Title: Entanglement Generation in Relativistic Quantum Fields

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Abstract: We present a general, analytic recipe to compute the entanglement that is generated between arbitrary, discrete modes of bosonic quantum fields by Bogoliubov transformations. Our setup allows the complete characterization of the quantum correlations in all Gaussian field states. Additionally, it holds for all Bogoliubov transformations. These are commonly applied in quantum optics for the description of squeezing operations, relate the modedecompositions of observers in different regions of curved spacetimes, and describe observers moving along non-stationary trajectories. We focus on a quantum optical example in a cavity quantum electrodynamics setting: an uncharged scalar field within a cavity provides a model for an optical resonator, in which entanglement is created by non-uniform acceleration. We show that the amount of generated entanglement can be magnified by initial single-mode squeezing, for which we provide an explicit formula. Applications to quantum fields in curved spacetimes, such as an expanding universe, are discussed.
Entanglement generation in relativistic quantum fields

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work in collaboration with

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Once upon a time in a Hilbert space far far away ...

- Bogoliubov transformations and entanglement

- Comparison of procedures for Fock states and Gaussian states

- Non-uniform cavity motion
Once upon a time in a Hilbert space far far away . . .

- Bogoliubov transformations and entanglement
- Comparison of procedures for Fock states and Gaussian states
- Non-uniform cavity motion:
  - construction of generic trajectories
  - entanglement generation within the cavity
(real, scalar) Bosonic quantum field $\phi$ with discrete spectrum:

$$\phi = \sum_n \left( a_n \phi_n + a_n^\dagger \phi_n^* \right)$$
Bogoliubov Transformations

(real, scalar) Bosonic quantum field $\phi$ with discrete spectrum:

$$\phi = \sum_n \left( a_n \phi_n + a_n^\dagger \phi_n^* \right)$$

Bogoliubov transformation: “In-Region” $\iff$ “Out-Region”

mode functions:

$$\phi_n = \sum_m \left( \alpha_{mn}^* \tilde{\phi}_m - \beta_{mn} \tilde{\phi}_m^* \right)$$

operators:

$$n^\dagger_m = \sum_m \left( n_{mm}^\dagger n_{mm}^0 - n_{mm} n_{mm}^0 \right)$$

coefficients:

$$\alpha_{mn} = \text{in-region} \quad \beta_{mn} = \text{out-region}$$
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coefficients:

$$\alpha_{mn} = (\tilde{\phi}_m, \phi_n), \quad \beta_{mn} = -(\tilde{\phi}_m, \phi_n^*)$$
Bogoliubov Transformations

(real, scalar) Bosonic quantum field $\phi$ with discrete spectrum:

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Studying Entanglement in Fock space

Transformation of Fock States

\[ |0\rangle \propto e^{W} |\tilde{0}\rangle \quad \text{with} \quad W = \frac{1}{2} \sum_{p,q} V_{pq} \tilde{a}^\dagger_p \tilde{a}^\dagger_q \]

act with creation operators

\[ a_n^\dagger = \sum_m \left( \alpha_{mn}^* \tilde{a}_m^\dagger + \beta_{mn} \tilde{a}_m \right) \quad \text{on} \quad |0\rangle \]

- Transform to out-region
Studying Entanglement in Fock space

Transformation of Fock States

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Usual Procedure

- Select in-region state
- Transform to out-region
- **Trace** out inaccessible modes
- Study entanglement properties
Gaussian States - covariance matrix formalism

\[ \sigma_{ij} = \langle X_i X_j + X_j X_i \rangle - 2 \langle X_i \rangle \langle X_j \rangle \]

quadratures: \[ X_{(2n-1)} = \frac{1}{\sqrt{2}} (a_n + a_n^\dagger) \quad \text{and} \quad X_{(2n)} = \frac{-i}{\sqrt{2}} (a_n - a_n^\dagger) \]
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Transforming initial state

\[ \sigma = \begin{pmatrix} C_{11} & C_{12} & C_{13} & \ldots \\ C_{21} & C_{22} & C_{23} & \ldots \\ C_{31} & C_{32} & C_{33} & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \Rightarrow \tilde{\sigma} = \begin{pmatrix} \tilde{C}_{11} & \tilde{C}_{12} & \tilde{C}_{13} & \ldots \\ \tilde{C}_{21} & \tilde{C}_{22} & \tilde{C}_{23} & \ldots \\ \tilde{C}_{31} & \tilde{C}_{32} & \tilde{C}_{33} & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \]
Tracing & Entanglement

