Title: Wild character varieties, meromorphic Hitchin systems and Dynkin diagrams

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Abstract: In 1987 Hitchin discovered a new family of algebraic integrable systems, solvable by spectral curve methods. One novelty was that the base curve was of arbitrary genus. Later on it was understood how to extend Hitchin's viewpoint, allowing poles in the Higgs fields, and thus incorporating many of the known classical integrable systems, which occur as meromorphic Hitchin systems when the base curve has genus zero. However, in a different 1987 paper, Hitchin also proved that the total space of his integrable system admits a hyperkahler metric and (combined with work of Donaldson, Corlette and Simpson) this shows that the differentiable manifold underlying the total space of the integrable system has a simple description as a character variety $\Hom(\pi_1(\Sigma), G)/G$ of representations of the fundamental group of the base curve $\Sigma$ into the structure group $G$. This misses the main cases of interest classically, but it turns out there is an extension. In work with Biquard from 2004 Hitchin's hyperkahler story was extended to the meromorphic case, upgrading the speakers holomorphic symplectic quotient approach from 1999. Using the irregular Riemann--Hilbert correspondence the total space of such integrable systems then has a simple explicit description in terms of monodromy and Stokes data, generalising the character varieties. The construction of such ``wild character varieties'', as algebraic symplectic varieties, was recently completed in work with D. Yamakawa, generalizing the author's construction in the untwisted case (2002-2014). For example, by hyperkahler rotation, the wild character varieties all thus admit special Lagrangian fibrations. The main aim of this talk is to describe some simple examples of wild character varieties including some cases of complex dimension 2, familiar in the theory of Painleve equations, although their structure as new examples of complete hyperkahler manifolds (gravitational instantons) is perhaps less well-known. The language of quasi-Hamiltonian geometry will be used and we will see how this leads to relations to quivers, Catalan numbers and triangulations, and in particular how simple examples of gluing wild boundary conditions for Stokes data leads to duplicial algebras in the sense of Loday. The new results to be discussed are joint work with R. Paluba and/or D. Yamakawa.
Wild character varieties, meromorphic Hitchin systems and Dynkin diagrams

P. Boalch, CNRS Orsay
(new parts are joint with
D. Yamakawa and/or R. Dijkgraaf)
Smooth ag. curve

Connections on v.biles / s
with regular singularities

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The Lax Project

Try to classify integrable systems with nice properties

- finite dimensional complex algebraic
- completely integrable Hamiltonian system $(M, \mathcal{X})$
- admits a Lax representation (any genus)

up to isomorphism (isogeny, deformation, ...)

Then look at different representations of each one
The Lax project

E.g. Look at isospectral deformations of rational matrix

\[ A(z) \]

\[ \kappa = \det(A(z) - \lambda) \implies \text{spectral curve} \]

\[ \mathcal{M}^* = \{ A \mid \text{orbits of polar parts fixed} \}/\mathbb{G} \quad \text{symplectic} \]

- lots of examples of such integrable systems
  
  Jacobi, Garnier, ....
The Lax project

Hitchin systems \((\text{fix } G = GL_n(G), \Sigma \text{ compact Riemann surface})\)

1. \(T^* \text{Bun}_G = \{ (V, \Phi) \mid V \text{ stable}, \Phi \in H^0(\text{End}(V \otimes \mathbb{R})) \}_{/\text{iso}}\)

2. \(\bigcup \mathcal{M}_{\text{H}} = \{ (V, \Phi) \mid \text{stable pair} \}_{/\text{iso}}\)

(\text{Higgs bundles})

- Hyperkahler: \(\mathcal{M}_{\text{H}} \equiv \mathcal{M}_{\text{OR}} \equiv \mathcal{M}_G = \text{Hom}(\pi_1(G), G)/G\)

\text{Higgs} \quad \text{Connections} \quad \text{character variety}
The Lax project

Vary $\Sigma \rightarrow$ isomodularity connection on spaces of connections

Hyperkahler: $\mathcal{M}_H \cong \mathcal{M}_R \cong \mathcal{M}_B = \text{Hom}(\pi_1(\Sigma), G)/G$

