Abstract: The index of rigidity was introduced by Katz as the Euler characteristic of de Rham cohomology of End-connection of a meromorphic connection on curve. As its name suggests, the index evaluates the rigidity of the connection on curve. Especially, in P^1 case, this index makes a significant contribution together with middle convolution. Namely Katz showed that regular singular connection on P^1 can be reduced to a rank 1 connection by middle convolution if and only if the index of rigidity is 2. After that, the work of Crawley-Boevey gave an interpretation of the index of rigidity and the Katz' algorithm from the theory of root system. Namely, he gave a realization of moduli spaces of regular singular connections on a trivial bundle as quiver varieties. In this setting the index of rigidity can be naturally computed by the Euler form of quiver, and the Katz algorithm can be understood as a special example of the theory of Weyl group orbits of positive roots of the quiver. I will give an overview of this story with a generalization to the case of irregular singular connections. Moreover, I will introduce an algebraic curve associated to a linear differential equation on Riemann surface as an analogy of the spectral curve of Higgs bundle. And compare some indices of singularities of differential equation and its associated curve, Milnor numbers and Komatsu-Malgrange irregularities. Finally as a corollary of this comparison of local indices, I will give a comparison between cohomology of the curve and de Rham cohomology of the differential equation and show the coincidence of the index of rigidity and the Euler characteristic of the associated curve.
On index of rigidity

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1. Katz' algorithm

\[ X : \text{proj. smooth curve} \]
\[ U \subset X : \text{open} \]
\[ j : U \to X \]
\[ L := (E_U, \nabla_U) : \text{connection on } U \]

**Def. (index of rigidity [Katz])**

\[ \text{rig}(L) := \chi(\ker (j^*)|_{\text{End}(L)}) \]
\[ = \frac{1}{2} \chi(1) \dim \mathcal{H}^1(X, j^*|\text{End}(L)) \]

\[ = 0 \]
**Fundamental Facts in $X=\mathbb{P}^1$ case**

**Thm. 1 (Katz, Bloch-Esnault)**
Suppose $L$ is irreducible.

$L$ is rigid $\iff$ $\text{rig} (L) = \lambda$

**Def. (Rigidity)**

$L$ is rigid $\iff$ If $\exists L', \forall x \in X \setminus \mathbb{V} \hat{K}_x$, $L \otimes \hat{K}_x = L' \otimes \hat{K}_x$

Field of formal Laurent series $\mathbb{C}(x)$

$\Rightarrow$ $L \cong L'$
**Fundamental Facts in $X = \mathbb{P}^1$ case**

**Thm. 1 (Katz, Bloch- Esnault)**

Suppose $L$ is irreducible.

$L$ is rigid $\iff$ $\text{rig}(L) = 2$

**Def. (Rigidity)**

$L$ is rigid $\overset{\text{def}}{\iff}$ $\exists L' s.t. L' \otimes \hat{k}_x \cong L \otimes \hat{k}_x \forall x \in X \setminus X' \setminus Y$

Field of formal Laurent

$\Rightarrow \ L \cong L'$


Thm 2. (Katz, Bloch-Esnault)
\[ \text{rig}(L) = \text{rig}(L_0) = \text{rig}(m_{Ca}(L)) \]

Fourier trans. \hspace{1cm} middle convolution.

(\underline{CF} (Eufer transform))
\[ I_x(t) := \frac{1}{2\pi i} \int_{x}^{x+i\infty} e^{-t^{2}-2\pi i t \alpha} \frac{\Gamma(\alpha)}{\Gamma(1+\alpha)} \frac{1}{\Gamma(\alpha)} \, dt = L^{-1} \mathcal{L}^{-1}(F) \]

Laplace trans.

