Time reversal invariant gapped boundaries of the double semion state

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Overview:

Two kinds of "fractionalization":

• Topological Order

• Symmetry Protection (SPT)

Symmetry-protected gapless edge modes
How to combine these?
Symmetry Enriched Topological phases

Anyons in 2D

Symmetry protection

Anyons in 2D with fractional symmetry
- FQHE (fractional charge)
- Spin liquid (fractional spin)
The question

2D SPT with discrete symmetry $\mathbb{Z}_2 \times \mathbb{Z}_2^T$ \hspace{1cm} \rightarrow \hspace{1cm} 2D SET \hspace{1cm} \text{Gauge } \mathbb{Z}_2$

- This is certainly one way to get a 2D SET
- But is the mapping one to one?
The question

2D SPT with discrete symmetry $\mathbb{Z}_2 \times \mathbb{Z}_2^T$ ➔ 2D SET

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The question

2D SPT with discrete symmetry $\mathbb{Z}_2 \times \mathbb{Z}_2^T$ \hspace{3cm} \rightarrow \hspace{3cm} 2D SET

Gauge $\mathbb{Z}_2$

- This is certainly one way to get a 2D SET
- But is the mapping one to one?
Phases with $\mathbb{Z}_2 \times \mathbb{Z}_2^T$ symmetry

- Ordinary Ising paramagnet, $T^2(\text{vortex}) = 1$
- Ordinary Ising paramagnet, $T^2(\text{vortex}) = -1$
- Twisted Ising paramagnet (Levin & Gu, ’12) $T^2(\text{vortex}) = 1$
- Twisted Ising paramagnet $T^2(\text{vortex}) = -1$
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Gaunging

Ordinary Ising paramagnet,
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→

Gauge $\mathbb{Z}_2$

Toric code
$(T^2 = 1)$

Ordinary Ising paramagnet,
$T^2(\text{vortex}) = -1$

→

Gauge $\mathbb{Z}_2$

Toric code
$T^2(\text{vortex}) = -1$

• Two different SET phases!
Gauging

Twisted Ising paramagnet, 
$T^2(\text{vortex}) = 1$

Twisted Ising paramagnet, 
$T^2(\text{vortex}) = -1$

Same bulk phase! 
(with different boundaries…)

Gauge $\mathbb{Z}_2$

Gauge $\mathbb{Z}_2$
What’s the difference?

Gauging the twisted Ising paramagnet: doubled semion model

\[ v = 1/2 \]

\[ v = -1/2 \]
Where can we see $T^2 = \pm 1$?

- Bulk: $T|s\rangle \Rightarrow |\bar{s}\rangle$

- (Gapped) boundary
  (Levin ‘13)

![Diagram showing bulk and boundary states](image)
Where can we see $T^2 = \pm 1$?

- Bulk
  \[ T |s\rangle \Rightarrow |\bar{s}\rangle \]

- (Gapped) boundary
  \[ T |s\rangle \Rightarrow |\bar{s}\rangle = |b \times s\rangle \]

- $T^2 = \pm 1$ at the boundary!
Bulk phases vs boundary conditions

• $T^2 = \pm 1$ at the boundary!

• Does this indicate distinct bulk phases
  • For the twisted Ising paramagnet? Yes
  • For the doubled semion model? No
Phases with $\mathbb{Z}_2 \times \mathbb{Z}_2^T$ symmetry

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Outline of the rest of the talk

- 2 kinds of Bose condensates = 2 kinds of boundaries ($T^2 = \pm 1$)
- Why gauging matters
- Explicit construction
- Trijunctions and symmetry-protected degeneracies
2 kinds of Bose condensates

\[ |\Psi_{\text{BEC}}\rangle = \sum_{N} (e^{i\theta})^N \sum_{x_i=\text{position of } i^{th} \text{ boson}} |x_1, \ldots x_{2N}\rangle \]

- Time reversal: \( \theta = 0, \pi \)
2 kinds of Bose condensates

semion

T

antisemion = semion \times boson
2 kinds of Bose condensates

• $T^2$ creates 2 bosons  \[ T^2 = e^{i\theta} \]
The importance of gauging

\[ \mathcal{L}_{\text{edge}} = \frac{2}{4\pi} \partial_x \phi_1 \partial_t \phi_1 - \frac{2}{4\pi} \partial_x \phi_2 \partial_t \phi_2 - g \cos (2\phi_1 - 2\phi_2 + \theta) \]

\( \mathbb{Z}_2^T : \begin{cases} 
\phi_1 \rightarrow \phi_2, & \phi_2 \rightarrow \phi_1 & T^2 = 1 \\
\phi_1 \rightarrow \phi_2, & \phi_2 \rightarrow \phi_1 + \pi & T^2 = -1 
\end{cases} \)

- 2 possible time-reversal transformations, depending on which condensate (i.e. what gapping term we add to the edge)
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\[ \mathbb{Z}_2 : \phi_1 \to \phi_1 + \pi , \quad \phi_2 \to \phi_2 \]

