A covariant perspective on singularity resolution

Francesca Vidotto

(with Carlo Rovelli, arXiv:1307.3228)

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PLAN OF THE TALK

- **Singularity resolution: a primer**
  Examples of non-singular cosmological models and their pathologies.
  Loop Quantum Cosmology: singularities are resolved by quantum effects.

- **Spinfoams: the covariant formulation of LQG**
  General philosophy and application to cosmology.
  Definition of the dynamics.

- **Maximal acceleration: consequences**
  Singularity resolution: discussion.
  Comments on Lorentz invariance in LQG.
NON-SINGULAR MODELS

- **Singularity resolution: how to**
  - Violating strong energy condition
  - Changing the way gravity couples to matter
  - Using a non-perfect fluid as a source
  - Adding scalar degrees of freedom to gravity
  - Writing a new action with higher-order invariants
  - ...

- **Instead, in Loop Quantum Cosmology singularities are removed by quantum effects.**
  Elements of the full theory, coding the relevant quantum effects, are imported into the study of symmetry-reduced models.

- **SU(2) group variables**
  - Minimal area gap
  - **Hamiltonian constraint**
  - Holonomy corrections
    \[
    \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3} \rho \left(1 - \frac{\rho}{\rho_c}\right)
    \]
$P(\psi) = |\langle W | \psi \rangle|^2$

- **Spinfoams theory is about**
  **TRANSITION AMPLITUDES**

In quantum gravity region of spacetime is a process. Amplitude = probability of a spacetime region to exist given a boundary (connected or disconnected).
This defines the dynamics.

- **EXTRACTING COSMOLOGY FROM THE FULL QUANTUM THEORY requires to formulate the questions appropriately**

  “Is there a bounce in Spinfoam Cosmology?”
2-complex $C$
(verticies, edges, faces)

$h_f = \prod_v h_{vf}$

**SUPERPOSITION PRINCIPLE**
Graph expansion

$W_C(h_i) = \int_{SU(2)} dh_v \prod_f \delta(h_f) \prod_v A(h_{vf})$

**LOCALITY**
Vertex expansion

$A(h_f) = \sum_{j_f} \int_{SL(2,\mathbb{C})} dg_e \prod_f (2j_f + 1) \text{Tr}_j [h_f Y^{-1} Y_j g_e g_e^{-1} Y]$}

**LORENTZ INVARIANCE**
Simplicity map

$Y_\gamma : H_j \rightarrow H_{j;\gamma j}$

$|j; m\rangle \mapsto |j, \gamma(j + 1); j, m\rangle$
A REMINDER OF THE CLASSICAL THEORY

- **Variables**
  \[ e = e_a dx^a \in \mathbb{R}^{(1,3)} \quad \text{and} \quad \omega = \omega_a dx^a \in sl(2, \mathbb{C}) \]

- **Action**
  \[ S[e, \omega] = \int B[e] \wedge F[\omega] \quad \text{where} \quad B = (e \wedge e)^* + \frac{1}{\gamma} (e \wedge e) \]

- **Boundary**
  gauge s.t. tetrads are diagonal
  \[ B^{0i} = K^i = \frac{1}{\gamma} e^o \wedge e^i \quad \text{and} \quad B^{ij} = L^i = e^o \wedge e^i \]

- **Simplicity constraint**
  \[ \vec{K} + \gamma \vec{L} = 0 \]

- **Lorentzian area**
  \[ A = \int_\mathcal{R} e^o \wedge e^i = \int_\mathcal{R} \gamma K^i = \int_\mathcal{R} L^i \]

\[ SL(2, \mathbb{C}) \rightarrow SU(2) \]
A REMINDER OF THE QUANTUM THEORY

Unitary irreducible reps of $SU(2)$ $|j; m\rangle \in \mathcal{H}_j$ and $SL(2, \mathbb{C})$ $|k, \nu; j, m\rangle \in \mathcal{H}_{k, \nu} = \bigoplus_{j=k,\infty} \mathcal{H}_{j, \nu}^j$

- $\gamma$-simple representations: $\nu = \gamma(k + 1)$

- $SU(2) \to SL(2, \mathbb{C})$ map: $Y_\gamma : \mathcal{H}_j \to \mathcal{H}_{j, \gamma j}$ with image s.t. $j = k$

  $|j; m\rangle \mapsto |(j, \gamma(j + 1)); j, m\rangle$

- Simplicity constraint $\vec{K} + \gamma \vec{L} = 0$ satisfied weakly on the image of $Y_\gamma$

  Boost generator \hspace{1cm} Rotation generator

- $L^i$ is the area operator: the Lorentzian area $A = \int_{\mathcal{R}} L^i$ has a minimal value!

  but $A = \int_{\mathcal{R}} \gamma K^i$ implies it exists a maximal acceleration!
LORENTZIAN AREA

\[ A = \int_R \gamma K^i = \int_R L^i \]

\[ \ell = 1/a \]

\[ A = \frac{\ell^2}{2} \eta = \frac{1}{2a^2} \eta \]

\( \eta \) is the boost parameter along the trajectory from P to P'

- Lorentzian area \( A_{min} = 4\pi G\hbar \)
- Max acceleration \( a_{max} = \sqrt{\frac{1}{8\pi G\hbar}} \)
- Min length \( \ell_{max} = \sqrt{8\pi G\hbar} \)
WEDGE AMPLITUDE

- motion of an accelerated observer in spacetime
- evolution of spacetime seen by an observer

- $R = \text{wedge}$

$$W(g, g', h) = \sum_j (2j + 1) \text{Tr}_j [Y^\dagger g' g^{-1} Y h]$$

- $g' g^{-1}$ boost between $P$ and $P'$ in the time gauge:

$$W(\eta, h) = \sum_j (2j + 1) \text{Tr}_j [Y^\dagger e^{i\eta K} Y h]$$

- Fourier transform:

$$W(\eta, j, m, m') = \langle j, m | Y^\dagger e^{i\eta K} Y | j, m' \rangle \quad m = m' = j \quad W(\eta, j) = \langle j, j | Y^\dagger e^{i\eta K} Y | j, j \rangle$$

[Bianchi '12]

Spinfoam & Cosmology

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COSMOLOGICAL SINGULARITIES

- Minimal distance from the horizon: \( \ell = R / \dot{R} \)
- Maximal acceleration: \( a \sim \sqrt{\frac{\dot{R}}{R}} \)
- Maximal energy density: \( \rho_{\text{max}} \sim \frac{3}{8\pi G} \frac{\dot{R}^2}{R^2} \bigg|_{\text{max}} = \frac{3}{8\pi G} \ell_{\text{min}}^{-2} = \frac{3}{\hbar(8\pi G)^2} \)

SPINFOAM: singularity are avoided!

