Some examples

\[ 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 + \ldots \]
\[ 1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + \ldots \]
\[ 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + \ldots \]
\[ 1 + 0 - 1 + 1 + 0 - 1 + 1 + 0 - 1 + \ldots \]

Divergent series are not bad! They are useful, and they converge faster than convergent series!
\[ s = \sum_{n=1}^{\infty} \left( 1 + 2 + 4 + 8 + \ldots \right) \\
= 1 + \sum_{n=1}^{\infty} \left( 2^n + 4^n + 8^n + \ldots \right) \]
\[S = \sum (1 + 2 + 4 + 8 \ldots)\]
\[= 1 + \sum (1 + 4 + 8 + 16 \ldots)\]
\[S = 1 + \sum (1 + 4 + 8 + 16 \ldots)\]
$S = \sum (1 + 2 + 4 + 8 \ldots)
= 1 + \sum (2 + 4 + 8 \ldots)
S = 1 + 2S
S = -1
\mathbb{Z}$
\[ s = \sum_{n=1}^{\infty} (1 + 2 + 4 + 8 + \ldots) \\
= 1 + S(2 + 4 + 8 + 16 + \ldots) \\
S = 1 + 2S \\
S = 1 + 2S \\
S = -1 \]
\[
S = \sum_{n=0}^{\infty} (1 + 2 + 4 + 8 + \cdots) \\
= 1 + 2S \\
S = 1 + 2S \\
S = -1
\]
\[
\begin{align*}
S &= \sum_{n=1}^{\infty} (1 + 2 + 4 + 8 + \ldots) \\
    &= 1 + \sum_{n=1}^{\infty} (2^n + 4^n + 8^n + \ldots) \\
S &= 1 + 2S \\
S &= -1 \\
S &= 1 + 3S \\
S &= -1 \\
\end{align*}
\]
\[ S = \sum_{n=0}^{\infty} (1 + 2 + 4 + 8 \ldots) \]
\[ = 1 + \sum_{n=0}^{\infty} (2 + 4 + 8 + 16 \ldots) \]
\[ S = 1 + 2 \left( \sum_{n=0}^{\infty} (1 + 2 + 4 + 8 \ldots) \right) \]
\[ = \frac{S}{2} + \frac{S}{1} \Rightarrow S = 2S \Rightarrow S = 0 \]
\[ S = 1 + 2S \Rightarrow S = -1 \]
The blackboard shows the following equations:

\[ S = 1 + 2 + 4 + 8 + \ldots \]
\[ S = 1 + S(2 + 4 + 8 + \ldots) \]
\[ S = 1 + 2S \]
\[ S = \frac{1}{1 - 2} = -1 \]

Another set of equations:

\[ S = 1 + 1 + 1 + \ldots \]
\[ S = 1 + S(1 + 1 + \ldots) \]
\[ S = 1 + 2 \]
\[ S = \infty \]
Some divergent series REALLY sum up to infinity--- this is perfectly OK!

\[ 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots = \zeta(1) \]
\[
\begin{align*}
\zeta(2) &= \sum_{n=1}^{\infty} \frac{1}{n^2} \\
\zeta(2) &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6} \\
\zeta(4) &= 1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90} \\
\zeta(6) &= 1 + \frac{1}{2^6} + \frac{1}{3^6} + \cdots = \frac{\pi^6}{945} \\
\zeta(3) &= \\
\end{align*}
\]
\[ s(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} \]
\[ s(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6} \]
\[ s(4) = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90} \]
\[ s(6) = 1 + \frac{1}{2^6} + \frac{1}{3^6} + \cdots = \frac{\pi^6}{945} \]
\[ f(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} \]
\[ f(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6} \]
\[ f(4) = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90} \]
\[ f(6) = \cdots = \frac{\pi^6}{945} \]
\[ f(3) = \cdots \]
\[ s(2) = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots = \frac{\pi^2}{12} \]
\[ s(4) = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90} \]
\[ s(6) = 1 + \frac{1}{2^6} + \frac{1}{3^6} + \cdots = \frac{\pi^6}{945} \]
\[ s(3) = - \frac{1}{2} \]
\[ s(\infty) = - \frac{1}{2} \]
Techniques for summing series

- Note: addition is not infinitely commutative
- Euler summation
- Borel summation
- Generic summation methods
- Note: addition is not infinitely associative
- Zeta summation
- Continued functions
Zeta summation