Thm 3. (Katz, Deligne, Anintin)
\( L \) is \textbf{rigid}
\[ \Rightarrow \text{can be reduced to rank} 1 \text{ conn. by} \]

Fourier transf. \& middle convolution.
8.2. Moduli sp. of linear ODE on $\mathbb{P}^1$ & quiver variety

$$\frac{dY}{dx} = \sum_{i=1}^{r} \frac{A_i}{x-a_i} Y \quad (A_i \in \text{Mat}(n, \mathbb{C})) : \text{Fuchsian ODE}$$

$$A_0 := -\sum_{i=1}^{r} A_i : \text{residue at } x=\infty$$

$C_i : \text{conj. class of } A_i \quad (\cong \text{local isom class at } x=a_i)$

$$M_c := \mathbb{P}\left[ \sum_{i=1}^{r} \frac{A_i}{x-a_i} \right] \quad \text{irred. } A_i \in C_i \quad \mathbb{P}\left[ \text{GL}(n, \mathbb{C}) \right]$$

(moduli sp. of Fuchsian ODE)
**Theorem (Crawley-Boevey)**

\[ \alpha \in \mathbb{Z}_{\geq 0}, \beta \in \mathbb{Z} \text{ s.t.} \]

\[ M_{c} \cong M_{\alpha}^{\text{reg}}(\alpha, Q) \]

\[ := \left[ M_{\alpha}^{\text{sta}}(\lambda) / \Pi_{i \in \mathbb{Q}} \text{GL}(\mathbb{C} \alpha_{i}, \alpha_{i}) \right] \]

**Theorem**

For generic \( C = (C_{0}, \ldots, C_{r}) \) ("eigen-val of \( C_{i} \) have no integer difference is enough")

\[ \text{Ker}(A) = g(\alpha) \quad (\forall A \in M_{c}) \]

**Here**

\[ g(\alpha) := \sum \alpha_{i}^{2} - \sum \alpha_{i}^{\text{reg}}(\alpha_{i}) : \text{Tits form of } \alpha \]

**Theorem**

\( (C.-B.) \)

\[ M_{c} \xrightarrow{\text{mid}} M_{c}^{\prime} \]

\[ \exists \alpha \in W_{Q}. \]

\[ M_{\alpha}^{\text{reg}}(\alpha, Q) \rightarrow M_{\alpha}^{\text{reg}}(\alpha_{Q}, Q) \]
Fundamental Facts & root system

I. $\text{rigid} \iff \text{rig} = 2$

$\text{rig}(A) = 2 \iff \dim M_c = 0$

\[
\begin{align*}
2 - \text{rig}(A) &= 2 - q(\alpha) \\
&= \dim M_\alpha^{\text{rig}}(\alpha, q) = \dim M_c.
\end{align*}
\]

II. $\text{rig}(\mathbb{Z}) = \text{rig}(M_c(\mathbb{Z}))$

III. $\mathbb{Z}$: rigid $\iff$ rank 1 conn. by midd. conv.

$\text{rig}(A) = q(\alpha) = q(M_c(\alpha)) = \text{rig}(M_c(\mathbb{Z}))$

$\Rightarrow \alpha$: pos. real root

$\Rightarrow \exists w \in W_\alpha \subseteq \mathbb{Z}$,

$w(\alpha) = \varepsilon$ (simple root)
Non-Fuchsian case: (Boalch, H.-Yamakawa, H.)

\[ H_0 = \left( \begin{array}{cc} \alpha_1 1_{n_0} & 0 \\ \alpha_2 1_{n_0} & 0 \end{array} \right) \tilde{x}^2 + \left( \begin{array}{c} R_{01} \\ R_{02} \end{array} \right) \tilde{x}^{-1} \]  
\[ R_{ij} \in M(n_0, \mathbb{C}) \]

\[ H_{\infty} = \left( \begin{array}{cc} \beta_1 1_{n_0} & 0 \\ \beta_2 1_{n_0} & 0 \end{array} \right) \tilde{x}^2 + \left( \begin{array}{c} R_{\infty 1} \\ R_{\infty 2} \end{array} \right) \tilde{x}^{-1} \]