Higgs connections character variety
The Lax project

Back to rational matrices:

- $A(z)dz$ is a meromorphic Higgs field (non trivial)
- $e^{-A(z)dz}$ is a meromorphic connection (trivial)

(i.e. classify hyperkahler manifolds with such extra structure)
The Lax project

\[ \text{wild} \]
\[ \text{nonabelian Hodge} \to \text{RHB} \]
\[ \text{mod. Higgs} \quad \text{mod. Connections} \quad \text{wild Character Variety} \]

**Theorem** Moduli spaces of meromorphic Higgs bundles often have such structure

- Nitsure, Bottacin, Markman - '95 ACIHS in Poisson sense
- PB. '99 Symplectic forms on \( \text{Mor} \cong \text{M}_B \) (mero. Atiyah-Bott/Goldman)
- Biquard-B. '01 Hyperkahler structure
- Algebraic approach to symplectic forms: Woodhouse '00, Krashen '01, B. '02, '09, '11, B.-Yamakawa '15
The Lax project

\[
\text{wild Hodge} \quad \text{RHB} \\
\text{M} \overset{\cong}{\rightarrow} \text{M} \overset{\cong}{\rightarrow} \text{M} \overset{\cong}{\rightarrow} \text{M} = \{ \text{monodromy \& Stokes data} \}
\]

\[
\text{mero. Higgs} \quad \text{mero. Connections} \quad \text{wild character variety}
\]

"Fission"
<table>
<thead>
<tr>
<th>Example</th>
<th>Higgs Integrable system</th>
<th>Connections (Sommerfeld system)</th>
<th>Monodromy/ Stokes</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_1 + A_2 z ) ( dz ) ( z )</td>
<td>Monakov</td>
<td>Dual Shlesinger</td>
<td>( G^* )</td>
</tr>
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<tr>
<td>$z^2$</td>
<td></td>
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$(A_1 + A_2 z) \frac{dz}{z}$

$\sum \frac{A_i}{z-a_i} \frac{dz}{z}$

$\text{Painlevé 6}$

$M_8 \cong \text{Fricke-Klein-Vag} \text{ surface}$

$2yz + z^2 + y^2 + z^2 + ax + by + cz = d$

$\cong d//T, \quad d = Sl_3^*, \quad \text{dim } 6-2\cdot 2 = 2$

$\cong S \times E \times E \times E \times G_2 / G_2, \quad \text{dim } 4\cdot 2 - 2\cdot 3 = 2$

$\cong G_2 \text{ representation of Painleve VI} \quad (\text{B.- Paluba, JAG '16})$
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</tr>
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<td>$(A_1 + A_2 z) \frac{dz}{z}$</td>
<td>Manakov</td>
<td>Schlesinger</td>
<td>$G^*/G$</td>
</tr>
<tr>
<td>$\sum \frac{A_i}{z-a_i} , dz$</td>
<td>Garnier (Classical Gaudin)</td>
<td>—</td>
<td>Painlevé 6</td>
</tr>
<tr>
<td>$2 \times 2 \quad 4 \text{poles}$</td>
<td></td>
<td></td>
<td>$x^2 + y^2 + z^2 + c^2 = d$</td>
</tr>
<tr>
<td>$(A_0 + A_1 z + A_2 z^2) \frac{dz}{z}$</td>
<td></td>
<td>Paynelevé 2</td>
<td>$M_0 \cong \text{Flaschka-Newell surface}$</td>
</tr>
<tr>
<td>$2 \times 2$</td>
<td></td>
<td></td>
<td>$xy^2 + x+y+z = b-b^{-1} \quad b \in \mathbb{C}*$</td>
</tr>
</tbody>
</table>

(${\text{New hyperkahler manifold, via Biquard-B. '01}}$)
Dynkin diagrams

Okamoto (‘80s):

\( P_6 \) has \( D_4 \) affine Coxeter group symmetry

\( P_2 \)  \(-\)  \( A_1 \)

\( M^* \cong D_4 \) ALE space/quotient variety \( \to \) \( M_{DR} \cong M_B \)

\( M^* \cong A_1 \) ALE space/Eguchi-Hanson \( \to \) \( M_{RE} \cong M_B \)

