Title: Critical Behavior of the Classical XY-model on Fractal Structures

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Abstract: There has been considerable interest in determining whether the universality hypothesis extends to systems which are of non-integer dimension or to systems which are scale invariant (fractals). Specifically research into these types of systems is concerned with determining the relevance of topological properties to their critical phenomena. We have performed Monte Carlo simulations for the XY model on three fractal lattices with different topological properties: the Sierpinski pyramid Menger sponge and Sierpinski carpet. We will give an overview of our results and show that while some properties such as the order of ramification are important in determining the critical behavior of these structures the fractal dimension is not.
Universality hypothesis

Critical behavior of a translationally-invariant system is determined by:
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2. spatial dimensionality, $d$
3. range of interactions
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Example:

liquid $\rightarrow$ gas $\rightarrow$ paramagnet $\rightarrow$ ferromagnet

Critical exponents

For translation-invariant systems, i.e. fractals?
Universality hypothesis

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2. spatial dimensionality, $d$
3. range of interactions

Example:

- liquid $\rightarrow$ gas $\rightarrow$ paramagnet $\rightarrow$ ferromagnet

$\Rightarrow$ same critical exponents

What about scale-invariant systems, i.e. fractals?
Topological properties of fractals

Hausdorff (fractal) dimension, $D$

$$n = s^D \Rightarrow D = \frac{\ln n}{\ln s}$$

order of ramification, $R$
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$$\Rightarrow R = \infty$$

$$\Rightarrow R_{\text{min}} = 2$$
Topological properties of fractals

Hausdorff (fractal) dimension, $D$

$n = s^D \Rightarrow D = \frac{\ln n}{\ln s} \Rightarrow L = 0$

order of ramification, $R$

lacunarity, $L$
Topological properties of fractals

Hausdorff (fractal) dimension, $D$

\[ n = s^D \Rightarrow D = \frac{\ln n}{\ln s} \]

\[ \Rightarrow L = 0 \]

order of ramification, $R$

\[ \Rightarrow L = 0.9984 \]

lacunarity, $L$

\[ \Rightarrow L = 3.924 \]
Previous research

(Gefen et al., 1980, 1984): RG techniques to examine Ising model, discrete $Z_2$ symmetry, on multiple fractal structures
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⇒ lower critical $d = 2$, and $D \leq d$
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XY model

Continuous $O(2)$ symmetry
XY model

Continuous $O(2)$ symmetry

Configuration of $2d$ spins, oriented in $xy$-plane within $[0, 2\pi)$

Hamiltonian of the system:

$$H = -J \sum_{\langle i,j \rangle} \cos (\theta_i - \theta_j)$$
**XY model**

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Hamiltonian of the system:

$$H = -J \sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j)$$

$J > 0$ - coupling constant

$\langle \cdots \rangle$ - nearest neighbours only

$\theta_i$ - angle variable on site $i$

Interesting spin excitations
XY model

- positive vortex

- negative vortex

Regular 2d systems: Berezinskii-Kosterlitz-Thouless (BKT) transition
XY model

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Regular 2d systems: Berezinskii-Kosterlitz-Thouless (BKT) transition

⇒ universal jump in helicity modulus $\gamma(T_c)/T_c = 2/\pi$ (Nelson & Kosterlitz, 1977)
XY model

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Regular 3d systems: continuous phase transition
## Fractal lattices

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M. Przedborski (Brock University)

Compute Ontario Research Day 2014
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MC simulations

SP: Metropolis algorithm (Metropolis et al., 1953)
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⇒ local update scheme

⇒ autocorrelation time $\tau \propto L^z$, with $z = 2$ for local updates

SC, MS: Wolff cluster algorithm (Wolff, 1989)

⇒ non-local updates, avoids critical slowing down
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Distribution of states $\{\Omega_i\}$ with $\rho(\Omega_i) \propto e^{-E(\Omega_i)/k_BT}$

⇒ Thermal average: $\langle A \rangle = \frac{1}{M} \sum_{i=1}^{M} A_i$

Open and closed boundary conditions:
MC simulations: \[ H = -J \sum \cos(\theta_i - \theta_j) \]

Heat capacity per site:
\[ C = \frac{1}{N} \frac{\langle H^2 \rangle - \langle H \rangle^2}{k_B T^2} \quad N \text{ - number of sites} \]

Linear magnetic susceptibility per site:
\[ \chi = \frac{\langle m^2 \rangle - \langle m \rangle^2}{k_B T} \quad m \text{ - magnetization per site} \]

Helicity modulus per site:
\[ \gamma = \left\langle \left( \frac{\partial^2 H}{\partial A^2} \right)_{A=0} \right\rangle - \frac{1}{k_B T} \left\langle \left( \frac{\partial H}{\partial A} \right)_{A=0}^2 \right\rangle + \frac{1}{k_B T} \left\langle \left( \frac{\partial H}{\partial A} \right)_{A=0} \right\rangle^2 \]
Results: Sierpiński pyramid

low-T values $\to 0$ as $N \to \infty$
Results: Sierpiński pyramid

low-T values → 0 as N → ∞
Results: Sierpiński pyramid

- low-T values $\rightarrow 0$ as $N \rightarrow \infty$
- $T_\gamma \rightarrow 0$ as $N \rightarrow \infty$
- for open BC, $\gamma = 0$ for all $N$

$\Rightarrow$ no phase transition in SP
Results: Sierpiński carpet

\[ \gamma \left( \frac{2 \pi \Phi_0^2}{k_B T} \right) \]

- \( N=4096, \text{closed} \)
- \( N=32768, \text{closed} \)
- \( N=4096, \text{open} \)
- \( N=32768, \text{open} \)
Results: Sierpiński carpet

- Graphs showing the relationship between $k_B T/J$ and $y/L^2$. The graph on the left compares different configurations: $N=256$, closed and open, $N=256$, closed, and $N=256$, open.
- The graph on the right plots $\gamma_{\text{MAX}}$ against $\ln N$.
Results: Menger sponge
Results: Menger sponge

\[ \gamma = \frac{2.7844}{(\ln N)^{1.41103}} \]
No finite-$T$ phase transition (BKT or continuous) in XY model on Sierpiński pyramid ($D = 2$, $R_{\text{min}} = 4$), Sierpiński carpet ($D = 1.8928$, $R = \infty$), or Menger sponge ($D = 2.7268$, $R = \infty$)
Conclusions

- No finite-T phase transition (BKT or continuous) in XY model on Sierpiński pyramid ($D = 2$, $R_{\text{min}} = 4$), Sierpiński carpet ($D = 1.8928$, $R = \infty$), or Menger sponge ($D = 2.7268$, $R = \infty$)

- Fractal dimension is not the deciding factor for whether the transition takes place

- Other topological properties (lacunarity) may influence critical behavior of fractal lattices