- Minimal volume - classically the conjugate variable is the Hubble rate
- LQC: holonomy corrections \( \rightarrow \) bounded Hubble rate! effective eqs: \( \ell_P \dot{R}/R \rightarrow \sin (\ell_P \dot{R}/R) \)

- Strong singularity are solved: big bang, big crunch, big rip...
- Maximal acceleration: it may have implications also for weak-singularity resolution.
- It results from the full theory, not in an approximation.
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- **SPINFOAM**: singularity are avoided!

- Minimal volume - classically the conjugate variable is the Hubble rate
- **LQC**: holonomy corrections \( \rightarrow \) bounded Hubble rate! effective eqs: \( \ell_P \dot{R} / R \rightarrow \sin (\ell_P \dot{R} / R) \)

- Strong singularity are solved: big bang, big crunch, big rip...
- Maximal acceleration: it may have implications also for weak-singularity resolution.
- It results from the full theory, not in an approximation.
- **Loop Quantum Gravity** predicts the existence of a minimal length and the existence of a maximal acceleration.

- The presence of a scale is not incompatible with Lorenz invariance: Lorentz transformations change **probabilities** but not **spectra**.
**SUMMARY**

- *Singularity resolution*
  Many models remove singularities, only in LQG this is achieved because quantum effects.

- *Spinfoam theory is useful in cosmology!*
  Appearance of a maximal acceleration: no singularities.

- *Lorentz invariance*
  Compatible with existence of a fundamental scale (min length, max acceleration).

Singularity resolution is tied to the existence of a discrete spectrum, but it is the spectrum of a spacetime, not a spacial quantity.
LORENTZIAN AREA

\[ A = \int_R \gamma K^i = \int_R L^i \]

\[ \ell = 1/a \]

\[ A = \frac{\ell^2}{2} \eta = \frac{1}{2a^2} \eta \]

- Lorentzian area \[ A_{\text{min}} = 4\pi G\hbar \]
- Max acceleration \[ a_{\text{max}} = \sqrt{\frac{1}{8\pi G\hbar}} \]
- Min length \[ \ell_{\text{max}} = \sqrt{8\pi G\hbar} \]
Qualitative Effective Dynamics in Bianchi IX Loop Quantum Cosmology

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Introduction
Phase space variables

We use the fiducial triads and co-triads to introduce a convenient parametrization of the phase space variables, $E_i^a, A_a^i$ given by

$$E_i^a = p_i L_i V_0^{-1} \sqrt{\text{det} q} \ e_i^a \quad \text{and} \quad A_a^i = c^i L_i^{-1} \omega_a^i.$$  \hspace{1cm} (4)

Thus, a point in the phase space is now coordinatized by eight real numbers $(p_i, c^i, \phi, p_\phi)$ with Poisson brackets given by

$$\{c^j, p_i\} = 8\pi G \delta_j^i \quad \{\phi, p_\phi\} = 1.$$ \hspace{1cm} (5)

The relation between the new and the old variables is

$$p_i = a_j a_k L_j L_k,$$

with $i \neq j \neq k \neq i$, $L_i$ fiducial lengths, $V_0 = L_1 L_2 L_3$. 

Introduction
Bianchi Metrics

We analyzed the numerical solutions of the effective equations that come from the improved LQC dynamics of the Bianchi I, II and IX models. We choose a massless scalar field as the matter source.

- **Bianchi I**,

\[
\text{ds}^2 = -N^2 \text{d}t^2 + a_1^2(t) \text{d}x^2 + a_2^2(t) \text{d}y^2 + a_3^2(t) \text{d}z^2
\]  

(1)

- **Bianchi II**,  

\[
\text{ds}^2 = -N^2 \text{d}t^2 + a_1^2(t) (\text{d}x - \alpha \text{d}y)^2 + a_2^2(t) \text{d}y^2 + a_3^2(t) \text{d}z^2
\]  

(2)

where \(\alpha\) is a switch (Bianchi I \(\alpha = 0\), Bianchi II \(\alpha = 1\)).

- **Bianchi IX** metric can be construct from the fiducial co-triads  

\[
\begin{align*}
\omega_a^1 &= \sin \beta \sin \gamma (\text{d}\alpha)_a + \cos \gamma (\text{d}\beta)_a, \\
\omega_a^2 &= -\sin \beta \cos \gamma (\text{d}\alpha)_a + \sin \gamma (\text{d}\beta)_a, \\
\omega_a^3 &= \cos \beta (\text{d}\alpha)_a + (\text{d}\gamma)_a.
\end{align*}
\]  

(3)

with physical co-triads \(\omega_a^i = a_i(t) \omega_a^i\) and 3-metric \(q_{ab} := \omega_a^i \omega_b^i\). The spacetime metric is \(g_{\mu\nu} = -n_\mu n_\nu + q_{\mu\nu}\).
We use the fiducial triads and co-triads to introduce a convenient parametrization of the phase space variables, $E^a_i$, $A^i_a$ given by

$$E^a_i = p_i L_i V_0^{-1} \sqrt{q} |q|^i e^a_i$$ and $$A^i_a = c^i L_i^{-1} \omega^i_a.$$ (4)

Thus, a point in the phase space is now coordinatized by eight real numbers $(p_i, c^i, \phi, \rho_\phi)$ with Poisson brackets given by

$$\{c^j, p_j\} = 8\pi G\gamma \delta^j_i$$ $$\{\phi, \rho_\phi\} = 1.$$ (5)

The relation between the new and the old variables is

$$p_i = a_j a_k L_j L_k.$$ 

with $i \neq j \neq k \neq i$, $L_i$ fiducial lengths, $V_0 = L_1 L_2 L_3$. 