- Hakuhara-Tsuritani normal forms
- Level-\( n_0 \) normal forms

Suppose that

\[ Y = \left( \frac{A_{2}^{(o)}}{\tilde{x}^2} + \frac{A_{1}}{\tilde{x}} + A_{2}^{(m)} \right) Y \text{ has HTL-normal forms } H_0, H_{\infty}, \]

\[ A_{2}^{(o)} \tilde{x}^2 + A_{1} \tilde{x} \sim H_0, -A_{2}^{(m)} \tilde{x} - A_{1} \tilde{x} \sim H_{\infty} \]

\[ \exists \ G_{(o)}^{(o)} \in GL(n, \mathbb{C}) \text{ s.t.} \]

\[ G_{(o)}^{(o)} \left( A_{2}^{(o)} \tilde{x}^2 + A_{1} \tilde{x} \right) G_{(o)}^{(o)-1} = \left( \begin{array}{cc} \alpha_1 & 0 \\ \alpha_2 & 0 \end{array} \right) \tilde{x}^2 + \left( \begin{array}{c} R_{01} \\ R_{02} \end{array} \right) \tilde{x}^{-1} \]

Under the coordinate change, we may assume \( G_{(o)} = \text{Id} \), i.e.

\[ -A_{2}^{(m)} = \left( \begin{array}{cc} \beta_1 & 0 \\ \beta_2 & 0 \end{array} \right), \]  
\[ -A_{1} = \left( \begin{array}{cc} R_{\infty 1} & 0 \\ R_{\infty 2} & 0 \end{array} \right) \]
**Quiver Picture**

\[ \begin{array}{c}
\text{Note: Because } p^* = G^{-1} \in GL(n, \mathbb{C}), \\
\text{we need open condition } \det (p^*_{ij}) \neq 0
\end{array} \]
Non-Fuchsian case: (Boalch, H. Tanaka, H.)

\[ H_0 = \left( \begin{array}{cc} \alpha_1 & \text{In}_{n_1} \\ \alpha_2 & \text{In}_{n_2} \end{array} \right) \tilde{x}^2 + \left( \begin{array}{c} R_{0,1} \\ R_{0,2} \end{array} \right) \tilde{x}^1 \quad \text{Rij} \in \mathbb{M}(n \times n) \]

\[ H_{10} = \left( \begin{array}{cc} \beta_1 & \text{In}_{n_1} \\ \beta_2 & \text{In}_{n_2} \end{array} \right) \tilde{x}^2 + \left( \begin{array}{c} R_{0,1} \\ R_{0,2} \end{array} \right) \tilde{x}^1 \quad \text{Hakuhara-Tauritin} \]

- Level-1 normal forms

Suppose that

\[ \mathbf{Y} = \left( \begin{array}{c} A_2 \left( \frac{\alpha_1}{\alpha_2} + \frac{\alpha_1}{\alpha_2} \right) \\ - \frac{\alpha_1}{\alpha_2} + \frac{\alpha_1}{\alpha_2} \end{array} \right) \mathbf{Y} \text{ has HTL-normal forms } H_0, H_{10}, \]

\[ A_2 \left( \frac{\alpha_1}{\alpha_2} + \frac{\alpha_1}{\alpha_2} \right) \tilde{x}^2 + \frac{\alpha_1}{\alpha_2} \tilde{x}^1 \cong H_0, \quad - A_2 \left( \frac{\alpha_1}{\alpha_2} + \frac{\alpha_1}{\alpha_2} \right) \tilde{x}^2 - \frac{\alpha_1}{\alpha_2} \tilde{x}^1 \cong H_{10} \]

\[ \exists \mathbf{G}^{(e)} \in \text{GL}(n, \mathbb{C}) \text{ s.t.} \]

\[ \left( \begin{array}{c} \mathbf{G}^{(e)} \left( A_2 \left( \frac{\alpha_1}{\alpha_2} + \frac{\alpha_1}{\alpha_2} \right) \tilde{x}^2 + \frac{\alpha_1}{\alpha_2} \tilde{x}^1 \right) \mathbf{G}^{(e)}^{-1} = \left( \begin{array}{cc} \alpha_1 & \text{In}_{n_1} \\ \alpha_2 & \text{In}_{n_2} \end{array} \right) \tilde{x}^2 + \left( \begin{array}{c} R_{0,1} \\ R_{0,2} \end{array} \right) \tilde{x}^1 \end{array} \right) \]

