a(\lambda) \mu_A(\lambda) &= 0 \\
\mu_b(\lambda) \mu_B(\lambda) &= 0 \\
\mu_c(\lambda) \mu_C(\lambda) &= 0
\end{align*}
\]

Consider \( \lambda' \) such that \( \nu(\lambda') \neq 0 \)

From decompositions (1)-(3)
\[
\begin{align*}
\mu_a(\lambda') &= 0 \text{ or } 2\nu(\lambda') \\
\mu_b(\lambda') &= 0 \text{ or } 2\nu(\lambda') \\
\mu_c(\lambda') &= 0 \text{ or } 2\nu(\lambda')
\end{align*}
\]

But then the RHS of decomposition (4) is
\[
\begin{align*}
0, \frac{2}{3}\nu(\lambda'), \frac{4}{3}\nu(\lambda'), 2\nu(\lambda') \\
\neq \nu(\lambda')
\end{align*}
\]

\[ \text{CONTRADICTION} \]
Aside: justifying preparation noncontextuality by local causality

\[
\frac{I}{2} = \frac{1}{2}\sigma_a + \frac{1}{2}\sigma_A \\
= \frac{1}{2}\sigma_b + \frac{1}{2}\sigma_B \\
= \frac{1}{2}\sigma_c + \frac{1}{2}\sigma_C \\
= \frac{1}{3}\sigma_a + \frac{1}{3}\sigma_b + \frac{1}{3}\sigma_c \\
= \frac{1}{3}\sigma_A + \frac{1}{3}\sigma_B + \frac{1}{3}\sigma_C.
\]

By preparation noncontextuality

\[
\nu(\lambda) = \frac{1}{2}\mu_a(\lambda) + \frac{1}{2}\mu_A(\lambda) \\
= \frac{1}{2}\mu_b(\lambda) + \frac{1}{2}\mu_B(\lambda) \\
= \frac{1}{2}\mu_c(\lambda) + \frac{1}{2}\mu_C(\lambda) \\
= \frac{1}{3}\mu_a(\lambda) + \frac{1}{3}\mu_b(\lambda) + \frac{1}{3}\mu_c(\lambda).
\]

PNC for \( I/2 \) can be justified by local causality

But PNC for \( \sigma_x \) cannot be justified by local causality.
Also,

Any bipartite Bell-type proof of nonlocality $\rightarrow$ proof of preparation contextuality

(proof due to Jon Barrett)
Measurement contextuality

New definition versus traditional definition
Recall: the traditional notion of noncontextuality:

\[ |\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle \]

\[ \chi_1(\lambda), \chi_2(\lambda), \chi_3(\lambda) \]

\[ |\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle \]

\[ \chi_1(\lambda), \chi_2'(\lambda), \chi_3'(\lambda) \]
This is equivalent to assuming:

\[ \{ |\psi_1\rangle\langle\psi_1|, I - |\psi_1\rangle\langle\psi_1| \} \]

measure \[ |\psi_1\rangle \]

coarse-grain \[ |\psi_2\rangle \]

and \[ |\psi_3\rangle \]

\[ \chi_1(\lambda) \]

\[ \chi^{-1}(\lambda) \]

measure \[ |\psi_2\rangle \]

coarse-grain \[ |\psi'_2\rangle \] and \[ |\psi'_3\rangle \]

\[ \chi_1(\lambda) \]

\[ \chi^{-1}(\lambda) \]
But recall that the most general representation was

\[ \{ \prod_k \} \rightarrow M \rightarrow \begin{array}{c} \xi_{P_1}(\lambda) \rightarrow \lambda \\ \xi_{P_2}(\lambda) \rightarrow \lambda \\ \xi_{P_3}(\lambda) \rightarrow \lambda \end{array} \]

Therefore:

traditional notion of noncontextuality = revised notion of noncontextuality for projective measurements and outcome determinism for projective measurements
So, the new definition of noncontextuality is not simply a generalization of the traditional notion.

