$D$ constant depth + logical

$D$ dimensions $\Rightarrow \Theta_D$
constant depth & logical dimensions $\Rightarrow \Theta_D$ (Bravyi-König)

Pastawski-Yoshida
Pastawski-Beverland-König

Subsystem
No string operators
\[ \{ \sum_{n,k,d} \} \text{ std code} \]

code block \[ \sum_{i=1}^{m} UR \overline{U} \]
(Bravyi-König) subsystem
NO string operators
TQFT

[[n, k, d]] stabilizer code

Code block \[ U_1 R_i U_1^{-1} \]

\[ |R_i| \leq d \]
\[ \{\mathbf{I}_{n}, \mathbf{K_{d}}\} \text{ std code} \]

code block

\[ |R_{i}| < d \]

\[ \bigcup_{1}^{m} \mathbf{U}_{R_{i}} \mathbf{U}_{R_{i}}^{\dagger} \text{ correctable} \]
[[n, k, d]] stabilizer code

code block $\prod_{i=1}^{m} U_i R_i U R$

$|R_i| = d$

logical unitary $U = \prod_{i=1}^{m} U_i \otimes I$
[[n, k, d]] stack code

code block \( \prod_{i=1}^{m} U R \cdot U \bar{R} \)

\( |R_i| < d \)

Logical unitary \( U = \prod_{i=1}^{m} U_i \otimes I \)

\( U \in O_{m-1} \)

\( \mathcal{P}_0 = \text{id, Pauli} \)

\( \mathcal{P}_1 = \text{Clifford} \) x phase
\[ \mathcal{P}_e = \{ U : \forall u, uPUU^{-1} \in \mathcal{R}_{d-1}, \forall p \in \mathcal{R}_3 \} \]
\[ P_e = \{ U : UPU^{-1} \in P_{k-1}, \forall P \in P_3 \} \]

Idea: Cleaning Lemma
\[ P_e = \{ U : UPU^{-1} = P_{e-1}, VPUV^{-1}P = P_3 \} \]

Idea: Cleaning Lemma

R correctable

Logical Pauli $P \rightarrow P' = PS$, Set 6
\[ P_e = \{ U : UPU^{-1} < P_{k-1}, \forall P \in P \} \]

**Idea: Cleaning Lemma**

- \( R \) correctable
- Logical Implication: \( P \rightarrow P' = PS, S \subseteq F \)
- \( P' \) unsupported on \( R \)
  \[ P' = I \otimes P \]
$R$ correctable

Logical rank $p \rightarrow p' = pS$, $S_{estab}$

$p'$ unsupported on $R$ $p' = I_p \otimes P$

Logical operator $U$ supported on $R$
$P = \text{any logical Pauli, unsupported on } R$

$U = U_1 \otimes I$

$P = I \otimes \overline{P}$
\( P = \text{any logical Pauli, unsupported on R} \)

\[
U = U_1 \otimes I \quad UPU^{-1} = P
\]

\[
P = I \otimes P
\]
$P$ = any logical Pauli, unsupported on $R$

$U = U_1 \otimes I$
$U P U^{-1} = P$

$P = I \otimes \overline{P}$  $\Rightarrow$  $U = \text{id}_P \times \text{phase}$

Support of $U$ is $R_1 U R_2 (U \overline{U})$

$U = U_1 \otimes \overline{U}_2 \otimes I$
$P = \text{any logical Pauli, unsupported on } R$

$U = U_1 \otimes I \quad UPU^{-1} = P$

$P = I_1 \otimes P \quad \Longrightarrow \quad U = \text{Id}_q \times \text{phase}$

\underline{Support of } U \text{ is } R_1UR_z(\tilde{U} \tilde{N})$

$U = U_1 \otimes U_z \otimes I \quad P \text{ cleaned on } R_z$

$P = P_1 \otimes I_z \otimes P$
$U E U^{-1} P^{-1} = \text{logical}$

and supported on $R$.

$= id x \text{phase}$

$U E U^{-1} = \text{phase } P$

$P^2 = U E U^{-1} = (\text{phase})^2 P^2 = \text{phase } = \pm 1$
\[ U P U^{-1} = \text{logical} \quad \text{and supported on } R, \]

\[ = 1_d \times \text{phase} = \]

\[ U P U^{-1} = \text{phase } P \]

\[ P^2 = U P U^{-1} = (\text{phase })^2 P^2 \quad \Rightarrow \quad \text{phase} = \pm 1 \]

\[ \Rightarrow \quad 1 \text{ is in } \mathbb{Q} \]
\[ U_{\text{logical}} = U_{i} \otimes U_{\text{Pauli}} \cdot T \]

\[ P_{\text{cleaniton}} \otimes R_{3} \]

\[ UPU^{-1} = \text{local, supported} \]

\[ UPU^{-1} = \text{phase} \times \text{Pauli} \]
logical

\[ U \text{ clean on } \mathbb{R}^3 \]

\[ UPU^{-1} \text{ supported on } \mathbb{R}^{1,1} \]

\[ UPU^{-1} = \text{ phase } \times \text{ Pauli } \Rightarrow \text{ is logical} \]
\[ U \cdot C \rightarrow C' \]

\[ Q \text{ Reed-Muller codes} \]