Under the coordinate change, we may assume \( \mathbf{G}^{(e)} = \text{Id} \), i.e.

\[ - A_2^{(e)} = \left( \begin{array}{ccc} \beta_1 & \beta_2 & \beta_3 \\ \beta_2 & \beta_3 \end{array} \right), \quad - A_1 = \left( \begin{array}{ccc} R_{0,1} & 0 \\ 0 & R_{0,2} \end{array} \right) \]
Note 2. Dimension vector $\alpha$ should live in sublattice.
\[ \alpha \in \mathcal{L} := \{ \alpha \in \mathbb{Z}^\mathbb{Z} \mid \sum_{i=1}^{2} \alpha_{0,i} = \sum_{j=1}^{3} \alpha_{0,j} \} \]

\[ \alpha_0.1 + \alpha_0.2 + \alpha_{10.3} \]

**Rank of ODE**

Note 3. Full Weyl group does not act on this $L$.

\[ W_{\alpha} \not\subset L \]

\[ W_{\alpha} : \langle \varepsilon_{0,i} + \varepsilon_{0,j} : i,j = 1,2,3 \rangle \subset L \]
\textbf{Thm} (Bolach, H- Yamakawa, H.)

\[ H = (H_0, \ldots, H_r) \text{ : "unramified" HTL-normal forms.} \]

\[ \exists Q = (Q_0, Q_1), \exists L \subset \mathbb{Z}^{Q_0}, \exists \alpha \in L, \exists \lambda \in C^{Q_0} \text{ s.t.} \]

\[ M_H \xrightarrow{\exists} M^\text{reg}_{\lambda}(Q, \alpha) : \text{open dense} \]

\textbf{Thm}

\[ M^\text{mem}_{\lambda} \xrightarrow{\text{middle convolution}} M_{H_i} \]

\[ \Downarrow \]

\[ M^\text{reg}_{\lambda}(Q, \alpha) \rightarrow M^\text{reg}_{\lambda}(Q, \mu(\alpha)) \]

\[ \exists \bar{L} : \text{root lattice of Kac-Moody type s.t.} \]

\[ \exists \bar{L} : \bar{L} \rightarrow L \subset \mathbb{Z}^{Q_0} \]

\[ \bar{W}_L \rightarrow W^\text{mc}_{Q} < W_{Q} \]
Classification of middle convolution orbits.

\[ r_{iq} = 2 \]

\[ \text{only 1- orbit} \quad (\text{\textit{cf.} Katz' algorithm}) \]

\[ r_{iq} = 0 \]

(Fundamental set in \( \mathbb{Z} \))

\[
\begin{align*}
&0 \quad 2 \quad 3 \\
&1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \\
&1 \quad 2 \quad 3 \\
&0 \quad 1 \\
&\end{align*}
\]
\[ r_i q = -2 \]

Here \( a, b \in \mathbb{Z} \).
And MORE!!
3. Ramified irregular singularities of ODEs & singularities of plane curve germs

For
\[ f(t) \in \mathbb{C}[t^{-1}], \quad q \in \mathbb{Z}_{>0} \]

\[ E_{f^g} : \mathbb{C}(\mathbb{C}^{x^g}) \langle \partial_x \rangle - \text{module} \]

\[ E_{f^g} := \mathbb{C}(\mathbb{C}^{x^g}) \text{ as } \mathbb{C}(\mathbb{C}^{x^g}) - \text{vec. sp} \]

\[ J \cdot m := \frac{f(x^g)}{x^g} \cdot m \quad \text{for } m \in E_{f^g} \]