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \]

\[ \zeta(1) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots = \infty \]

\[ \zeta(0) = 1 + 1 + 1 + 1 + 1 + \ldots = -\frac{1}{2} \]

\[ \zeta(-1) = 1 + 2 + 3 + 4 + 5 + 6 + \ldots = -\frac{1}{12} \]
Zeta summation

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$$\zeta(1) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... = \infty$$

$$\zeta(0) = 1 + 1 + 1 + 1 + 1... = -\frac{1}{2}$$

$$\zeta(-1) = 1 + 2 + 3 + 4 + 5 + 6... = -\frac{1}{12}$$
\[ S(2) = \frac{1}{1 - \frac{1}{4}} \]
\[ S(3) = 1 + \frac{1}{4} + \frac{1}{9} + \cdots = \frac{7}{4} \]
\[ S(4) = 1 + \frac{1}{4} + \frac{1}{9} + \cdots = \frac{11}{8} \]
\[ S(5) = \cdots \]
\[ f(x) = \frac{1}{1-x}, \quad x \neq 1 \]

\[
\begin{align*}
\sum_{n=0}^{\infty} x^n &= \frac{1}{1-x} \\
\sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{\pi^2}{6} \\
\sum_{n=1}^{\infty} \frac{1}{n^4} &= \frac{\pi^4}{90} \\
\sum_{n=1}^{\infty} \frac{1}{n^6} &= \frac{\pi^6}{945} \\
\sum_{n=1}^{\infty} \frac{1}{n^8} &= \frac{\pi^8}{93555}
\end{align*}
\]
\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \]

\[ \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad \Re x > 0 \]

\[ \Gamma(-1.3) \]

\[ f(x) = \frac{1}{1-x}, \quad x \neq 1 \]

\[ f(\infty) = \lim_{n \to \infty} x^n, \quad |x| < 1 \]

\[ \zeta(2) = \frac{\pi^2}{6} \]

\[ \zeta(2) = \frac{2}{\pi^2} \]

\[ \zeta(4) = \frac{1}{15} \]

\[ \zeta(6) = -\frac{1}{2} \]
\[ \Gamma(z) = (z-1)! \]

\[ \Gamma(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad x_n \times 2^0 \]

\[ \Gamma(-1.3) \]

\[ f(z) = \frac{1}{1-x} \quad x \neq 1 \]

\[ g(z) = \sum_{n=0}^{\infty} x^n \quad |x| < 1 \]

\[ f(2) = 2^2 + \frac{2}{2} + \frac{2}{2} + \cdots = \frac{2^2}{2} \]

\[ g(4) = 1 + \frac{1}{4} + \frac{1}{4} + \cdots = \frac{1}{2} \]

\[ g(6) = \cdots \]

\[ g(\infty) = -\frac{1}{2} \]
\[
\Gamma(n) = (n-1)!
\]
\[
\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt
\]
\[
f(x) = \begin{cases} \frac{1}{1-x} & x \neq 1 \\ \sum_{n=0}^{\infty} x^n & |x| < 1 \end{cases}
\]
\[ \Gamma(x) = \frac{\Gamma(x-1)}{x} \]

\[ \Gamma(x) = \int_0^\infty t^{x-1}e^{-t} \, dt \quad \text{Re} \, x > 0 \]

\[ f(t) = \frac{\Gamma(t+1)}{t} \]

\[ f(\infty) = \frac{1}{|x| < 1} \]

\[ f(\infty) = \frac{1}{|x| < 1} \]