Introduction

Constraints

The choice of the physical triads and connections has fixed the Gauss and vector constraint and only remain the Hamiltonian constraint

$$C_H = \int_V \left[ \frac{NE^a_i E^b_j}{16\pi G\sqrt{|q|}} \left( \epsilon^{ij}_k F^{ab}_k - 2(1 + \gamma^2)K^i_{[a} K^j_{b]} \right) + N\mathcal{H}_{\text{matt}} \right] d^3x$$

(6)

with

$$F^{ab}_k = 2\partial_{[a} A^k_{b]} + \epsilon^{ij}_k A^i_a A^j_b$$

and

$$\mathcal{H}_{\text{matt}} = \frac{p^2_0}{2\sqrt{|q|}}.$$

- The curvature is calculated from the connection and not from the holonomies, $F^{ab}_k = 2\partial_{[a} A^k_{b]} + \epsilon^{ij}_k A^i_a A^j_b$.
- The connection is calculated from the holonomies

$$A^k_a = \lim_{\mu_k \to 2\bar{\mu}_k} \sum_k \frac{1}{2L_k L_k} \left( h^{(k)}_k - (h^{(k)}_k)^{-1} \right) \Rightarrow A^k_a = \frac{\sin(\bar{\mu}_k c_k)}{\bar{\mu}_k L_k} \omega^k_a$$

with

$$\bar{\mu}_1 = \lambda \sqrt{\frac{p_1}{p_2p_3}}, \quad \bar{\mu}_2 = \lambda \sqrt{\frac{p_2}{p_1p_3}}, \quad \bar{\mu}_3 = \lambda \sqrt{\frac{p_3}{p_1p_2}}.$$

The value of $\lambda$ is chosen such that $\lambda^2 = 4\sqrt{3\pi}(\gamma p_1^2)$. 
Introduction

Constraints

The choice of the physical triads and connections has fixed the Gauss and vector constraint and only remain the Hamiltonian constraint

\[ C_H = \int_V \left[ \frac{NE_i^a E_j^b}{16\pi G|q|} \left( \epsilon^{ij}_k F_{ab}^k - 2(1 + \gamma^2) K_{[a}^i K_{b]}^j \right) + N H_{\text{matt}} \right] d^3 x \]  

(6)

with \( F_{ab}^k = 2 \partial_{[a} A_{b]}^k + \epsilon_{ij}^k A_a^i A_b^j \) and \( H_{\text{matt}} = \frac{p_3^2}{2\sqrt{|q|}} \).

- The curvature is calculated from the connection and not from the holonomies, \( F_{ab}^k = 2 \partial_{[a} A_{b]}^k + \epsilon_{ij}^k A_a^i A_b^j \).
- The connection is calculated from the holonomies

\[
A_a = \lim_{l_k \to 2\mu_k} \sum_k \frac{1}{2l_k L_k} \left( h^{(k)}_k - (h^{(k)}_k)^{-1} \right) \quad \Rightarrow \quad A_a^k = \frac{\sin(\mu_k c_k)}{\mu_k L_k} \omega_a^k
\]

with

\[
\mu_1 = \lambda \sqrt{\frac{p_1}{p_2 p_3}}, \quad \mu_2 = \lambda \sqrt{\frac{p_2}{p_1 p_3}}, \quad \mu_3 = \lambda \sqrt{\frac{p_3}{p_1 p_2}}.
\]

The value of \( \lambda \) is chosen such that \( \lambda^2 = 4\sqrt{3\pi \gamma P_1} \).
Introduction
Effective constraint

\[ \mathcal{H}_{BII} = \frac{p_1 p_2 p_3}{8\pi G \gamma^2 \lambda^2} \left[ \sin \bar{\mu}_1 c_1 \sin \bar{\mu}_2 c_2 + \sin \bar{\mu}_2 c_2 \sin \bar{\mu}_3 c_3 + \sin \bar{\mu}_3 c_3 \sin \bar{\mu}_1 c_1 \right] \]
\[ + \frac{1}{8\pi G \gamma^2} \left[ \frac{\alpha (p_2 p_3)^{3/2}}{\lambda \sqrt{p_1}} \sin \bar{\mu}_1 c_1 - (1 + \gamma^2) \left( \frac{\alpha p_2 p_3}{2p_1} \right)^2 \right] - \frac{p_\phi^2}{2} \approx 0 \]

\[ \mathcal{H}_{BIX} = -\frac{p_1 p_2 p_3}{8\pi G \gamma^2 \lambda^2} \left[ \sin \bar{\mu}_1 c_1 \sin \bar{\mu}_2 c_2 + \sin \bar{\mu}_2 c_2 \sin \bar{\mu}_3 c_3 + \sin \bar{\mu}_3 c_3 \sin \bar{\mu}_1 c_1 \right] \]
\[ - \frac{\text{i}}{8\pi G \gamma^2 \lambda} \left( \frac{(p_1 p_2)^{3/2}}{\sqrt{p_3}} \sin \bar{\mu}_3 c_3 + \frac{(p_2 p_3)^{3/2}}{\sqrt{p_1}} \sin \bar{\mu}_1 c_1 + \frac{(p_3 p_1)^{3/2}}{\sqrt{p_2}} \sin \bar{\mu}_2 c_2 \right) \]
\[ - \frac{\text{i}^2 (1 + \gamma^2)}{32 \pi G \gamma^2} \left[ 2(p_1^2 + p_2^2 + p_3^2) - \left( \frac{p_1 p_2}{p_3} \right)^2 - \left( \frac{p_2 p_3}{p_1} \right)^2 - \left( \frac{p_3 p_1}{p_2} \right)^2 \right] \]
\[ + \frac{p_\phi^2}{2} \approx 0 \]