\[ [2^m-1,1,37] \]

\[ [7,1,37] \]

\[ \bar{z} = z \otimes 7 \]

\[ \bar{x} = x \otimes 7 \]

\[ \text{XXXII} \]

\[ \text{IIIxxx} \]
$$U^\dagger U' = \text{logical and supported on } R \dagger$$

$$= \text{id} \otimes \text{phase} = 1 \otimes \text{phase}$$

$$U^\dagger U' = \text{phase } \rho = (\text{phase})^2 \rho^2 \longrightarrow \text{phase} = \pm 1$$

$$\Rightarrow \text{is in } \theta$$
\[ 2^{m-1} + 2^{m-2} + 2^{m-3} + \ldots + 1 = m \text{ correctable sets} \]
\[ 2^{m-1} + 2^{m-2} + 2^{m-3} + \cdots + 1 = m \text{ correctable sets} \]

\[ \mathbf{\Theta}_{m-1} \] 
\[ \text{diag} \left( 1, e^{\frac{\pi i}{7m-2}} \right) \in P_{m-1} \]

\[ [215, 1, 37] \]
\[ 2^m \times \left( \frac{\pi}{2m-2} \right) e^{\frac{\pi}{2m-2}} \in \mathbb{P}_{m-1} \]

\[ \text{Subsystem codes (Pastawski-Yoshida)} \]

\[ [15, 1, 37] \]
Logical ops:

- "bare": logical only
- "dressed": act on logical + gauge
Logical Ops:

- "bare" logical only
- "dressed" act on logical + gauge

Cleaning Lemma

bare'' logical only

dressed': act on logical + gauge

Cleaning Lemma

bare(R) + dressed
Logical ops: 

- bare" logical only
- dressed" act on logical + gauge

Cleaning Lemma: \[ |\text{bare}(R) + |\text{dressed}(R')| = 2K \]
\( r \text{ correctable} \implies \text{laressed } (R) = 0 \implies \lambda (R^\dagger) = 2 \text{ (save) } \)
\[ \text{correctable} \Rightarrow \text{laressed } l \Rightarrow l(1R^9) = 0 \Rightarrow l \text{ cleanable (laorealgal) are can be cleaned on } R \text{ with stats} \]
\[ \text{rectangle} \Rightarrow \text{dressed}(R) = 0 \Rightarrow \text{dressed}(R^\mathbb{C}) = 2 \mathbb{K} \]

- davi cLEANABLE (dale logical)
- \(|R| < d\)
- \(\text{dressed} (R) = 0 \Rightarrow \text{dressed} (R^\mathbb{C}) = 2 \mathbb{K}\)
- \(\text{dressed cleanable} \) (can clean on \(R\) with gauge generator)
Coded block: \( U R_i U R \)  \( i = 1 \)

Dressed logical \( U \) supported on

\( R_1 \) is correctable

\( R_2 \); \( P_m \) are dressed cleanable
Coded block = \( UR_i UR \)

\( R_1 \) is correctable

\( R_2, P_m \) are dressed cleanable

Dressed logical \( U \) supported on

\( = U_{\log} \oplus U_{\text{gauge}} \)

\( \Rightarrow U_{\log} \in P_{m-1} \)
supported on correctable \( R \)

\[ \Rightarrow U = \ldots \]
U supported on correctable \( R \)

\[ \Rightarrow \quad U = \text{phase \times \text{dly}} \]

\( R \), dressed cleanable

\[ U = U_1 \otimes I = U_{\text{log}} \otimes U_{\text{gauge}} \]

Dressed Pauli logical op

\[ P = I \times \overline{P} \]

\[ \overline{U} \cdot U = \overline{P} = U_{\text{log}} \otimes U_{\text{log}} \otimes U_{\text{gauge}} \otimes U_{\text{gauge}} \]
Union Lemma:

Dimensions

Geom local stab generators
Local subsystem code

Gauge generators are geom local

Union of separated dressed cleanable sets is dressed cleanable
\( UPU'P' \)

gate has range \( r \)  
depth \( h \)  
supported  
\( \delta \)  
\( \eta \)  
\( rh \)
D \dim \quad \text{Fattened triangulation}

\[ \Rightarrow \quad D+1 \]

\text{corrected regions}

\text{(stabilisation)}

---

D+1 \text{dressed cleanable sub-system} \quad \mathcal{P}_D^{D+1}
gate has range \( r \)
depth \( h \)
supported
\[ \delta_y \cdot rh \]

1 core cleanable + 0 dressed cleanable \( \Rightarrow P_y \)
Distance $\approx \text{dist}(0, E) \cdot \log n$

$[1, 1, 5, 1, 3, 1]$

Subsystem codes

$K$-local + $g$ $g$
Distance $\approx \text{const}(0, \infty)$ log $n$

[$[15, 1, 3]$]

Subsystem codes (Pastawski-Hayden)

$K$ logical + $g$ gauge
Distance \geq \exp (D, \Sigma) \log n

Subsystem codes (Pastaw)
K+ logical + g-gauge
Distance $\geq \text{antist}(0, \varepsilon) \log\left[\frac{1}{15,1,3}\right]$

Subsystem codes (Pastaw)

$K$ logical + $g$ gauge

[Diagram of quantum circuit or code]

Twisted triangulation