- 2 possible time-reversal transformations are related by \( \mathbb{Z}_2 \) transformation
- If \( \mathbb{Z}_2 \) is global, they are different (2 distinct SPT's)
- If \( \mathbb{Z}_2 \) is gauged, this is just a gauge transformation and they must be the same (only 1 SET!)
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Doubled semion model

(Levin & Wen)

\[ H = - \sum_v A_v + \sum_P B_P \]

• Vertices:
  \[ A_v = \prod \sigma^z \]

• Plaquettes:
  \[ B_p = \prod \sigma^x \times \text{Phase terms} \]

\[ \begin{align*}
  &\begin{array}{c}
  \text{Vertex:}
  \end{array} \\
  &\begin{array}{c}
  \text{Plaquette:}
  \end{array}
\end{align*} \]
Doubled semion model

\[ H = - \sum_v A_v + \sum_P B_P \]

\[ |\Psi_0\rangle = \sum_{C=\{\text{loops}\}} (-1)^{N_{\text{loops}}} |C\rangle \]

\[ \quad \quad - \quad + \quad + \quad + \ldots \]
Doubled semion model

\[ B_p = \prod \sigma^x \times \text{Phase terms} \]

- **Boson pair creation**: (squares to +1)
  - \( \square = \sigma^z \)

- **Boson condensation (1)**:

  \[
  H = - \sum_v A_v + h \sum_P B_P - J \sum_e \sigma^z_e \\
  \text{as } J/h \to \infty \Rightarrow - \sum_e \sigma^z_e
  \]

Trivial ground state with no loops
Gapped boundary 1

- Bose condensed region (no loops)
- Semions at boundary: no degeneracy
- Doubled semion region \((-1)^{N_{\text{loops}}\bar{N}}\)

\[ Z \quad A_v \quad B_p \]
Gapped boundary 1

Bose condensed region (no loops)

Semions at boundary: no degeneracy

Doubled semion region \((-1)^{N_{\text{loops}}}\)

\(Z\)

\(A_v\)

\(B_p\)
Gapped boundary 2

\[ B_{P_1} \sigma_{12}^{\bar{z}} B_{P_2} \]

- squares to +1
- Boson pair creation/annihilation: (-1)
- Boson hopping: +1
Gapped boundary 2

Bose condensed region (now with loops!)

Doubled semion region \((-1)^{N_{\text{loops}}}\)
The semion at the edge

- Problem: now we can’t terminate the semion string without breaking T, unless we also remove some terms from the Hamiltonian!

- Semions+ $T = $ Kramers degeneracy!

- Can show: $T^2 = -1$
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The semion at the edge

- Problem: now we can’t terminate the semion string without breaking $T$, unless we also remove some terms from the Hamiltonian!

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Part 3: Tri-junctions

\[ T^2 = 1 \text{ semions} \]
\[ \alpha = 0 \quad \alpha = \pi \]
\[ T^2 = -1 \text{ semions} \]

Gapped, T-invariant boundary
Tri-junctions

- Time reversal relates 2 states which are not connected by a local operator (but by an extended semion string)
- Tri-junctions with time-reversal have degeneracy (even without semions)
Tri-junctions

\[ |A\rangle \text{ or } |\overline{A}\rangle = g|A\rangle \quad \text{BEC 1} \]
\[ |B\rangle \text{ or } |\overline{B}\rangle = g|B\rangle \quad \text{BEC 2} \]

Gapped edges:
\[ |AB\rangle + |\overline{AB}\rangle \]
\[ |A\overline{B}\rangle + |\overline{AB}\rangle \]
Tri-junctions

\[ |A\rangle \equiv |\phi_1 - \phi_2 = 0\rangle, \quad |B\rangle \equiv |\phi_1 - \phi_2 = \pi/2\rangle \]

\[ |\overline{A}\rangle \equiv |\phi_1 - \phi_2 = \pi\rangle, \quad |\overline{B}\rangle \equiv |\phi_1 - \phi_2 = -\pi/2\rangle \]

\[ \mathbb{Z}_2^T : \begin{cases} \phi_1 \to \phi_2, \quad \phi_2 \to \phi_1 & |A\rangle \to |A\rangle, \quad |B\rangle \to |\overline{B}\rangle \\ \phi_1 \to \phi_2, \quad \phi_2 \to \phi_1 + \pi & |A\rangle \to |\overline{A}\rangle, \quad |B\rangle \to |B\rangle \end{cases} \]
Tri-junctions

\[ T(|AB\rangle + |\overline{AB}\rangle) = |AB\rangle + |\overline{AB}\rangle \]

- 2-fold degeneracy for each pair of domain walls between BEC1 and BEC2

- Operator mapping between them: \( |B\rangle \rightarrow g|B\rangle \)
  tunnels a semion between the two boundaries
Summary

• 2 SPT’s -> 1 SET

• 2 “different” kinds of gapped boundary for the doubled semion model

• …which are in the same phase but lead to different time-reversal transformations of boundary semions

• (Gapped) domains between these 2 gapped boundaries have an extra T-protected degeneracy