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Tracing out inaccessible modes, e.g., all but 2 & 3

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where \( \tilde{C}_{mn} = \sum_{i,j} M_{mi} C_{ij} M_{nj}^T \)

Tracing & Entanglement

Tracing out inaccessible modes, e.g., all but 2 & 3

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where \( \tilde{C}_{mn} = \sum_{i,j} M_{mi} C_{ij} M_{nj}^T \)

Compute entanglement measure of choice

e.g., (logarithmic) Negativity \( \mathcal{N} \), Gaussian Contangle, Entropy of Entanglement, etc.

Inertial cavity of width $\delta = b - a$

Uncharged scalar field $\phi$

solutions: $\{\phi_n\}: \quad \phi = \sum_n (\phi_n a_n + \phi^*_n a_n^\dagger), \quad \{a_m, a_n^\dagger\} = \delta_{mn}$
Quantum fields in accelerated cavities
Quantum fields in accelerated cavities

Repeat quantisation procedure in Rindler coordinates

solutions: \( \{ \phi_n \} : \quad \phi = \sum_n (\phi_n \tilde{a}_n + \phi_n^* \tilde{a}_n^\dagger) \), \[ [\tilde{a}_m, \tilde{a}_n^\dagger] = \delta_{mn} \]
Quantum fields in accelerated cavities

Finite duration of acceleration:
acceleration stops at fixed coordinate time $\eta = \eta_1$
"In-Region": separable state of modes $k, k'$ in single cavity

Bogoliubov coefficients: perturbative in $h := \frac{2b-a}{b+a}$

$$\alpha_{mn} = \alpha_{mn}^{(0)} + \alpha_{mn}^{(1)} + O(h^2), \quad \beta_{mn} = \beta_{mn}^{(1)} + O(h^2)$$

Vacuum: $\Lambda_{nm}^{(1)} = \frac{\beta_{nm}^{(1)}}{\beta_{nm}^{(1)}}$.

"Out-Region": Trace out all other modes:

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Entanglement generation in relativistic quantum fields
Results - Generation of Entanglement

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$$\alpha_{mn} = \alpha_{mn}^{(0)} + \alpha_{mn}^{(1)} + O(h^2), \quad \beta_{mn} = \beta_{mn}^{(1)} + O(h^2)$$

Vacuum: $\mathcal{N}_{vac} = |\beta_{kk'}^{(1)}|$

Symmetric single-mode squeezing:

“Out-Region”: Trace out all other modes: entanglement creation

Results - Generation of Entanglement

“In-Region”: separable state of modes $k, k'$ in single cavity

Bogoliubov coefficients: perturbative in $h := \frac{2b-a}{b+a}$

$$\alpha_{mn} = \alpha_{mn}^{(0)} + \alpha_{mn}^{(1)} + O(h^2), \quad \beta_{mn} = \beta_{mn}^{(1)} + O(h^2)$$

Vacuum: $\mathcal{N}_{vac} = |\beta_{kk'}^{(1)}|$

Symmetric single-mode squeezing:

$$\mathcal{N}_s = \sqrt{\text{Re}(G_{kk'}^{*}\beta_{kk'}^{(1)})^2 + [\text{Im}(G_{kk'}^{*}\beta_{kk'}^{(1)}) \cosh(s) - \text{Im}(G_{kk'}^{*}\alpha_{kk'}^{(1)}) \sinh(s)]^2}$$

“Out-Region”: Trace out all other modes: entanglement creation

Results - Generation of Entanglement

“In-Region”: separable state of modes $k, k'$ in single cavity

\[ \mathcal{N} / \hbar \]

- $\mathcal{N}_{s=1}$
- $\mathcal{N}_{\text{vac}}$

“Out-Region”: Trace out all other modes: entanglement creation


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Entanglement generation in relativistic quantum fields
Summary & Conclusion

Covariance matrix formalism...

- useful for quantum fields with *discrete spectrum*
- results can be obtained *analytically* for given $\alpha$'s & $\beta$'s
- entanglement completely determined for Gaussian states