Warm up: zeta regularized determinants

- Given a sequence

\[ 0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty \quad \text{spec(\Delta)} \]

How one defines \( \prod \lambda_i = \det \Delta \)?

- Define the spectral zeta function:

\[
\zeta_\Delta(s) = \sum \frac{1}{\lambda_i^s}, \quad \Re(s) \gg 0
\]

Assume: \( \zeta_\Delta(s) \) has meromorphic extension to \( \mathbb{C} \) and is regular at 0.

- Zeta regularized determinant:

\[
\prod \lambda_i := e^{-\zeta_\Delta'(0)} = \det \Delta
\]
Holomorphic determinants

- Example: For Riemann zeta function, $\zeta'(0) = -\log \sqrt{2\pi}$. Hence

\[ 1 \cdot 2 \cdot 3 \cdot \ldots = \sqrt{2\pi}. \]

- Usually $\Delta = D^* D$. The determinant map $D \leadsto \sqrt{\det D^* D}$ is not holomorphic. How to define a holomorphic regularized determinant? This is hard.

- Quillen’s approach: based on determinant line bundle and its curvature, aka holomorphic anomaly.
Curved noncommutative tori $A_\theta$

$$A_\theta = C(\mathbb{T}_\theta^2) = \text{universal } C^*\text{-algebra generated by unitaries } U \text{ and } V$$

$$VU = e^{2\pi i \theta} UV.$$ 

$$A_\theta^\infty = C^\infty(\mathbb{T}_\theta^2) = \left\{ \sum_{m,n \in \mathbb{Z}} a_{m,n} U^m V^n : (a_{m,n}) \text{ Schwartz class} \right\}.$$
Differential operators \( \delta_1, \delta_2 : \mathcal{A}_\theta^\infty \to \mathcal{A}_\theta^\infty \)

\[
\begin{align*}
\delta_1(U) &= U, & \delta_1(V) &= 0 \\
\delta_2(U) &= 0, & \delta_2(V) &= V
\end{align*}
\]

Integration \( \varphi_0 : \mathcal{A}_\theta \to \mathbb{C} \) on smooth elements:

\[
\varphi_0\left( \sum_{m,n \in \mathbb{Z}} a_{m,n} U^m V^n \right) = a_{0,0}.
\]

Complex structures: Fix \( \tau = \tau_1 + i \tau_2, \quad \tau_2 > 0 \). Dolbeault operators

\[
\partial := \delta_1 + \tau \delta_2, \quad \partial^* := \delta_1 + \bar{\tau} \delta_2.
\]
Conformal perturbation of the metric (Connes-Tretkoff)

- Fix $h = h^* \in A_\theta^\infty$. Replace the volume form $\varphi_0$ by $\varphi : A_\theta \to \mathbb{C}$,

$$\varphi(a) := \varphi_0(ae^{-h}).$$

- It is a twisted trace (KMS state):

$$\varphi(ab) = \varphi(b\Delta(a)),$$

where

$$\Delta(x) = e^{-h}xe^h.$$
Perturbed Dolbeault operator

- Hilbert space $\mathcal{H}_\varphi = L^2(A_\theta, \varphi)$, \textit{GNS} construction.

- Let $\partial_\varphi = \delta_1 + \tau \delta_2 : \mathcal{H}_\varphi \rightarrow \mathcal{H}^{(1,0)}$,

  $\partial_\varphi^* : \mathcal{H}^{(1,0)} \rightarrow \mathcal{H}_\varphi$.

  and $\triangle = \partial_\varphi^* \partial_\varphi$, \textit{perturbed non-flat Laplacian}.
Scalar curvature for $A_\theta$

- Gilkey-De Witt-Seeley formulae in spectral geometry motivates the following definition:

The scalar curvature of the curved nc torus $(A_\theta, \tau, h)$ is the unique element $R \in A_\theta^\infty$ satisfying

$$\text{Trace}(a\Delta^{-s})|_{s=0} + \text{Trace}(aP) = \varphi_0(aR), \quad \forall a \in A_\theta^\infty,$$

where $P$ is the projection onto the kernel of $\Delta$.

- In practice this is done by finding an asymptotic expansion for the kernel of the operator $e^{-t\Delta}$, using Connes’ pseudodifferential calculus for nc tori.
Final formula for the scalar curvature (Connes-Moscovici; Fathizadeh-K)