For sharp measurements, it is a revision of the traditional notion.
Local determinism:
We ask: Does the outcome depend on space-like separated events (in addition to local settings and $\lambda$)?

Local causality:
We ask: Does the probability of the outcome depend on space-like separated events (in addition to local settings and $\lambda$)?
Local determinism:
We ask: Does the outcome depend on space-like separated events (in addition to local settings and $\lambda$)?

Local causality:
We ask: Does the probability of the outcome depend on space-like separated events (in addition to local settings and $\lambda$)?

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Traditional notion of measurement noncontextuality:
We ask: Does the outcome depend on the measurement context (in addition to the observable and $\lambda$)?

The revised notion of measurement noncontextuality:
We ask: Does the probability of the outcome depend on the measurement context (in addition to the observable and $\lambda$)?
Local determinism:
We ask: Does the outcome depend on space-like separated events (in addition to local settings and λ)?

Local causality:
We ask: Does the probability of the outcome depend on space-like separated events (in addition to local settings and λ)?

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Traditional notion of measurement noncontextuality:
We ask: Does the outcome depend on the measurement context (in addition to the observable and λ)?

The revised notion of measurement noncontextuality:
We ask: Does the probability of the outcome depend on the measurement context (in addition to the observable and λ)?
traditional notion of noncontextuality = revised notion of noncontextuality for projective measurements and outcome determinism for projective measurements

No-go theorems for previous notion are not necessarily no-go theorems for the new notion!

In face of contradiction, could give up outcome determinism
However, one can prove that

[diagram: preparation noncontextuality → outcome determinism for projective measurements]
However, one can prove that

preparation noncontextuality $\rightarrow$ outcome determinism for projective measurements

Proof
We’ve established that

preparation noncontextuality → outcome determinism for sharp measurements

Therefore:

measurement noncontextuality and preparation noncontextuality

measurement noncontextuality and outcome determinism for sharp measurements
We’ve established that

preparation noncontextuality  →  outcome determinism for sharp measurements

Therefore:

measurement noncontextuality and preparation noncontextuality  →  Traditional notion of noncontextuality
We’ve established that

preparation noncontextuality \rightarrow \text{outcome determinism for sharp measurements}

Therefore:

measurement noncontextuality and preparation noncontextuality \rightarrow \text{Traditional notion of noncontextuality}

no-go theorems for the traditional notion of noncontextuality can be salvaged as no-go theorems for the generalized notion
Measurement-based proof of contextuality

(i.e. of the impossibility of a noncontextual realist model of quantum theory)
Proof of contextuality for unsharp measurements in 2d

\[ M_a \leftrightarrow \{ \Pi_a, \Pi_A \} \]
\[ M_b \leftrightarrow \{ \Pi_b, \Pi_B \} \]
\[ M_c \leftrightarrow \{ \Pi_c, \Pi_C \} \]

\( \Pi_x \) projects onto \( \psi_x \)

\[ \Pi_a + \Pi_A = I \]
\[ \Pi_b + \Pi_B = I \]
\[ \Pi_c + \Pi_C = I \]

\[ \Pi_a \Pi_A = 0 \]
\[ \Pi_b \Pi_B = 0 \]
\[ \Pi_c \Pi_C = 0 \]
Proof of contextuality for unsharp measurements in 2d

\[
\begin{align*}
M_a & \leftrightarrow \{\Pi_a, \Pi_A\} \\
M_b & \leftrightarrow \{\Pi_b, \Pi_B\} \\
M_c & \leftrightarrow \{\Pi_c, \Pi_C\} \\
\Pi_x \text{ projects onto } & \psi_x
\end{align*}
\]

\[
\begin{align*}
\Pi_a + \Pi_A &= I \\
\Pi_b + \Pi_B &= I \\
\Pi_c + \Pi_C &= I \\
\Pi_a \Pi_A &= 0 \\
\Pi_b \Pi_B &= 0 \\
\Pi_c \Pi_C &= 0
\end{align*}
\]