with lapse \( N = \sqrt{p_1 p_2 p_3} \).
Introduction
Effective Bianchi IX with Inverse Triad Corrections

\[ \mathcal{H}^{(2)}_{BIX} = \frac{V^4 A(V) h^6(V)}{8\pi GV_c^2 \gamma^2 \lambda^2} \left( \sin \bar{\mu}_1 c_1 \sin \bar{\mu}_2 c_2 + \sin \bar{\mu}_1 c_1 \sin \bar{\mu}_3 c_3 + \sin \bar{\mu}_2 c_2 \sin \bar{\mu}_3 c_3 \right) \]

\[ - \frac{\partial A(V) h^4(V)}{4\pi GV_c^4 \gamma^2 \lambda} \left( p_1^2 p_2^2 \sin \bar{\mu}_3 c_3 + p_2^2 p_3^2 \sin \bar{\mu}_1 c_1 + p_1^2 p_3^2 \sin \bar{\mu}_2 c_2 \right) \]

\[ - \frac{\alpha^2 (1 + \gamma^2) A(V) h^4(V)}{8\pi GV_c^4 \gamma^2} \times \]

\[ \left( 2V[p_1^2 + p_2^2 + p_3^2] - \left[ (p_1 p_2)^4 + (p_1 p_3)^4 + (p_2 p_3)^4 \right] \frac{h^6(V)}{V_c^6} \right) \]

\[ + \frac{h^6(V) V^2}{2V_c^6} p_\phi^2 \approx 0 \]

with \( \lambda^2 = 4\sqrt{3\pi} \gamma \ell^2, \ V_c = 2\pi \gamma \ell^2, \ \alpha = (2\pi^2)^{1/3} \), lapse \( N = 1 \) and

\[ h(V) = \sqrt{V + V_c} - \sqrt{|V - V_c|} \]

\[ A(V) = \frac{1}{2V_c}(V + V_c - |V - V_c|) \]
Big Volume Limit
Scale Factors with and without Inverse Triad Corrections
Isotropic Limit

Flat FRW ($k = 0$) Limit

Density

Time
Bianchi I Limit
Kasner Exponents

Kasner Exponents

+ k₁ + k₂ + k₃
k₁
k₂
k₃

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Vacuum Bianchi IX
Maximal Density without Inverse Triad Corrections
Vacuum Bianchi IX
Maximal Density with Inverse Triad Corrections
Bianchi IX
Classical Limit

The universe can be seen as a particle moving in a potential that presents reflections at exponential walls. The classical potential in the classical constraint, in terms of Misner variables is

\[ W = \frac{1}{2} e^{-4\Omega} \left( e^{-4\beta_+} - 4 e^{-\beta_+} \cosh \sqrt{3}\beta_- + 2 e^{-2\beta_+} [\cosh 2\sqrt{3}\beta_- - 1] \right), \quad (7) \]

where \( \Omega = -\frac{1}{3} \log V \) and the anisotropies \( \beta_\pm \) are defined via

\[ a_1 = e^{-(\beta_+ + \sqrt{3}\beta_-)/2}, \quad a_2 = e^{-(\beta_+ - \sqrt{3}\beta_-)/2}, \quad a_3 = e^{-\Omega - \beta_+} \quad (8) \]

The modified potential as a function of \( p_i \) is

\[ W_{\text{eff}} = -\frac{V^2 A(V) h^4(V)}{V_c^4} \left( p_1^2 + p_2^2 + p_3^2 - \left[ (p_1 p_2)^4 + (p_1 p_3)^4 + (p_2 p_3)^4 \right] \frac{h^6(V)}{2VV_c^6} \right) \]

For a simple case, when \( \beta_- = 0 \) and \( \beta_+ \rightarrow -\infty \), the potentials are

\[ W \sim \frac{1}{2} e^{-4\Omega - 4\beta_+} \quad \text{and} \quad W_{\text{eff}} \sim \frac{1}{2V_c^9} e^{-52\Omega - 4\beta_+} \quad (9) \]

The \( \beta_+ \)-dependency of both potentials is the same. Thus, we have an exponential wall for the modified potential, too.
Bianchi IX
Effective Potential with Inverse Triad Corrections

Modified potential with $\beta_- = 0$ ($a_1 = a_2$).
Results

- All solutions have a bounce. In other words, singularities are resolved. In Bianchi IX, there is an infinite number of bounces and recollapses.
- The classical point-like and cigar-like singularities are resolved.
- We could not show numerically the resolution of the barrel-like singularities because showing this implies a fine-tuning in the initial conditions, but we studied the limit of this kind of singularities, and there is nothing that indicates that they are not resolved, too.
- Bianchi I and therefore the isotropic case $k=0$ are limiting cases of Bianchi IX, but they are not contained within Bianchi IX. While the isotropic FRW $k=1$ is contained within Bianchi IX only if the inverse triad corrections are not included.
Results

- If the inverse triad corrections are not included, then the geometric scalars \((\theta, \sigma^2, \rho)\) are not absolutely bounded. But if the inverse triad corrections are included then, on each solution, the geometric scalars are bounded but there is not an absolute bound for all the solutions.

- The potential wall does not disappear and we have, potentially, chaotic behavior near the classical singularity. However, if we start from large volumes there will be a lower bound for volume. Since there are no large anisotropies near the smallest allowed volume, the solutions will not exhibit chaotic behavior.

