We demonstrate that the many-body localized phase is characterized by the existence of infinitely many local conservation laws. We argue that many-body eigenstates can be obtained from product states by a sequence of nearly local unitary transformation, and therefore have an area-law entanglement entropy, typical of ground states. Using this property, we construct the local integrals of motion in terms of projectors onto certain linear combinations of eigenstates [1]. The local integrals of motion can be viewed as effective quantum bits which have a conserved $z$-component that cannot decay. Thus, the dynamics is reduced to slow dephasing between distant effective bits. For initial product states, this leads to a characteristic slow power-law decay of local observables, which is measurable experimentally, as well as to logarithmic in time growth of entanglement entropy [2,3]. We support our findings by numerical simulations of random-field XXZ spin chains. Our work shows that the many-body localized phase is locally integrable, reveals a simple entanglement structure of eigenstates, and establishes the laws of dynamics in this phase.

Ergodicity and its breaking

System explores full phase space
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Ergodicity breaking

In phase transitions
Ergodicity and its breaking

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System explores full phase space

Ergodicity breaking

In phase transitions  In bear habitats
Many-body localization problem

When/how does ergodicity break in many-body systems?
Do interactions destroy localization?
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New experimental systems
Isolated & quantum-coherent. Tunable interactions and disorder

- Cold atoms, optical lattices
- Polar molecules
- Spin systems (NV-centers in diamond)
New experimental systems
Isolated & quantum-coherent. Tunable interactions and disorder

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Three-Dimensional Anderson Localization of Ultracold Matter
S. S. Kondov, W. R. McGehee, J. J. Zirbel, B. DeMarco

EXP: Ecole Normale, Florence, Urbana
New experimental systems

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REPORTS

Three-Dimensional Anderson Localization of Ultracold Matter

S. S. Kondov, W. R. McGehee, J. J. Zirbel, B. DeMarco

Interplay of disorder and interactions in an optical lattice Hubbard model

S. S. Kondov, W. R. McGehee, B. DeMarco

(Dated: May 28, 2013)

EXP: Ecole Normale, Florence, Urbana

Studying many-body localization experimentally now possible
Entanglement propagation in ergodic systems

Delocalized systems

Light-cone-like spreading of correlations

Lieb, Robinson’72, Hastings’04, Calabrese, Cardy’05
Kim, Huse’13
Entanglement propagation in ergodic systems

Delocalized systems

Light-cone-like spreading of correlations

Initial product state

Linear growth of entanglement entropy

\[ S_{ent} = vt \]

(even when transport is diffusive)

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Kim, Huse'13

Experimental observation: Cold atoms  Cheneau et al'12 Trapped ions  Jurcevic et al'14
A quantum many-body system

Condensed matter physicists
A quantum many-body system

Condensed matter physicists

System in a **ground state**

**But there is much more...**

This talk: Highly excited many-body states
A simple model of many-body localization

\[ E_i \]

\[ t \]

Spinless interacting 1D fermions \( \approx \) Random-field XXZ spin-1/2 chain

\[ H = \sum_i E_i c_i^+ c_i + \sum_i c_i^+ c_{i+1} + h.c. + \sum_i n_i n_{i+1} \]

\[ H = \sum_i h_i S_i^z + J_{\perp} \sum_i (S_i^+ S_{i+1}^- + h.c.) + J_z \sum_i S_i^z S_{i+1}^z \]

Use disorder as tuning parameter \( h_i \in [-W; W] \)
A simple model of many-body localization

Jordan-Wigner

Spinless interacting 1D fermions $\approx$ Random-field XXZ spin-1/2 chain

$H = \sum_i E_i c_i^+ c_i + \sum_i t c_i^+ c_{i+1} + h.c. + V \sum_i n_i n_{i+1}$

$H = \sum_i h_i S_i^z + J_{\perp} \sum_i (S_i^+ S_{i+1}^- + h.c.) + J_z \sum_i S_i^z S_{i+1}^z$

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*Jordan-Wigner*

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Use disorder as tuning parameter $h_i \in [-W; W]$}

Many-body localization at strong disorder (numerics)

Oganesyan, Huse’07, Prosen et al’08, Pal, Huse’10, Monthus, Garel’10,..
Entanglement propagation in localized systems

\[ |\psi_0\rangle \rightarrow \text{Time evolution} \rightarrow |\psi(t)\rangle = e^{-iHt} |\psi_0\rangle \]

\[ L \quad R \]
Entanglement propagation in localized systems

\[ |\psi_0\rangle \quad \text{Time evolution} \quad |\psi(t)\rangle = e^{-i\mathcal{H}t} |\psi_0\rangle \]