\[ f(x) = \frac{1}{x} \quad x \neq 0 \]
\[ f(x) = \begin{cases} \frac{1}{1-x} & x \neq 1 \\ \sum_{n=0}^{\infty} x^n & |x| < 1 \end{cases} \]
\[ s = \sum_{n=0}^{\infty} \left( 1 + 2 + 4 + 8 \cdots \right) \]
\[ = 1 + s \left( 2 + 4 + 8 + 16 \cdots \right) \]
\[ = 1 + 2s \left( 1 + 2 + 4 + 8 \cdots \right) \]
\[ s = 1 + 2s \]
\[ s = -1 \]
\[
S = \sum_{n=1}^{\infty} (1 + 2 + 4 + 8 + \ldots) \\
= 1 + \sum_{n=1}^{\infty} (2 + 4 + 8 + 16 + \ldots) \\
= 1 + 2 \sum_{n=1}^{\infty} (1 + 2 + 4 + 8 + \ldots) \\
= 1 + 2S \\
S = 1 + 2S \\
S = -1
\]
\[ S = \sum_{n=0}^{\infty} (4^n + 8^n \cdot 2^n) \]
\[ = 1 + \sum_{n=0}^{\infty} (2^n + 4^n) \cdot 2^n \]
\[ S = 1 + 2S \]
\[ S = -1 \]
\[ S = \sum_{n=0}^{\infty} (1 + 2 + 4 + 8 \ldots) \]
\[ = 1 + \frac{S}{2} (1 + 2 + 4 + 8 \ldots) \]
\[ S = 1 + 2S \]
\[ S = -1 \]
\[ S = \sum_{n=0}^{\infty} \left( 1 + 2^n + 4^n + 8^n \ldots \right) \]
\[ = 1 + S \left( 2 + 4 + 8 + 16 + \ldots \right) \]
\[ S = 1 + 2S \sum_{n=0}^{\infty} \left( 1 + 2^n + 4^n + 8^n + \ldots \right) \]
\[ S = 1 + 2S \]
\[ S = -1 \]
\[
\begin{align*}
S &= S(1 + 2 + 4 + 8 + \ldots) \\
   &= 1 + S(2 + 4 + 8 + 16 + \ldots) \\
   &= 1 + S(1 + 2 + 4 + 8 + \ldots) \\
S &= 1 + 2S \\
S &= -1
\end{align*}
\]
\[ s = \sum_{n=0}^{\infty} (1 + 2^n + 4^n + \cdots) \]
\[ = 1 + \sum_{n=0}^{\infty} (2^n + 4^n + 8^n + \cdots) \]
\[ = 1 + 2 \left( \sum_{n=0}^{\infty} (1 + 2^n + 4^n + \cdots) \right) \]
\[ S = 1 + 2 \times S \]
\[ S = -1 \]
\[ s = \sum_{n=1}^{\infty} \left( \frac{1}{n} + \frac{1}{2n} + \frac{1}{3n} + \cdots \right) \]
\[ = 1 + \sum_{n=1}^{\infty} \left( \frac{1}{n} + \frac{1}{2n} + \frac{1}{3n} + \cdots \right) \]
\[ S = 1 + 2 \sum_{n=1}^{\infty} \left( \frac{1}{n} + \frac{1}{2n} + \cdots \right) \]
\[ S = \infty \]
\[ S = 1 + 2 \sum_{n=1}^{\infty} \left( \frac{1}{n} + \frac{1}{2n} + \cdots \right) \]
\[ S = \infty \]
\[ \frac{\partial V}{\partial x} = -F \]
Zeta summation

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$$\zeta(1) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots = \infty$$

$$\zeta(0) = 1 + 1 + 1 + 1 + 1 + \ldots = -\frac{1}{2}$$

$$\zeta(-1) = 1 + 2 + 3 + 4 + 5 + 6 + \ldots = -\frac{1}{12}$$
\[ H = p^2 + \frac{x^2}{4} + \frac{1}{4} \]
\[ E_x \]

\[
H = \frac{p^2}{2m} + \frac{x^2}{4} + \frac{1}{4} \frac{x^4}{4}
\]

\[ H \psi = E \psi \]

\[
\left( -\frac{d^2}{dx^2} + \frac{x^2}{4} + \frac{x^4}{4} \right) \psi(x) = E \psi(x)
\]

\[ \psi(x) \to 0 \quad \text{as} \quad x \to \pm \infty \]
$$\begin{align*}
E_x & \quad H = p^2 + \frac{x^2}{4} + \frac{x^4}{4} \\
\text{H4} & = E_4 \\
(-\frac{d^2}{dx^2} + \frac{x^2}{4} + \frac{x^4}{4}) \psi(x) = \epsilon \psi(x) \\
\psi(x) & \rightarrow \pm \infty \\
\text{HP} & \Rightarrow E P_0 + E P_1 + E P_2 \\
E(\epsilon) & = \sum q_n \epsilon_n
\end{align*}$$
\[ H = \frac{p^2}{2} + \frac{x^2}{4} + \frac{x^4}{4} \]

\[ H^4 = E_4 \]

\[ (-\frac{d^2}{dx^2} + \frac{x^2}{4} + \frac{x^4}{4}) \psi(x) = E \psi(x) \]

\[ \psi(x) \rightarrow 0 \text{ as } x \rightarrow \pm \infty \]

\[ E = E_0 + E_1 + E_2 \]

\[ E(x) = \sum q_n e^{in} \]

\[ a_n \sim C 3^n n! (-1)^{n+1} \]

\[ a_n \rightarrow \infty \text{ as } n \rightarrow \infty \]
\[ H = p^2 + \frac{x^2}{4} + \frac{x^4}{4} \]

\[ H \psi = \epsilon \psi \quad (\frac{-d^2}{dx^2} + \frac{x^2}{4} + \frac{x^4}{4}) \psi(x) = \epsilon \psi(x) \]

\[ \psi(x) \to 0 \quad x \to \pm \infty \]

\[ E \rightarrow E_0 + E_1 + E_2 \quad \cdots \]