- The initial conditions at bounce allow us to have good control on the solutions, but we do not necessarily need to impose them at the bounce.
\[ H_{BII} = \frac{p_1 p_2 p_3}{8\pi G \gamma^2 \lambda^2} \left[ \sin \tilde{\mu}_1 c_1 \sin \tilde{\mu}_2 c_2 + \sin \tilde{\mu}_2 c_2 \sin \tilde{\mu}_3 c_3 + \sin \tilde{\mu}_3 c_3 \sin \tilde{\mu}_1 c_1 \right] \\
+ \frac{1}{8\pi G \gamma^2} \left[ \frac{\alpha (p_2 p_3)^{3/2}}{\lambda \sqrt{p_1}} \sin \tilde{\mu}_1 c_1 - (1 + \gamma^2) \left( \frac{\alpha p_2 p_3}{2p_1} \right)^2 \right] - \frac{p_\phi^2}{2} \approx 0 \]

\[ H_{BIX} = -\frac{p_1 p_2 p_3}{8\pi G \gamma^2 \lambda^2} \left( \sin \tilde{\mu}_1 c_1 \sin \tilde{\mu}_2 c_2 + \sin \tilde{\mu}_2 c_2 \sin \tilde{\mu}_3 c_3 + \sin \tilde{\mu}_3 c_3 \sin \tilde{\mu}_1 c_1 \right) \\
- \frac{i \gamma}{8\pi G \gamma^2 \lambda} \left( \frac{(p_1 p_2 p_3)^{3/2}}{\sqrt{p_3}} \sin \tilde{\mu}_1 c_3 + \frac{(p_2 p_3)^{3/2}}{\sqrt{p_1}} \sin \tilde{\mu}_3 c_3 + \frac{(p_3 p_1)^{3/2}}{\sqrt{p_2}} \sin \tilde{\mu}_2 c_2 \right) \\
- \frac{i \gamma^2}{32\pi G \gamma^2} \left[ 2(p_1^2 + p_2^2 + p_3^2) - \left( \frac{p_1 p_2}{p_3} \right)^2 - \left( \frac{p_2 p_3}{p_1} \right)^2 - \left( \frac{p_3 p_1}{p_2} \right)^2 \right] \\
+ \frac{p_\phi^2}{2} \approx 0 \]

with lapse \( N = \sqrt{p_1 p_2 p_3} \).
Abstract

- Loop quantum cosmology (LQC) adapts key insights of LQG to cosmology (cf. [1] and/or M. Bojowald’s talk for a review).
- We discuss how different operators in Bianchi I and isotropic LQC can be ordered in such a way as to ensure consistency between the two models.
- The operators we consider include the Hamiltonian constraint, as well as observables such as volume, directional Hubble rates, expansion, and shear, which are central to investigations of singularity resolution.
Outline

- Mathematical Formalism
- Projector, Embedding, Hamiltonian constraint
- The $L$ matrix
- QISO and QFLRW conditions
- Issues of operator ordering
- The curvature operator
- $\hat{f}_{ij}$, $\hat{g}_{ij}$, $\hat{h}_{ij}$, and ordering conditions
- Generalized solutions
- Volume
- Expansion, shear, and directional Hubble rates
- Sign issues
- Summary and Future Work
Mathematical Formalism

- $A_a^i, \tilde{E}_i^a$: 
  $$A_a^i \equiv \Gamma_a^i + \gamma K_a^i$$  
  $$\tilde{E}_i^a = \det(e_b^i)e_i^a$$

- Variables for the homogeneous, but anisotropic Bianchi I (BI) model [2]: $c^i, p_i$
  $$A_a^i = c^i e_a^i$$  
  $$\tilde{E}_i^a = p_i \det(\tilde{e}_j^b)\tilde{e}_i^a$$  
  $$\{ c^i, p_j \} = 8\pi G \gamma \delta_j^i$$

- Variables for the homogeneous, isotropic Friedmann–Lemaître–Robertson–Walker (FLRW) model [3]: $c, p$
  $$A_a^i = ce_a^i$$  
  $$\tilde{E}_i^a = p \det(\tilde{e}_j^b)\tilde{e}_i^a$$  
  $$\{ c, p \} = \frac{8\pi G \gamma}{3}$$
Projector, Embedding, HC

- In [2], a projector (a surjective map)
  \[ P : \mathcal{H}_{BI} \to \mathcal{H}_{FLRW} \]
  was introduced which intertwines the Hamiltonian constraints of the two models:
  \[ P \circ C_{FLRW} = C_{BI} \circ P \]

- The adjoint of \( P \) then defines an embedding (an injective map)
  \[ \iota \equiv P^\dagger : \mathcal{H}_{FLRW} \to \mathcal{H}_{BI} \]
  which also intertwines the Hamiltonian constraints:
  \[ C_{FLRW}^\dagger \circ \iota = \iota \circ C_{BI}^\dagger \]
  – a dynamical embedding.

- This shows that the FLRW model is dynamically equivalent to the sector consisting of the image of \( \iota \).
The $L$ matrix (for Bianchi I)

\[ L^i_j = \delta^i_j \frac{1}{|p_j|^{1/2}} \sum_{k,l} \epsilon_{jkl} c_k c_l \]

\[ L^{1}_1 = \frac{c_2 c_3}{2|p_1|} \]

\[ (L^C)^i_j = \delta^i_j \frac{1}{|p_j|^{1/2}} \sum_{k,l} \epsilon_{jkl} \alpha_k \alpha_l \]

\[ (L^C)^{1}_1 = \frac{\alpha_2 \alpha_3}{2|p_1|} \]

\[ \alpha_i \equiv c_i + i4\pi G' \gamma \frac{|p_1 p_2 p_3|^{1/2}}{p_i} \]

\[ (A^C)^i_a \equiv A^i_a + i4\pi G \gamma c^i_a \]

where $L^i_j(x)$ is defined by:

\[ F^i \equiv dA^i + \epsilon^i_{jk} A^j \wedge A^k =: L^i_j \Sigma^j \]

and where $(L^C)^i_j(x)$ is defined by:

\[ (F^C)^i \equiv d(A^C)^i + \epsilon^i_{jk} (A^C)^j \wedge (A^C)^k =: (L^C)^i_j \Sigma^j \]

Classically, $L := F/\Sigma$, and therefore the ordering of $\hat{L}$ depends only on the (non-trivial) ordering of $\hat{F}$. 
QISO and QFLRW conditions