**Theorem (Hukuhara-Turrittin-Levelt)**

\[ M : \text{fin. dim. } \mathbb{C}(\mathbb{C}^{x^g}) \langle \partial_x \rangle - \text{module} \]

\[ M \cong \bigoplus_{i} \left( E_{f_{i^{g}}} \otimes \frac{R_{i}}{x^{g}} \right) \quad \text{for } R_{i} \in M(n^2, \mathbb{C}) \]

Here \( E_{f_{i^{g}}} \) is free & \( E_{f_{i^{g}}} \neq E_{f_{j^{g}}} \quad (i \neq j) \)
Def. (Local associated curve)
For $E_{\mathbf{g}}$, define
\[
C_{\mathbf{g}}(x, y) := \prod_{k=1}^{g} \left( y - \frac{\frac{2\pi i k}{x} y^k}{x} \right)
\]

Corresp. ODE to $E_{\mathbf{g}}$ has sol. $\exp \left( \int \frac{\frac{2\pi i k}{x} y^k}{x} \, dx \right)$

\[
\mathbf{u}_k(y) := \int e^{\frac{2\pi i k}{x} y^k} \, dx
\]

\[
= \int e^{F_k(x,y)} \, dx.
\]

Note: Locus of $C_{\mathbf{g}}$ is critical points of phase functions $F_k(x, y)$
Thm. Turn $y$ around $\infty$ with very small radius, critical points of $f_2(x, y) \Leftrightarrow \frac{\partial}{\partial x} f_2(x, y) = 0$ make a knot which is isomorphic to that of $C f_2$ around $(x, y) = (0, \infty)$.

Moreover, critical values $z = f_2(x, y)$ also make a knot on $x - z$ space which is isomorphic to that of $C f_2$ around $(x, y) = (0, \infty)$.

Thm (Milnor number vs Komoia-Malgrange irregularity)

$\mu(C f_2)_{(0, \infty)} = (2p+q-1)(q-1) - \text{Irr(End}(E f_2))$

Here $p = \deg \alpha^r \frac{f_2}{x}$
Global curve

$X$: compact Riemann surface
$
\nabla$: meromorphic connection on trivial bundle $\mathcal{L}/X$

$\nabla = \omega - A(\omega) \, d\omega$

$a_1, \ldots, a_r$: poles of $A(\omega)$

$\xi \in T^*X$: canonical 1-form

$C_A := \text{det}(\xi - A(\omega))_0 \subseteq T^*X$

$\Sigma_A \subseteq \overline{T^*X} := \text{Hyp}(0_X \oplus k_X)$

$\Sigma_\infty := \overline{T^*X} / T^*X$
Assumption

1. HTL-decomp of $\forall a \in a_i$

   $$\oplus (\mathbb{H}_{f,j} \otimes \frac{R_j}{\sigma \eta_i})$$

   Suppose $R_i \in M(1, \mathbb{C})$

2. $\mathcal{C}_A$ may have sing. pts. at $c(A, x) = (a_i, \infty)$

   Suppose $\mathcal{C}_A$ has no other sing. pts.
\[ \mathcal{M}(\overline{C_A})_{(a_1, \infty)} = -\text{rk} \left( \text{End}(\nabla) \right) - \text{Irr} \left( \text{End}(\nabla) \right) + 2(\text{rk}(A_{a_1}) - 1)(C_A, \Sigma_{\infty})(\infty, a_1^{\infty}) \]

Cf. (Deligne's Milnor formula)

\[ \mathcal{M}(X/S, x) = (-1)^{\dim X} \text{dim} \Phi_{\text{eff}}(x) + (-1)^{\text{Swan}} \Phi_{\text{eff}}(x). \]

Thm. \( C_A \): normalization of \( \overline{C_A} \)

\[ j : X \setminus \{ a_1, \ldots, a_r \} \to X \]

\[ h^i(C_A, C) = h^i(X, j_*(\text{End}(\nabla))) \quad \forall i \]

\[ \Rightarrow \quad X(C_A) = \chi_{\text{Dr}}(j_*(\text{End}(\nabla))) = \text{rig}(\nabla) \]