**Theorem:** The scalar curvature of $(A_\theta, \tau, k)$, up to an overall factor of $\frac{\pi}{\tau^2}$, is equal to

$$
R_1(\log \Delta)(\triangle_0(\log k)) +
R_2(\log \Delta_{(1)}, \log \Delta_{(2)})(\delta_1(\log k)^2 + |\tau|^2\delta_2(\log k)^2 + \tau_1 \{\delta_1(\log k), \delta_2(\log k)\}) +
iW(\log \Delta_{(1)}, \log \Delta_{(2)})(\tau_2 [\delta_1(\log k), \delta_2(\log k)])
$$
where

\[ R_1(x) = -\frac{1}{2} - \frac{\sinh(x/2)}{x} \frac{1}{\sinh^2(x/4)}. \]

\[ R_2(s, t) = (1 + \cosh((s + t)/2)) \times \]
\[ \frac{-t(s + t) \cosh s + s(s + t) \cosh t - (s - t)(s + t + \sinh s + \sinh t + \sinh(s + t))}{st(s + t) \sinh(s/2) \sinh(t/2) \sinh^2((s + t)/2)}, \]

\[ W(s, t) = -\frac{(-s - t + t \cosh s + s \cosh t + \sinh s + \sinh t - \sinh(s + t))}{st \sinh(s/2) \sinh(t/2) \sinh((s + t)/2)}. \]
Holomorphic determinants

- **Logdet is not a holomorphic function.** How to define a holomorphic determinant $\det : \mathcal{A} \to \mathbb{C}$. 

- Quillen’s approach: based on determinant line bundle and its curvature, aka holomorphic anomaly.

- Recall: Space of Fredholm operators:

  $$F = \text{Fred}(H_0, H_1) = \{ T : H_0 \to H_1; \ T \text{ is Fredholm} \}$$

  $$K_0(X) = [X, F], \quad \text{classifying space for K-theory}$$
The determinant line bundle

- Let $\lambda = \wedge^{\text{max}}$ denote the top exterior power functor.

- **Theorem (Quillen) 1)** There is a holomorphic line bundle $\text{DET} \to F$ s.t.
  $$(\text{DET})_T = \lambda(\ker T)^* \otimes \lambda(\ker T^*)$$
Cauchy-Riemann operators on $\mathcal{A}_\theta$

- Families of spectral triples

$$\mathcal{A}_\theta, \quad \mathcal{H}_0 \oplus \mathcal{H}^{0,1}, \quad \begin{pmatrix} 0 & \bar{\partial}^* + \alpha^* \\ \bar{\partial} + \alpha & 0 \end{pmatrix},$$

with $\alpha \in \mathcal{A}_\theta$, $\bar{\partial} = \delta_1 + \tau \delta_2$.

- Let $\mathcal{A} =$ space of elliptic operators $D = \bar{\partial} + \alpha$.

- Pull back DET to a holomorphic line bundle $\mathcal{L} \to \mathcal{A}$ with

$$\mathcal{L}_D = \lambda(\text{Ker}D^*) \otimes \lambda(\text{Ker}D^*).$$
From det section to det function

- If $\mathcal{L}$ admits a \textit{canonical global holomorphic frame} $s$, then

$$\sigma(D) = \text{det}(D)s$$

defines a holomorphic determinant function $\text{det}(D)$. A canonical frame is defined once we have a canonical flat holomorphic connection.
Quillen’s metric on $\mathcal{L}$

- Define a metric on $\mathcal{L}$, using regularized determinants. Over operators with $\text{Index}(D) = 0$, let

$$||\sigma||^2 = \exp(-\zeta_\Delta'(0)) = \det \Delta, \quad \Delta = D^* D. \quad (1)$$

- Prop: This defines a smooth Hermitian metric on $\mathcal{L}$.

- A Hermitian metric on a holomorphic line bundle has a unique compatible connection. Its curvature can be computed from

$$\bar{\partial} \partial \log ||s||^2, \quad (2)$$

where $s$ is any local holomorphic frame.
Connes’ pseudodifferential calculus

- To compute this curvature term we need a powerful pseudodifferential calculus, including logarithmic pseudos.
- Symbols of order $m$: smooth maps $\sigma : \mathbb{R}^2 \to A^\infty_\theta$ with

$$||\delta^{(i_1,i_2)} \partial^{(j_1,j_2)} \sigma(\xi)|| \leq c(1 + |\xi|)^{m-j_1-j_2}.$$ 

The space of symbols of order $m$ is denoted by $S^m(A_\theta)$. 
Classical symbols

- Classical symbol of order $\alpha \in \mathbb{C}$:

$$\sigma \sim \sum_{j=0}^{\infty} \sigma_{\alpha-j} \quad \text{ord} \sigma_{\alpha-j} = \alpha - j.$$ 

$$\sigma(\xi) = \sum_{j=0}^{N} \chi(\xi) \sigma_{\alpha-j}(\xi) + \sigma^N(\xi) \quad \xi \in \mathbb{R}^2.$$ 

- We denote the set of classical symbols of order $\alpha$ by $S_{cl}^{\alpha}(A_{\theta})$ and the associated classical pseudodifferential operators by $\Psi_{cl}^{\alpha}(A_{\theta})$. 
A cutoff integral

- Any pseudo $P_\sigma$ of order $<-2$ is trace-class with

$$\text{Tr}(P_\sigma) = \varphi_0 \left( \int_{\mathbb{R}^2} \sigma(\xi) d\xi \right).$$