- Quantum isotropy condition: $L^i_j(x)$ is constant, and pure trace:
  \[ L^i_j(x) = L^i_j(y) \quad \forall x, y \]  
  (homogeneity)
  \[ L^i_j(x) - \frac{1}{3} \delta^i_j \text{tr} L(x) = 0 \quad \forall i, j, x \]  
  (isotropy)

- Quantum FLRW condition: $(L^C)^i_j(x)$ is constant and pure trace, and:
  \[ \text{Re} \left( \text{tr} L^C \right) + \frac{3(8\pi G \gamma)^2}{8} = \frac{8(\text{Im} \left( \text{tr} L^C \right))^2}{3(8\pi G \gamma)^2} \]
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  \]
Issues of operator ordering

- There are many important quantum operators.
- Chiefly among them, we are concerned with the curvature operator, $\hat{F}$.
- $\hat{F}$ has several different components, so we have an operator ordering ambiguity.
- We attempt to solve this ambiguity via relating the BI and FLRW theories to each other at the quantum level.
The curvature operator $\hat{F}$

(figure from [2])

\[ \Delta \ell_{P_1}^2 = 4 \sqrt{3 \pi} G \gamma \ell_{P_1}^2 \]

\[ \bar{\mu}_i = \sqrt{\frac{P_i}{P_1 P_2 P_3}} \Delta \ell_{P_1}^2 \]

& cyc.

\[ F_{ab}^k \equiv 2 \lim_{\square_{ij} \to \text{point}} \text{tr} \left( \frac{h_{\square_{ij}} - I}{A r_{\square_{ij}}} \tau^k \right) \hat{e}_a \hat{e}_b \]

\[ \hat{F}_{ab}^k = \sum_{ij} 2 \lim_{A r_{\square_{ij}} \to \Delta \ell_{P_1}^2} \text{tr} \left( \frac{h_{\square_{ij}} - I}{A r_{\square_{ij}}} \tau^k \right) \hat{e}_a \hat{e}_b = \sum_{ij} \epsilon_{ij} \left( \frac{\sin \bar{\mu}_i c_i \hat{e}_a}{\bar{\mu}_i} \right) \left( \frac{\sin \bar{\mu}_j c_j \hat{e}_b}{\bar{\mu}_j} \right) \]

\[ = \sum_{ij} \epsilon_{ij} \hat{e}_a \hat{e}_b |k \hat{p}_k| \hat{f}_{ij} \sin(\bar{\mu}_i c_i) \hat{g}_{ij} \sin(\bar{\mu}_j c_j) \hat{h}_{ij} = \hat{\Sigma}_{ab} \hat{L}^k \]
\( \hat{f}_{ij}, \hat{g}_{ij}, \hat{h}_{ij} \)

- The previous line implies that:
  \[
  2\hat{L}^1 \equiv \hat{f}_{23}\sin(\hat{\mu}_2c_2)\hat{g}_{23}\sin(\hat{\mu}_3c_3)\hat{h}_{23} + (2 \leftrightarrow 3) \quad \text{& cyc.}
  \]

so that, in the BI model, \( \hat{L}_i^j = \delta_i^j \hat{L}^i \).

- We can use \( \hat{L}_i^j \) to impose ‘quantum isotropy’ in BI LQC. However, the ordering parameters (\( \hat{f}_{ij}, \hat{g}_{ij}, \hat{h}_{ij} \)) still need to be fixed.

- Deriving (\( \hat{f}_{ij}, \hat{g}_{ij}, \hat{h}_{ij} \)) is non-trivial (many terms of nested difference equations).
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Deriving (\( \hat{f}_{ij}, \hat{g}_{ij}, \hat{h}_{ij} \)) is non-trivial (many terms of nested difference equations).
Ordering conditions I

- $\text{Im } i$ is the sector of quantum isotropy as long as
  \[
  \hat{f}_{12}, \hat{g}_{12}, \hat{h}_{12}, \hat{f}_{21}, \hat{g}_{21}, \hat{h}_{21}
  \]
  all depend on $p_3$ only (\& cyc.) (condition A).

- This is because the conjugate ($k^{th}$) $p$, which is related to the respective $\lambda$, is not effected by either of the ($i^{th}$ & $j^{th}$) exponential (shift) operators, $s$, in which $\hat{F}$ is quadratic.

- When this condition is satisfied, the ordering functions commute with all of the other factors, so there is no further ambiguity in the definition of $\hat{L}_j$. 

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- When this condition is satisfied, the ordering functions commute with all of the other factors, so there is no further ambiguity in the definition of \( \hat{L}_j \).
Ordering conditions II

Is there any ordering choice for $\hat{L}_j$ such that it is intertwined by $i$?

Yes. The condition on the ordering parameters is then stronger (because intertwining also involves the FLRW quantum theory in which, e.g., $p_3$ no longer commutes with $c_1$ and $c_2$). Then one solution is given by:

\[ \hat{f}_{12} = \hat{f}_{21} = \hat{g}_{12} = \hat{g}_{21} = \text{sgn}(p_3), \quad \hat{h}_{ij} = 1. \]
Ordering conditions III

\[ s_1^+ : (\lambda_1, \lambda_2, \lambda_3) \mapsto (\lambda_1', \lambda_2', \lambda_3') \equiv \left( \frac{\Lambda \pm 1}{\Lambda} \lambda_1, \lambda_2, \lambda_3 \right) \]

\[ g_{ij}(\lambda_1, \lambda_2, \lambda_3) = g_{s(i)s(j)}(\lambda_{s(1)}, \lambda_{s(2)}, \lambda_{s(3)}) \]

\[ h_{ij}(\lambda_1, \lambda_2, \lambda_3) = h_{s(i)s(j)}(\lambda_{s(1)}, \lambda_{s(2)}, \lambda_{s(3)}) \]

& cyc.