By definition

\[
\begin{align*}
\chi_a(\lambda) + \chi_A(\lambda) &= 1 \\
\chi_b(\lambda) + \chi_B(\lambda) &= 1 \\
\chi_c(\lambda) + \chi_C(\lambda) &= 1
\end{align*}
\]
Proof of contextuality for unsharp measurements in 2d

\[ M_a \leftrightarrow \{ \Pi_a, \Pi_A \} \]
\[ M_b \leftrightarrow \{ \Pi_b, \Pi_B \} \]
\[ M_c \leftrightarrow \{ \Pi_c, \Pi_C \} \]

\[ \Pi_x \text{ projects onto } \psi_x \]

\[ \Pi_a + \Pi_A = I \]
\[ \Pi_b + \Pi_B = I \]
\[ \Pi_c + \Pi_C = I \]

\[ \Pi_a \Pi_A = 0 \]
\[ \Pi_b \Pi_B = 0 \]
\[ \Pi_c \Pi_C = 0 \]

By definition

\[ \chi_a(\lambda) + \chi_A(\lambda) = 1 \]
\[ \chi_b(\lambda) + \chi_B(\lambda) = 1 \]
\[ \chi_c(\lambda) + \chi_C(\lambda) = 1 \]

By outcome determinism for sharp measurements

\[ \chi_a(\lambda)\chi_A(\lambda) = 0 \]
\[ \chi_b(\lambda)\chi_B(\lambda) = 0 \]
\[ \chi_c(\lambda)\chi_C(\lambda) = 0 \]

Thus, \{\psi(\lambda), \chi(\lambda)\}
M \equiv \text{implement one of } M_a, M_b \text{ and } M_c \text{ with prob. } 1/3 \text{ each, register only whether first or second outcome occurred}
M ≡ implement one of $M_a$, $M_b$ and $M_c$ with prob. 1/3 each, register only whether first or second outcome occurred

$$M \leftrightarrow \{\frac{1}{3} \Pi_a + \frac{1}{3} \Pi_b + \frac{1}{3} \Pi_c, \frac{1}{3} \Pi_A + \frac{1}{3} \Pi_B + \frac{1}{3} \Pi_C\}$$
M \equiv \text{implement one of } M_a, M_b \text{ and } M_c \text{ with prob. } 1/3 \text{ each, register only whether first or second outcome occurred}

M \leftrightarrow \left\{ \frac{1}{3} \Pi_a + \frac{1}{3} \Pi_b + \frac{1}{3} \Pi_c, \frac{1}{3} \Pi_A + \frac{1}{3} \Pi_B + \frac{1}{3} \Pi_C \right\}

M \leftrightarrow \left\{ \frac{1}{3} \chi_a(\lambda) + \frac{1}{3} \chi_b(\lambda) + \frac{1}{3} \chi_c(\lambda), \frac{1}{3} \chi_A(\lambda) + \frac{1}{3} \chi_B(\lambda) + \frac{1}{3} \chi_C(\lambda) \right\}
M ≡ implement one of $M_a$, $M_b$ and $M_c$ with prob. $1/3$ each, register only whether first or second outcome occurred

\[ M \leftrightarrow \{ \frac{1}{3} \Pi_a + \frac{1}{3} \Pi_b + \frac{1}{3} \Pi_c, \frac{1}{3} \Pi_A + \frac{1}{3} \Pi_B + \frac{1}{3} \Pi_C \} = \{ \frac{1}{2} I, \frac{1}{2} I \} \]

\[ M \leftrightarrow \{ \frac{1}{3} \chi_a(\lambda) + \frac{1}{3} \chi_b(\lambda) + \frac{1}{3} \chi_c(\lambda), \frac{1}{3} \chi_A(\lambda) + \frac{1}{3} \chi_B(\lambda) + \frac{1}{3} \chi_C(\lambda) \} \]