(Ex. 3, 0706.2639)
Spaces from graphs/quivers

Kronheimer '89: If \( \Gamma \) an affine ADE Dynkin graph, then
\[
\dim V_i \sim \text{minimal null root of } \gamma
\]
\[
\text{Rep}(\Gamma, V) \sslash H \text{ is cSdim } 2
\]

\[
\text{Rep}(\Gamma, V) = \text{Hom}(V_1, V_2) \oplus \text{Hom}(V_2, V_1)
\]
\[
a \quad b
\]
\[
\cong T^* \text{Hom}(V_1, V_2) \quad \text{(symplectic)}
\]

\( H := \text{GL}(V_1) \times \text{GL}(V_2) \) acts on \( \text{Rep}(\Gamma, V) \)
with moment map \( \mu(a, b) = (ab, -ba) \)

Additive/Nakajima quiver variety:
\[
\text{Rep}(\Gamma, V) \sslash H = \mu^{-1}(\lambda)/H \quad (\lambda \in \mathbb{C}^\Gamma \subset \text{Lie}(H)^*)
\]
Suppose $\Gamma = \circ \circ$ or $\circ \circ \circ$ etc.

Then what is $\text{Rep}^*(\Gamma, \nu)$?

**Multiplicative version**

\[ B(V_1, V_2) : \]

\[ \Gamma = \begin{array}{ccc}
0 & a & V_1 \\
\circ & 0 & \circ \\
b & \circ & 0
\end{array} \]

\[ \text{Rep}^*(\Gamma, \nu) = \left\{ (a, b) \mid 1 + ab \text{ invertible} \right\} \]

\[ \cap \text{ "invertible representations"} \]

\[ \text{Rep}(\Gamma, \nu) \]

**Thm (VandenBergh '04)** $\text{Rep}^*(\Gamma, \nu)$ is a "multiplicative" (or "quasi") Hamiltonian $\mathfrak{h}$-space

with

group valued moment map

\[ \mu(a, b) = (1 + ab, (1 + ba)^{-1}) \in \mathfrak{h} \]

E.g. Multi. Quiver Var. $\left( \begin{array}{ccc}
0 & 1 & 2 \\
1 & 0 & 1 \\
2 & 1 & 0
\end{array} \right) \cong \left\{ xyz + z^2 + y^2 + z^2 = ax + by + cz + d \right\}$
6. Haec ergo teneatur definitio signorum ( ), inter quae indices ordine a sinistra ad dextram scribere constitui; atque indices hoc modo clausulis inclusi in posterum denotabunt numerum ex istis indicibus formatum. Ita a simplicissimis calibus inchoando, habeimus:

\[(a) = a\]
\[(a, b) = ab + x\]
\[(a, b, c) = abc + c + a\]
\[(a, b, c, d) = abcd + cd + ad + ab + x\]
\[(a, b, c, d, e) = abcde + cde + ade + abe + abc + e + c + a\]

cuncta.

"Euler's continuant polynomials"
Stokes structures

(Shih 1975, Deligne 1978, Malgrange 1980 ...)

Stokes diagram with Stokes directions

Halo at $\infty$ with singular directions

Subdominant solutions $u_i + h(u_{i+1})$

$U_{\infty} \cong \{ x y z + x + y + z = b - b^{-1} \}$

$\cong \left\{ (p_1, ..., p_m) \in \left( \mathbb{P}^1 \right)^m \left| \begin{array}{c}
\rho_i \equiv \rho_{i+1} \mod 6 \\
\frac{(\rho_1 - \rho_2) (\rho_2 - \rho_3) (\rho_4 - \rho_5) (\rho_5 - \rho_6) (\rho_6 - \rho_1)}{(\rho_2 - \rho_3) (\rho_4 - \rho_5) (\rho_6 - \rho_1)} = b^2 \end{array} \right. \right\}/PSL_2(\mathbb{C})$
Wild Character Varieties

Fix $G$ (e.g. $GL_n(\mathbb{C})$)

Wild Riemann surface $(\Sigma, \underline{q}, \underline{Q}) \Rightarrow \text{wild character variety}$

$\Sigma$ compact Riemann surface with marked points
$\underline{q} = (q_1, \ldots, q_m)$
and irregular types
$\underline{Q} = Q_1, \ldots, Q_m$

$\Sigma^0 = \Sigma \setminus \underline{q}$

$M^\text{naive}$

$M^\text{DR} = \{ \text{Alg. connections on } G\text{-bundles on } \Sigma^0 \}
\text{isom.}
\rho \cong dQ_i + \Lambda_i \frac{dz_i}{z_i}
\text{holom.}$

E.g.