\[ \lambda_i \equiv \frac{\text{sgn}(p_i) \sqrt{|p_i|}}{(4\pi \gamma \sqrt{\Delta_l f_{P1}^3})^{1/3}} \]

\[ \Lambda \equiv \lambda_1 \lambda_2 \lambda_3 \]

\[ s \in S\{1,2,3\} \]

\[ i, j = 1, 2 \text{ case:} \]

\[ \frac{\text{sgn}(\Lambda - 1)}{\text{sgn}(\Lambda)} \frac{g_{12} \circ s_1^-}{g_{12}} (\lambda) \frac{h_{12} \circ s_2^+ \circ s_1^-}{h_{12}} (\lambda) + \frac{\text{sgn}(\Lambda + 1)}{\text{sgn}(\Lambda)} \frac{g_{12} \circ s_1^+}{g_{12}} (\lambda) \frac{h_{12} \circ s_2^- \circ s_1^+}{h_{12}} (\lambda) = \frac{g(\Lambda - 1) + g(\Lambda + 1)}{g(\Lambda)} \]

and

\[ \frac{\text{sgn}(\Lambda - 1)}{\text{sgn}(\Lambda)} \frac{g_{12} \circ s_1^-}{g_{12}} (\lambda) \frac{h_{12} \circ s_2^- \circ s_1^-}{h_{12}} (\lambda) = \frac{g(\Lambda - 1) h(\Lambda - 2)}{g(\Lambda) h(\Lambda)} \]

\[ \frac{\text{sgn}(\Lambda + 1)}{\text{sgn}(\Lambda)} \frac{g_{12} \circ s_1^+}{g_{12}} (\lambda) \frac{h_{12} \circ s_2^+ \circ s_1^+}{h_{12}} (\lambda) = \frac{g(\Lambda + 1) h(\Lambda + 2)}{g(\Lambda) h(\Lambda)} \]

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& cyc.

\[ g_{ij}(\lambda_1, \lambda_2, \lambda_3) = g_{s(i)s(j)}(\lambda_{s(1)}, \lambda_{s(2)}, \lambda_{s(3)}) \]

\[ h_{ij}(\lambda_1, \lambda_2, \lambda_3) = h_{s(i)s(j)}(\lambda_{s(1)}, \lambda_{s(2)}, \lambda_{s(3)}) \]

\[ \lambda_i \equiv \frac{\text{sgn}(p_i) \sqrt{|p_i|}}{(4\pi \gamma \sqrt{\Delta l_1^3})^{1/3}} \]

\[ \Lambda \equiv \lambda_1 \lambda_2 \lambda_3 \]

\[ s \in S_{\{1,2,3\}} \]

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\[ \text{and} \]

\[ \frac{\text{sgn}(\Lambda - 1)}{\text{sgn}(\Lambda)} \frac{g_{12} \circ s_1^-}{g_{12}} (\lambda) \frac{h_{12} \circ s_2^- \circ s_1^-}{h_{12}} (\lambda) = \frac{g(\Lambda - 1)}{g(\Lambda)} \frac{h(\Lambda - 2)}{h(\Lambda)} \]

\[ \frac{\text{sgn}(\Lambda + 1)}{\text{sgn}(\Lambda)} \frac{g_{12} \circ s_1^+}{g_{12}} (\lambda) \frac{h_{12} \circ s_2^+ \circ s_1^+}{h_{12}} (\lambda) = \frac{g(\Lambda + 1)}{g(\Lambda)} \frac{h(\Lambda + 2)}{h(\Lambda)} \]

& cyc.
Generalized solutions

There exist a general family of solutions, which are solutions to:

\[
g_{ij}(\lambda) = \text{sgn}(\Lambda)g(\Lambda)\gamma(\lambda)
\]
\[
h_{ij}(\lambda) = h(\Lambda)\eta(\lambda)
\]

with the following constraints on \(\gamma\) and \(\eta\):

\[
\gamma \circ s_1^- (\lambda) \frac{\eta \circ s_2^- \circ s_1^- (\lambda)}{\eta} = 1
\]
\[
\gamma \circ s_1^+ (\lambda) \frac{\eta \circ s_2^+ \circ s_1^+ (\lambda)}{\eta} = 1
\]
\[
\gamma \circ s_1^- (\lambda) \frac{\eta \circ s_2^- \circ s_1^- (\lambda)}{\eta} = 1
\]
\[
\gamma \circ s_1^+ (\lambda) \frac{\eta \circ s_2^- \circ s_1^+ (\lambda)}{\eta} = 1
\]

& cyc.
Volume: $\hat{V}$

The embedding intertwines the volume of the fiducial cell, $\hat{V} \sim a^3$:

\[
(\hat{V} \psi)(\lambda) = (V(\lambda)) (\psi)(\lambda) = (V(\lambda)) \psi(V(\lambda)) \\
(\psi)(\lambda) = (\hat{V} \psi)(V(\lambda)) = (V(\lambda)) \psi(V(\lambda))
\]

\[
\lambda \equiv (\lambda_1, \lambda_2, \lambda_3) \\
V(\lambda) \equiv 2\lambda_1\lambda_2\lambda_3
\]

Actually, **ANY** function $f(\hat{V})$ is intertwined:

\[
(\psi)(\lambda) \equiv \psi(V(\lambda)) \\
(\psi)(\lambda) = (f(\hat{V}) \psi)(V(\lambda)) = f(V(\lambda)) \psi(V(\lambda)) \\
f(\hat{V})(\psi)(\lambda) = f(V(\lambda))(\psi)(\lambda) = f(V(\lambda)) \psi(V(\lambda))
\]

\[
\therefore \quad \psi \circ f(\hat{V}) = f(\hat{V}) \circ \psi \quad \forall f(\hat{V})
\]
Expansion, shear, and directional Hubble rates

\[ B_{ab} \equiv \nabla_b \xi_a \]
\[ \xi_a \xi^a = -1 \]
\[ \theta \equiv B^{ab} h_{ab} \]
\[ H_i = \frac{\dot{a}_i}{a_i} \]
\[ H^i_j \equiv e^i_a e^b_j \nabla_b n^a =: H_i \delta^i_j \]
\[ \theta \equiv \text{tr} H = H_1 + H_2 + H_3 \]
\[ h_{ab} \equiv g_{ab} + \xi_a \xi_b \quad \text{(Wald)} \]
\[ \sigma_{ab} \equiv B_{(ab)} - \frac{1}{3} \theta h_{ab} \]
\[ \sigma_{ij} \equiv H_{(ij)} - \frac{1}{3} \theta \delta_{ij} \]