- Anderson-localized: \( S_{\text{ent}}(t) \leq \text{const} \)

- Many-body localized: slow growth of entanglement

\[ S_{\text{ent}}(t) \propto \log t \]

Bardarson, Pollmann, Moore’12
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- Many-body localized: slow growth of entanglement

\[ S_{\text{ent}}(t) \propto \log t \]

- "Glassy" dynamics, extremely long time scales

- Entanglement extensive in system size, non-thermal

Ergodicity restored? Slow particle transport??

Bardarson, Pollmann, Moore’12
The mechanism of entanglement growth: Toy model

\[ |\psi_0\rangle = \frac{1}{2}(c_1^+ + c_2^+)(c_3^+ + c_4^+)|0\rangle \]
The mechanism of entanglement growth: Toy model

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Assume weak interactions

Eigenstate \[ |\alpha\beta\rangle = c_\alpha^+ c_\beta^+ |0\rangle + O(e^{-x/\xi}) \]

Energy: \[ E_{\alpha\beta} = E_\alpha + E_\beta + C_{\alpha\beta} V e^{-x/\xi} \]

Reduced density matrix

\[
\rho(t) = \frac{1}{2} \begin{bmatrix}
1 & \cos \omega t \\
\cos \omega t & 1
\end{bmatrix}
\]

\[ \omega \sim \frac{V}{\hbar} e^{i/\xi} \]

Serbyn, Papic, Abanin PRL '13
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\[ \rho(t) = \frac{1}{2} \begin{bmatrix} 1 & \cos \omega t \\ \cos \omega t & 1 \end{bmatrix} \]

\[ \omega \sim \frac{V}{\hbar} e^{x/\xi} \quad t_{\text{deph}} \sim \frac{\hbar}{V} e^{x/\xi} \]

Interaction-induced dephasing → entanglement generation

Serbyn, Papic, Abanin PRL '13
Case of many particles

**Hypothesis:** Eigenstates at small $V$ are "close" to non-interacting eigenstates

Non-interacting

$$c_{\alpha_1}^+ c_{\alpha_2}^+ ... c_{\alpha_i}^+ ... c_{\alpha_N}^+ |0\rangle$$

Interacting

$$|\{\alpha\}\rangle = |\alpha_1 \alpha_2 ... \alpha_i ... \alpha_N\rangle$$

Energy: perturbation theory in $V$

$$E_{\{\alpha\}} = \sum E_{\alpha_i} + V \sum C_{\alpha_i \alpha_j} e^{-\frac{|R_i - R_j|}{\xi}} + ...$$

1-body energy interactions 2-body 3,4...-body interactions

Interactions of far-away particles are exponentially small
The laws of entanglement growth

Serbyn, Papic, Abanin PRL’13

Initial product state is a superposition of many eigenstates
The laws of entanglement growth

Initial product state is a superposition of many eigenstates

Degrees of freedom a distance $x$ away get etangled at

$$ t_{\text{depht}} \sim \frac{\hbar}{V} e^{x/\xi} $$

$$ S_{\text{ent}}(t) = C \xi \log \frac{Vt}{\hbar} $$

$C$ depends on diagonal entropy of initial state
The laws of entanglement growth

Serbyn, Papic, Abanin PRL’13

Initial product state is a superposition of many eigenstates

\[ x \]

Degrees of freedom a distance \( x \) away get entangled at

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\[ S_{\text{ent}}(t) = C \xi \log \frac{Vt}{\hbar} \]

\( C \) depends on diagonal entropy of initial state

Saturated value: set by diagonal entropy

\[ S_{\text{ent}}(\infty) = S_{\text{diag}} \ll S_{\text{Thermal}} \]

Predicted: propagation speed, dependence on disorder, interactions, initial state

Confirmed by numerics
Numerical analysis

\[ S_{\text{ent}}(t) \propto \xi \log \frac{Vt}{\hbar} \]

Interaction dependence

Saturated entanglement extensive in system size

Disorder dependence

\[ C = \frac{S_{\text{ent}}(\infty)}{S_{\text{diag}}} \]
Localized phase at strong interactions (e.g., spins)?
Is entanglement growth universal?
Localized phase at **strong interactions** (e.g., spins)?
Is entanglement growth universal?
Localized phase at \textbf{strong interactions (e.g., spins)}? Is entanglement growth universal?