\[ E(\epsilon) \sim \sum_{n=0}^{\infty} a_n \epsilon^n \quad a_n \sim C \frac{3^n n! (-1)^{n+1}}{n!} \]

\[ a_n \sim n \to \infty \]
\[ H = \frac{x^2}{4} + \frac{x^4}{4} \]

\[ (-\frac{d^2}{dx^2} + \frac{x^2}{4} + \frac{x^4}{4}) \psi(x) = \epsilon \psi(x) \]

\[ \psi(x) \rightarrow 0 \quad \text{as} \quad x \rightarrow \pm \infty \]

\[ \text{HP} \Rightarrow E P_0 + E P_1 + E P_2 \quad \cdots \cdots \]

\[ E(\epsilon) \sim \sum_{n=0}^{\infty} \alpha_n e^n \]

\[ \alpha_n \sim C \cdot 3^n n! (-1)^{n+1} \quad \text{as} \quad n \rightarrow \infty \]

\[ S(\epsilon) \rightarrow \text{Ans. i.e. } E(\epsilon) \]
What if we don’t know all the terms in the series??

WE USE CONTINUED FUNCTIONS!
$$H = p^2 + \frac{x^2}{4} + \frac{x^4}{4}$$

$$\psi(x) \to 0 \quad \text{as} \quad x \to \pm \infty$$

$$E(\epsilon) \sim \sum_{n=0}^{\infty} a_n \epsilon^n$$

$$a_n \sim C \frac{\epsilon^n}{n!}$$

$$a_n \to \infty \quad \text{as} \quad n \to \infty$$

Ans. i.e. $E(\epsilon)$
\[ f(x) = \sum_{n=0}^{\infty} a_n x^n \quad |x| < R \]
\[ f(x) = \sum_{n=0}^{\infty} a_n x^n \quad |x| < R \]

\[ = b_0 e^{b_1 x} e^{b_2 x} e^{b_3 x} e^{b_4 x} \]
\[ f(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n} x_k \quad |x_k| < R \]
\[ f(x) = \sum_{n=0}^{\infty} \frac{1}{a_0 + n} \left| x \right| < R \]

\[ = b_0 e^{b_1 x} e^{b_2 x} \]

\[ a_0 \rightarrow b_0, \quad a_1 \rightarrow b_1, \quad a_2 \rightarrow b_2 \]
Continued exponentials

\[ a_0 e^{a_1 z} e^{a_2 z} \ldots \]

\[ \sum_{n=0}^{\infty} c_n z^n \]

\[ c_0 = a_0, \]
\[ c_1 = a_1 a_0, \]
\[ c_2 = a_0 a_1 a_2 + \frac{1}{2} a_0 a_1^2, \]
\[ c_3 = a_0 a_1 a_2 a_3 + \frac{1}{2} a_0 a_1 a_2^2 + a_0 a_1^2 a_2 + \frac{1}{6} a_0 a_1^3 \]
\[(n-1)!\]
\[\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}\]
\[\text{Re} x > 0\]
\[\text{Cavell}\]
\[x \to x^2\]
\[\text{exists}\]
\[\frac{1}{n!} \sim \frac{n^n}{n^n e^n} \sim e\]
\[f(\infty) = \frac{1}{1-x}\]
\[f(\infty) = \frac{2}{1-x}\]
\[f(\infty) = \frac{3}{1-x}\]
\[\lim_{n \to \infty} \frac{n!}{n^n} = \frac{1}{e}\]
\[f(-2) = ?\]
\[ f(x) = \lim_{n \to \infty} \frac{1}{a_0 x^n} \quad |x| < R \]

\[ = b_0 e^{b_1 x} e^{b_2 x} e^{b_3 x} \ldots \]

\[ a_0 \rightarrow b_0 \]
\[ a_1 \rightarrow b_1 \]
\[ a_2 \rightarrow b_2 \]

\[ b_n = 1 \]

\[ \sum_{n=0}^{\infty} \frac{(n-1)}{n!} x^n \]

**Converges if** \(|x| < \frac{1}{e}\)**.*)
\[ f(x) = \lim_{n \to \infty} \frac{1}{a_n} x^n \quad |x| < R \]

\[ = b_0 + b_1 x + b_2 x^2 + \ldots \]

\[ a_0 \to b_0 \]
\[ a_1 \to b_1 \]
\[ a_2 \to b_2 \]

\[ = \sum_{n=0}^{\infty} \frac{(n+1)!}{n!} x^n \]

converges if \(|x| < \frac{1}{e}\)
Example

\[ e \approx \sum_{n=0}^{\infty} \frac{(n + 1)^{n-1}}{n!} z^n \]
Region of convergence