$\tilde{M}$ ≡ ignore the system, flip a fair coin

$\tilde{M} \leftrightarrow \{ \frac{1}{2} I, \frac{1}{2} I \}$
M $\equiv$ implement one of $M_a$, $M_b$ and $M_c$ with prob. 1/3 each, register only whether first or second outcome occurred

\[
M \leftrightarrow \left\{ \frac{1}{3} \Pi_a + \frac{1}{3} \Pi_b + \frac{1}{3} \Pi_c, \frac{1}{3} \Pi_A + \frac{1}{3} \Pi_B + \frac{1}{3} \Pi_C \right\} = \left\{ \frac{1}{2} I, \frac{1}{2} I \right\}
\]

\[
M \leftrightarrow \left\{ \frac{1}{3} \chi_a(\lambda) + \frac{1}{3} \chi_b(\lambda) + \frac{1}{3} \chi_c(\lambda), \frac{1}{3} \chi_A(\lambda) + \frac{1}{3} \chi_B(\lambda) + \frac{1}{3} \chi_C(\lambda) \right\}
\]

$\tilde{M}$ $\equiv$ ignore the system, flip a fair coin

\[
\tilde{M} \leftrightarrow \left\{ \frac{1}{2} I, \frac{1}{2} I \right\}
\]

$\tilde{M}$ $\leftrightarrow \left\{ \frac{1}{2}, \frac{1}{2} \right\}$

By the assumption of measurement noncontextuality

$M \simeq \tilde{M} \rightarrow \left\{ \frac{1}{3} \chi_a + \frac{1}{3} \chi_b + \frac{1}{3} \chi_c, \frac{1}{3} \chi_A + \frac{1}{3} \chi_B + \frac{1}{3} \chi_C \right\} = \left\{ \frac{1}{2}, \frac{1}{2} \right\}$
M ≡ implement one of \( M_a, M_b, \) and \( M_c \) with prob. 1/3 each, register only whether first or second outcome occurred

\[
M \leftrightarrow \left\{ \frac{1}{3} \eta_a + \frac{1}{3} \eta_b + \frac{1}{3} \eta_c, \frac{1}{3} \eta_A + \frac{1}{3} \eta_B + \frac{1}{3} \eta_C \right\} = \left\{ \frac{1}{2} I, \frac{1}{2} I \right\}
\]

\[
M \leftrightarrow \left\{ \frac{1}{3} \chi_a(\lambda) + \frac{1}{3} \chi_b(\lambda) + \frac{1}{3} \chi_c(\lambda), \frac{1}{3} \chi_A(\lambda) + \frac{1}{3} \chi_B(\lambda) + \frac{1}{3} \chi_C(\lambda) \right\}
\]

\( \tilde{M} \equiv \) ignore the system, flip a fair coin

\[
\tilde{M} \leftrightarrow \left\{ \frac{1}{2} I, \frac{1}{2} I \right\}
\]

\[
\tilde{M} \leftrightarrow \left\{ \frac{1}{2}, \frac{1}{2} \right\}
\]

By the assumption of measurement noncontextuality

\[
M \simeq \tilde{M} \rightarrow \left\{ \frac{1}{3} \chi_a + \frac{1}{3} \chi_b + \frac{1}{3} \chi_c, \frac{1}{3} \chi_A + \frac{1}{3} \chi_B + \frac{1}{3} \chi_C \right\} = \left\{ \frac{1}{2}, \frac{1}{2} \right\}
\]

But \( \{0, 1\}, \{\frac{1}{3}, \frac{2}{3}\}, \{1, 0\}, \{\frac{2}{3}, \frac{1}{3}\} \neq \left\{ \frac{1}{2}, \frac{1}{2} \right\} \)

**CONTRADICTION**
The mystery of contextuality

There is a tension between

1) the dependence of representation on certain details of the experimental procedure

and

2) the independence of outcome statistics on those details of the experimental procedure