\textbf{YES!}
The key: MBL phase is integrable. Integrals of motion are local.
Constructing integrals of motion

\[ |\alpha\rangle_L \quad |\beta\rangle_i \quad |\gamma\rangle_R \]

A local integral of motion in terms of projectors onto eigenstates

\[ \hat{P}_i(\beta) = \sum_{\alpha, \gamma} |\alpha\beta\gamma\rangle \langle\alpha\beta\gamma| \]

\[ |\alpha\beta\gamma\rangle = \hat{O}_{Ri}\hat{O}_{Li}|\alpha\rangle_L \otimes |\beta\rangle_i \otimes |\gamma\rangle_R \]

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These operators are local. Integrals of motion by construction.
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These operators are local. Integrals of motion by construction.
Universal Hamiltonian of MBL phase

"Effective" spins $\frac{1}{2}$ $\tau_i^z$ with conserved z-projection (defined via projectors)

$[\tau^i_z, H] = 0$

$\tau^i_z$ support in a region of size $\sim \xi$

Hamiltonian depends only on $\tau^i_z$

$$H = \sum_i H_i \tau_i^z + \sum_{ij} H_{ij} \tau_i^z \tau_j^z + \sum_{ijk} H_{ijk} \tau_i^z \tau_j^z \tau_k^z + \ldots$$

$$H_{ij} \propto \exp(-|i - j|a/\xi)$$
Universal dynamics and entanglement growth

- No relaxation $\rightarrow$ no thermalization
  \[
  \langle \tau_x^i(t) \rangle = \text{Const}
  \]

- Interaction-induced dephasing
  \[
  \langle \tau_x^i(t) \rangle \rightarrow 0 \quad t \rightarrow \infty
  \]

- Steady non-thermal state

- Universal logarithmic growth of entanglement entropy (hard to measure)

- Any local observable would show oscillations and a power-law decay (easy to measure)
Interferometric signatures of many-body localization

Test spin

Modified spin-echo

Dephasing $\rightarrow$ power-law spin-echo decay $\bar{D}(t) \sim t^{-\xi \ln 2}$

A way to directly probe MBL in spin systems, cold atoms

Serbyn, Papic, Abanin+Lukin, Demler’14
The structure of many-body localized states

- Eigenstates obtained from product states by a sequence of local unitaries (quantum circuit of finite depth)

- Excited MBL eigenstates obey area law-entanglement (similar to ground states of gapped systems!)
Explicitly constructing integrals of motion

Effective spins: good for quantum information processing?
Yes, but how do they relate to physical spins?

Approach 1: Connect eigenstates to product states by local unitary $U$

$$\tau^i_z = U s^i_z U^+$$

Expand $\tau^i_z$ via physical operators

$$\tau^i_z = C_1 s^i_z + C_2 s^{i+1}_z + C_3 s^{i-1}_z + D_1 s^i_z s^{i+1}_x + ..$$

with A. Chandran, G. Vidal, in progress
Explicitly constructing integrals of motion

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$$\tau_z^i = C_1 s_z^i + C_2 s_{z+1}^i + C_3 s_{z-1}^i + D_1 s_z^i s_x^{i+1} + \ldots$$

Approach 2: Extension of a strong disorder renormalization group

Extract statistics of couplings and properties of integrals of motion

with A. Chandran, G. Vidal, in progress
Localization without quenched disorder?
(Muller’13, Hiveneers, de Roeck’13, Fisher, Grover’13..)

Two coupled spin chains, “fast” $O_i^z$ and “slow” $S_i^z$

$\lambda \ll 1$

$\uparrow \downarrow \uparrow \downarrow \uparrow \downarrow \uparrow \downarrow \uparrow \downarrow \uparrow$

$J \sim 1$

Papic, Stoudenmire, DA ’14
Localization without quenched disorder?
(Muller’13, Huveneers, de Roeck’13, Fisher, Grover’13..)

Two coupled spin chains, “fast” $\sigma_i^z$ and “slow” $S_i^z$

Papic, Stoudenmire, DA ‘14

Interaction $H_{\text{int}} = W \sum_i \sigma_i^z S_i^z$

Localization at $\lambda << 1$?

Preliminary:

Log-growth of entanglement

Signals localization??
Summary

- Ergodicity breaking in disordered systems: local conservation laws
- Universal dynamics: slow dephasing
- Logarithmic growth of entanglement entropy
- More to come...

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