$Q_i \in \mathfrak{t}(\mathbb{R}^n) \subset \mathfrak{g}(\mathbb{C}) \quad \text{tCG}$
Wild Character Varieties

Fix $G$ (e.g. $GL_n(C)$)

Wild Riemann surface $(\Sigma, q, \mathcal{Q}) \Rightarrow$ wild character variety

$\Sigma$ compact Riemann surface
with marked points
$q = (q_1, \ldots, q_m)$
and irregular types
$\mathcal{Q} = q_1, \ldots, q_m$

$\Sigma^0 = \Sigma \setminus q$

- at least for trivial Betti weights
- in general include parabolic extensions/weights $\Theta$

1. v.good: $D \cong dQ + \Lambda(z)\frac{dz}{z}$

2. good if v.good after some pullback $z = t^r$

$M^\text{naive}_{\text{BR}} = \{ \text{Alg connections on } G\text{-bundles on } \Sigma^0 \}$

$\Lambda \Rightarrow dQ_i + \Lambda_i\frac{dz_i}{z_i}$

$\text{holom.}$
Wild Character Varieties

Fix $G$ (e.g. $GL_n(C)$)

E.g. $(Disc, O, Q)$ $G = GL_2(C)$

$Q = A/\mathbb{Z}^k$, $A = (a, b)$ $a \neq b$

$k = 3$

$Q \Rightarrow$

- Centraliser group $H = T = (\mathbb{Z}^*)^2 \subset G$
- Singular directions $A$
  - Solutions involve $\exp(Q)$
    $Q = \text{diag}(q_1, q_2)$
  - Stokes diagram: plot growth of $\exp(q_1), \exp(q_2)$
Wild Character Varieties

Fix $G$ (e.g. $\text{GL}_n(C)$)

E.g. $(\text{Disc}, D, Q) \quad G = G\text{-}L_2(C)$

$Q = A/\mathbb{Z}^k, \quad A = (a, b) \quad a \neq b$

$k = 3$

$Q \Rightarrow$

- Centraliser group $H = T = \left( \begin{smallmatrix} \mathbb{Z} & 0 \\ 0 & \mathbb{Z} \end{smallmatrix} \right) \leq G$

- Singular directions $A$

Solutions involve $\exp(Q)$

$Q = \text{diag}(q_1, q_2)$

Stokes diagram: plot growth of $\exp(q_1), \exp(q_2)$
**Wild Character Varieties**

Fix $G$ (e.g. $GL_n(\mathbb{C})$)

E.g. $(\text{Disc}, 0, Q)$  
$G = GL_2(\mathbb{C})$  
$Q = A/\mathbb{Z}^k$,  
$A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$,  
$a \neq b$

**Stokes local system:**

- $G$ local system on $\tilde{\Delta}$
- Flat reduction to $H$ in $IH$
- Monodromy around $e(d)$ in $\mathcal{D}^{\text{core}}$

$IH$ halo/annulus

Extra punctures $e(d)$
Wild Character Varieties

Fix $G$ (e.g. $\text{GL}_n(\mathbb{C})$)

$E.g. \ (\text{Disc, O, Q}) \quad G = \text{GL}_2(\mathbb{C})$

$Q = \mathbb{A} / \mathbb{Z}^k \quad A = \begin{pmatrix} a & b \\ \end{pmatrix} \quad a \neq b$

Stokes local system:

- $G$ local system on $\tilde{\Delta}$
- flat reduction to $H$ in $\text{IH}$
- monodromy around $e(d)$ in $\text{IH}$