- All operators of physical interest can be constructed from \( L^i_j \).
- \( \theta \) = expansion (average Hubble rate); \( \sigma_{ab} \) = shear (gives measure of inhomogeneities);
- \( H^i_j \) = diagonal ‘expansion matrix’; \( H_i \) = directional Hubble rates [the ‘twist’ \( \omega \), the anti-symmetric part of \( B \), is zero].
Expansion, shear, and directional Hubble rates

\[ B_{ab} \equiv \nabla_b \xi_a \]
\[ \xi_a \xi^a = -1 \]
\[ h_{ab} \equiv g_{ab} + \xi_a \xi_b \] (Wald)
\[ \theta \equiv B^{ab} h_{ab} \]
\[ H_i = \frac{\dot{a}_i}{a_i} \]
\[ H_j^i \equiv \epsilon^i_a \epsilon^b_j \nabla_b n^a = : H_i \delta_j^i \]
\[ \theta \equiv \text{tr } H = H_1 + H_2 + H_3 \]

All operators of physical interest can be constructed from \( L_j^i \).

\( \theta \) = expansion (average Hubble rate); \( \sigma_{ab} \) = shear (gives measure of inhomogeneities); \( H_j^i \) = diagonal ‘expansion matrix’; \( H_i = \) directional Hubble rates [the ‘twist’ \( \omega \), the anti-symmetric part of \( B \), is zero].
Expansion, shear, and directional Hubble rates

- For BI, we have:
  \[ \sigma^2 = (H_1 - H_2)^2 + (H_2 - H_3)^2 + (H_3 - H_1)^2 \]
  \[ \theta^2 = \frac{1}{9} (H_1 + H_2 + H_3)^2 \]

  \[ \theta^2 = \gamma^{-2} (\det L)(\tr(L^{-1}))^2 \]
  \[ H_i^2 = \gamma^{-2} (\det L) L_i^{-2} \]
  \[ \sigma^2 \equiv \sigma_{ij} \sigma^{ij} = 2 \gamma^{-2} (\det L) (\tr(L^{-2}) - \tr(L)) \]

  (for FLRW, we have: \[ L = \gamma^2 H^2 \] \[ \theta^2 = \frac{1}{\gamma^2} L = H^2 \] and \[ \sigma^2 = 0 \].)

- We can use either \( L_i \) or \((L^c)_i\) — the two choices will turn out to lead to the same sector of quantum isotropy:
  \[ L^c \equiv e^{-V} L e^V \]
Sign issues

- Quantize these by replacing $L^i_j$ with $\hat{L}^i_j$: as long as these expressions are used, no matter what ordering of them is chosen, because $\hat{L}^i_j$ is intertwined, so will $|\theta|$, $\sigma^2$, and $|\hat{H}_i|$ be intertwined.

- Side note: Not only can the signs of $H_i$, and hence $\theta$, not be written in terms of $L^i_j$, but the corresponding operators seem to not even be defined.

- Specifically: $\text{sgn } H_i = (\text{sgn } c_i) (\text{sgn } p^i)$, and $\text{sgn } c_i$ is not an almost periodic function (∉ Cyl$_S$, nor even the larger Fleischhack space [Cyl$_F = \text{Cyl}_{S} \oplus \mathcal{V}$] [where $\mathcal{V}$ is the set of all functions on $\mathbb{R}$ vanishing at ±∞ and at 0] [4], because of LQG*)], and hence has no operator analogue in LQC.
Summary and Future Work

- We have found a possible general operator ordering that is consistent across both of the models, up to the noted parameters; all operators built from the $\hat{L}_i$ give intertwining, because the $\hat{L}_i$ themselves are intertwined.

- Future Goals: **Carry out the program in the full theory**, including implementation of the QFLRW condition, as well as comparing the dynamics (Hamiltonian constraint) and observables in BI/FLRW and in the full theory.

- Full spinfoam cosmology (SFC) $\leftrightarrow$ LQC $\leftrightarrow$ LQG integration.
Sign issues

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The curvature operator \( \hat{F} \)

(each link is spin \( \frac{1}{2} \))

\[ F_{ab} \equiv 2 \lim_{\Box_{i,j} \rightarrow \text{point}} \text{tr} \left( \frac{h_{\Box_{i,j}} - I}{A_{\Box_{i,j}}} \tau^k \right) \varepsilon^i_a \varepsilon^j_b \]

\[ \hat{F}_{ab} = \sum_{ij} 2 \lim_{A_{\Box_{i,j}} \rightarrow \Delta \ell_{Pl}^2} \text{tr} \left( \frac{h_{\Box_{i,j}} - I}{A_{\Box_{i,j}}} \tau^k \right) \varepsilon^i_a \varepsilon^j_b = \sum_{ij} \epsilon^i_{ij} \epsilon^j_{[a} \epsilon^i_{b]} \hat{p}_k \hat{f}_{ij} \sin(\hat{\mu}_i c_i) \hat{g}_{ij} \sin(\hat{\mu}_j c_j) \hat{h}_{ij} = \hat{\Sigma}_{ab} \hat{L}^k \]
Ordering conditions II

- Is there any ordering choice for $\hat{L}_j^i$ such that it is intertwined by $i$?

**Yes.** The condition on the ordering parameters is then stronger (because intertwining also involves the FLRW quantum theory in which, e.g., $p_3$ no longer commutes with $c_1$ and $c_2$). *Then one solution is given by:*

$$\hat{f}_{12} = \hat{f}_{21} = \hat{g}_{12} = \hat{g}_{21} = \text{sgn}(p_3) ; \quad \hat{h}_{ij} = 1 .$$