Introduction

Our goal is to describe shape by spectrum. There is a slight problem, though.

**Spectral Geometry is hard**

Reasons:
- The relationship between shape and spectrum is intractable
- The relationship between shape and spectrum is nonlinear

Intuition can be gained by:
- Using numerical methods
- Linearizing and inverting locally
Infinitesimal Inverse Spectral Geometry

Let $\mathbb{R}^{n_{dof}}$ parametrize the space of shapes $\mathcal{G}$. Suppose the first $n_{ev}$ eigenvalues of the studied operator(s) can be computed and stored in $\mathbb{R}^{n_{ev}}$. Define the spectral map:

$$\sigma : \mathbb{R}^{n_{dof}} \rightarrow \mathbb{R}^{n_{ev}}$$

Suppose $A, B \in \mathbb{R}^{n_{dof}}$ are the initial and target shapes.

**Goal:** Build a path $P(t)$ from $A$ to $B$ using $\sigma$
Infinitesimal Inverse Spectral Geometry

The path must be built using only:

- The target spectrum $\sigma(B)$
- The behaviour of $\sigma$ near the current shape $P(t)$

We can not guarantee that $\sigma(B)$ will be reached. A milder condition is that

$$|\sigma(P(t_2)) - \sigma(B)| \leq |\sigma(P(t_1)) - \sigma(B)| \text{ for } t_2 > t_1$$
Gradient Descent

Simplest implementation of the above criteria. Define 
\[ v_\sigma = \sigma(B) - \sigma(P). \]

\[ \frac{d}{dt} P(t) = -\frac{1}{2} \text{grad} |v_\sigma|^2 \]

Let \( \mathcal{J}(P) \) be the Jacobian matrix of \( \sigma \) at \( P \). Then the above can be rewritten as

\[ \frac{d}{dt} P = \mathcal{J}^T(P)v_\sigma \]

Problem: No notion of local inversion.
Pseudoinverse Method

Let \( \mathcal{J}^+ \) denote the pseudoinverse of the Jacobian matrix of \( \sigma \). An improvement to the gradient method is given by:

\[
\frac{d}{dt} p = \mathcal{J}^+(p) v_{\sigma}
\]

This conceptually realizes the idea of local inversion of the spectral map.
Numerical Implementation

- Integrate numerically until the desired spectrum is achieved or the distance to the target spectrum no longer decreases. (Finite element solver: FreeFem++)
- Compare the final spectrum to $\sigma(B)$
- Compare the final shape to $B$
  - Requires a metric on the space of shapes
  - Requires a well understood space of shapes
Star-Shaped Domains with Bandlimited Boundary

Domains in $\mathbb{R}^2$ star-shaped about the origin:

$$r(\phi) = a + b \exp \left( C_0 + \sum_{i=1}^{N} [C_i \cos(i\phi) + S_i \sin(i\phi)] \right)$$

**Figure**: Examples of star-shaped domains with bandlimited boundary.
Star-Shaped Domains with Bandlimited Boundary

A metric on this set of shapes is given by the Hausdorff distance between compact subsets of the plane. Let $d(x, y)$ be the distance between $x, y \in \mathbb{R}^2$. Let $X, Y \subset \mathbb{R}^2$ be compact.

$$d_G(X, Y) = \max \left\{ \sup_{x \in X} \inf_{y \in Y} d(x, y), \sup_{y \in Y} \inf_{x \in X} d(x, y) \right\}$$

This distance can be minimized over the isometries of the plane (translations, rotations and reflexions) to yield a distance on isometry equivalence classes of shapes. Denote this distance by $d_{[G]}(\cdot, \cdot)$. 
Example 1

Figure: Example of success for $n_{dof} = 11, n_{ev} = 31$. 
Example 2

Figure: Example of success for $n_{dof} = 11$, $n_{ev} = 25$. 
Results

The algorithm is run for random pairs \((A, B)\) of initial and target shapes. Four possible outcomes:

- Found the target shape and target spectrum \(\rightarrow\) Success
- Failed to find both the shape and the spectrum \(\rightarrow\) Failure, local optimum
- Found the right spectrum but the wrong shape \(\rightarrow\) Failure, potential counterexample
- Found the right shape, but wrong spectrum \(\rightarrow\) Failure, numerical artifact

Success rates can be computed
Figure: Success Rate for $n_{dof} = 11$ and $n_{ev} = 40$ as a function of the starting shape distance $d_{[G]}(A, B)$. 
Figure: Success Rate for $d_{G}(A, B)$ between 0 and $d$ for $n_{dof} = 11$ and various $n_{ev}$ as a function of the starting shape distance $d_{[G]}(A, B)$. 

Success Rates
Success Rates

Figure: Success Rate for $d_g(A, B)$ between 0 and $d$ for $n_{dof} = 11$ and various $n_{ev}$ as a function of the starting shape distance $d_{[Y]}(A, B)$. 
**Figure**: Proportion of isometric runs among isospectral ones for $n_{dof} = 11$. 
Summary

- Local reconstruction of shape from spectrum is possible
- Long distance reconstruction is hard, but sometimes possible
- Counterexamples to our program seem rare
Conjectures:

- The pseudoinverse approach works in general
- The studied space of shapes is special
The determinant line bundle

- Let $\lambda = \Lambda^\text{max}$ denote the top exterior power functor.

- **Theorem (Quillen)** 1) There is a holomorphic line bundle $\text{DET} \to F$ s.t.
  \[(\text{DET})_T = \lambda(\text{Ker}T)^* \otimes \lambda(\text{Ker}T^*)\]

2) There map $\sigma : F_0 \to \text{DET}$

  \[\sigma(T) = \begin{cases} 
  1 & T \text{ invertible} \\
  0 & \text{otherwise} 
  \end{cases}\]

is a holomorphic section of $\text{DET}$ over $F_0$. 

Cauchy-Riemann operators on $\mathcal{A}_\theta$

- Families of spectral triples

$$\mathcal{A}_\theta, \quad \mathcal{H}_0 \oplus \mathcal{H}^{0.1}, \quad \begin{pmatrix} 0 & \bar{\partial}^* + \alpha^* \\ \bar{\partial} + \alpha & 0 \end{pmatrix},$$

with $\alpha \in \mathcal{A}_\theta$, $\bar{\partial} = \delta_1 + \tau \delta_2$.

- Let $\mathcal{A} =$ space of elliptic operators $D = \bar{\partial} + \alpha$. 