- For $\text{ord}(P) \geq -2$ the integral is divergent, but, assuming $P$ is classical, and of non-integral order, one has an asymptotic expansion as $R \to \infty$

$$\int_{B(R)} \sigma(\xi) d\xi \sim \sum_{j=0, \alpha-j+2 \neq 0}^{\infty} \alpha_j(\sigma) R^{\alpha-j+2} + \beta(\sigma) \log R + c(\sigma),$$

where $\beta(\sigma) = \int_{|\xi|=1} \sigma_{-2}(\xi) d\xi = \text{Wodzicki residue of } P$ (Fathizadeh).
The Kontsevich-Vishik trace

▶ The cut-off integral of a symbol $\sigma \in S^\alpha_{cl}(\mathcal{A}_\theta)$ is defined to be the constant term in the above asymptotic expansion, and we denote it by $\int \sigma(\xi) d\xi$.

▶ The canonical trace of a classical pseudo $P \in \Psi^\alpha_{cl}(\mathcal{A}_\theta)$ of non-integral order $\alpha$ is defined as

$$\text{TR}(P) := \varphi_0 \left( \int \sigma_P(\xi) d\xi \right).$$

▶ NC residue in terms of TR:

$$\text{Res}_{z=0} \text{TR}(A Q^{-z}) = \frac{1}{q} \text{Res}(A).$$
Logarithmic symbols

- Derivatives of a classical holomorphic family of symbols like $\sigma(AQ^{-z})$ is not classical anymore. So we introduce the Log-polyhomogeneous symbols:

$$
\sigma(\xi) \sim \sum_{j \geq 0} \sum_{l=0}^{\infty} \sigma_{\alpha-j,l}(\xi) \log^l |\xi| \quad |\xi| > 0,
$$

with $\sigma_{\alpha-j,l}$ positively homogeneous in $\xi$ of degree $\alpha - j$.

- Example: $\log Q$ where $Q \in \Psi^q_c(A_\theta)$ is a positive elliptic pseudodifferential operator of order $q > 0$.

- Wodzicki residue: $\text{Res}(A) = \varphi_0(\text{res}(A))$,

$$
\text{res}(A) = \int_{|\xi|=1} \sigma_{-2,0}(\xi) d\xi.
$$
Variations of LogDet and the curvature form

- Recall: for our canonical holomorphic section $\sigma$,

$$\|\sigma\|^2 = e^{-\zeta_{\Delta,\alpha}'(0)}$$

- Consider a holomorphic family of Cauchy-Riemann operators $D_w = \bar{\partial} + \alpha_w$. Want to compute

$$\bar{\partial} \partial \log \|\sigma\|^2 = \delta_w \delta_w \zeta_{\Delta}'(0) = \delta_w \delta_w \frac{d}{dz} \text{TR}(\Delta^{-z})|_{z=0}.$$
The second variation of logDet

- Prop 1: For a holomorphic family of Cauchy-Riemann operators $D_w$, the second variation of $\zeta'(0)$ is given by:

$$\delta_{\bar{w}} \delta_{\bar{w}} \zeta'(0) = \frac{1}{2} \varphi_0 \left( \delta_{\bar{w}} D \delta_{\bar{w}} \text{res}(\log \Delta \ D^{-1}) \right).$$

- Prop 2: The residue density of $\log \Delta \ D^{-1}$:

$$\sigma_{-2,0}(\log \Delta \ D^{-1}) = \frac{(\alpha + \alpha^*) \xi_1 + (\bar{\tau} \alpha + \tau \alpha^*) \xi_2}{(\xi_1^2 + 2 \Re(\tau) \xi_1 \xi_2 + |\tau|^2 \xi_2^2)(\xi_1 + \tau \xi_2)} - \log \left( \frac{\xi_1^2 + 2 \Re(\tau) \xi_1 \xi_2 + |\tau|^2 \xi_2^2}{|\xi^2|} \right) \frac{\alpha}{\xi_1 + \tau \xi_2},$$

and

$$\delta_{\bar{w}} \text{res}(\log(\Delta)D^{-1}) = \frac{1}{2\pi \Im(\tau)} (\delta_{\bar{w}} D)^*.$$
Curvature of the determinant line bundle

- **Theorem** (A. Fathi, A. Ghorbanpour, MK.): The curvature of the determinant line bundle for the noncommutative two torus is given by

  \[ \delta_w \delta_w \zeta'(0) = \frac{1}{4\pi \Im(\tau)} \varphi_0 (\delta_w D(\delta_w D)^*) . \]

- **Remark:** For \( \theta = 0 \) this reduces to Quillen’s theorem (for elliptic curves).
A holomorphic determinant a la Quillen

- Modify the metric to get a flat connection:

\[ ||s||_f^2 = e^{||D-D_0||^2} ||s||^2 \]

- Get a flat holomorphic global section. This gives a holomorphic determinant function

\[ \text{det}(D, D_0) : \mathcal{A} \to \mathbb{C} \]

It satisfies

\[ |\text{det}(D, D_0)|^2 = e^{||D-D_0||^2} \text{det}_\zeta(D^* D) \]