$\tilde{\Delta}$

$\circ e(d)$ extra punctures

$\text{IH}$ halo/annulus
Wild Character Varieties

Fix $G$ (e.g. $\text{GL}_n(\mathbb{C})$)

E.g. $(\text{Disc}, 0, Q)$, $G = \text{GL}_2(\mathbb{C})$

$Q = \mathbb{A}/\mathbb{Z}^2$, $A = (a, b)$, $a \neq b$

Stokes local system:
- $G$ local system on $\tilde{\Delta}$
- Flat reduction to $H$ in $\mathcal{H}$
- Monodromy around $e(d)$ in $\mathcal{H}$

Topological data that the multisucession approach to Stokes data gives

\begin{align*}
\{ \text{Connections with irregular type } &Q \} \\ \iff & \{ \text{Stokes local systems} \}
\end{align*}
Wild Character Varieties

Fix $G$ (e.g. $GL_n(C)$)

E.g. $(\text{Disc}, O, Q)$  $G = GL_2(C)$

$Q = \mathbb{A}/\mathbb{A}^k$,  $A = (a, b)$  $a \neq b$

Basepoints $b_1, b_2$

$\widetilde{\mathcal{T}} = \mathcal{T}_1 \setminus (\mathcal{H}, \{b_1, b_2\})$

$\tilde{\mathcal{U}}_{b_1} = \text{Hom}_\mathbb{C}(\widetilde{\mathcal{T}}, G)$

$= \left\{ \rho : \widetilde{\mathcal{T}} \to G \mid \rho(a) \in H, \rho(\mathcal{A}) \in \text{Stab} \text{ and } \mathbf{A} \right\}$

$\tilde{\Delta}$

$\mathcal{H}$ halo/annulus

$e(d)$ extra punctures
**Wild Character Varieties**  

Fix $G$ (e.g. $\text{GL}_n(K)$)

E.g. $(\text{Disc}, O, Q)$, $G = \text{GL}_2(K)$

$Q = A/\mathbb{Z}^k$, $A = (a, b)$, $a \neq b$

- Basepoints $b_1, b_2$

\[ \tilde{T} = \tilde{T}_1(\mathcal{D}, \mathcal{S}, b_1, b_2) \]

\[ \tilde{U}_B = \text{Hom}_G(\tilde{T}, G) \]

\[ = \left\{ \rho : \tilde{T} \to G \mid \begin{array}{c} \rho(b_1) \in H \\ \rho(b_2) \in \text{Stab}_K \forall a \in A \end{array} \right\} \]

**Thm** (arXiv 0203141)

\[ \tilde{U}_B \text{ is a quasi-Hamiltonian } G \times H \text{ space} \]
Wild Character Varieties

Fix $G$ (e.g. $GL_n(\mathbb{C})$)

E.g. $(\text{Disc}, G, Q) \quad G = G\text{-}L_2(\mathbb{C})$

$Q = A/\mathbb{Z}^k, \quad A = (a, b) \quad a \neq b$

Basepoints $b_1, b_2$

$I\hat{\Delta} = \prod_i (a_i, b_1, b_2)$

$I\hat{\Delta}_B = \text{Hom}_G(I\hat{\Delta}, G)$

$\begin{cases}
\rho: I\hat{\Delta} \to G \\
\rho(a_i) \in H \\
\rho(b_1) \in \text{Stab}_i \\
\rho(b_2) \in \text{Stab}_i \\
\forall i \in A
\end{cases}$

Thm (arXiv 0303.xxxx)

$I\hat{\Delta}_B$ is a quasi-Hamiltonian $G \times H$ space
Wild Character Varieties

Fix $G$ (e.g. $GL(n, \mathbb{C})$)

E.g. $(\text{Disc}, 0, Q) \quad G = GL_2(\mathbb{C})$

$Q = A/2^k \quad A = (a \ b) \quad a \neq b$

Thm (arXiv 0203-****)

$A(Q) = G \times (U_+ \times U_-)^k \times H$ is a quasi-Hamiltonian $G \times H$ space ("fission space")

$$(C, S, h) \quad S = (s_1, \ldots, s_{2k}) \quad \text{odd/even} \in U_+/-$$

Moment map $\mu(C, S, h) = (C h s_{2k} \cdots s_2 s_1, C, h^{-1}) \in G \times H$

Cor. $B(Q) := A(Q) // G$ is a quasi-Hamiltonian $H$-space

$= \mu_{G^{-1}(1) / G}$

$= \mathcal{M}_B((1p', 0, Q))$

$\cong \{ (S, h) \in (U_+ \times U_-)^k \times H \mid h s_{2k} \cdots s_2 s_1 = 1 \}$
**Wild Character Varieties**

**Cor.**

\[ \{ (S, h) \in (U^- U^+) \times H \mid hS_{z_k} \ldots S_2 S_1 = 1 \} \text{ is a quasi-Hamiltonian H-space} \]

\[ \cong \{ (S_2, \ldots, S_{2k-1}) \mid S_{2k-1} \ldots S_2 S_1 \in G^0 = U^- H U^+ < G \} \]

\[ \cong \{ (S_2, \ldots, S_{2k-1}) \mid (S_{2k-1} \ldots S_2 S_1)_1 \neq 0 \} \quad \text{(Gauss)} \]

E.g. \( k=2 \)

\[
\begin{pmatrix}
(1 \ a_1) & (1 \ 0) \\
(0 \ 1) & (b_1 \\
\end{pmatrix}_{11} = 1 + ab
\]

So \( B(q) \cong B(U) \) of Van den Bergh

\[ \mu = h^{-1} = (1 + ab \ , \ (1 + ba)^{-1}) \]

**Lemma**

\[
\begin{pmatrix}
(1 \ a_1) & (1 \ 0) \\
(0 \ 1) & (b_1 \\
\end{pmatrix} \ldots \begin{pmatrix}
(1 \ a_r) & (1 \ 0) \\
(0 \ 1) & (b_r) \\
\end{pmatrix}_{11} = (a_1, b_1, \ldots, a_r, b_r)
\]

- Euler's continuants are group valued moment maps
**Wild Character Varieties**

Cor. 
\[ \{ (s, h) \in (u_n u_-)^k \times H \mid h s_2 h \cdots s_3 s_2 = 1 \} \text{ is a quasi-Hamiltonian } H\text{-space} \]

\[ \cong \{ (s_2, \ldots, s_{2k-1}) \mid s_{2k-1} \cdots s_3 s_2 \in G^0 = U^- U_+ U \} \]

\[ \cong \{ (s_2, \ldots, s_{2k-1}) \mid (s_{2k-1} \cdots s_3 s_2)_{\text{tr}} \neq 0 \} \quad (\text{Gauss}) \]

\[ \cong \{ a, b \in \text{Rep}(\Gamma, V) \mid (a_1, b_1, \ldots, a_{k-1}, b_{k-1}) \neq 0 \} \]

\[ \cong: \text{Rep}^*(\Gamma, V) \]

\[ \Gamma = \begin{array}{c}
\begin{array}{c}
\vdots \\
\end{array}
\end{array} \]

\[ V = \bigotimes \]

\[ \left[ \text{Similarly for } V = U_1 \oplus U_2 \text{ any dimension } \right] \]

\[ \text{(2009-2015) } \]

\[ \Gamma \text{ any "fission graph"} \]

\[ \mu(a_1, \ldots, b_{k-1}) = ([a_1, b_1, \ldots, a_{k-1}, b_{k-1}], (b_{k-1}, \ldots, b_1, a_1)^{-1}) \]
Fission graphs (arXiv 0806 appendix C) \( G = GL(V) \)

\[
Q = A_r/2^r + \cdots + A_1/2 \\
= A_r w^r + \cdots + A_1 w \\
(A_i \in \mathbb{C}) \\
w = \frac{1}{2}
\]

\( r = 3 \):

```
  \[ \begin{array}{c}
    \text{"fission tree"}
  \end{array} \]
```

\( r = 2 \): get all complete k-partite graphs

\( \text{e.g.} \):

\( Q = \text{diag}(q_1, \ldots, q_n) \Rightarrow \text{nodes} = \{1, \ldots, n\} \), \# edges \( i \leftrightarrow j = \deg_{w}(q_i; -q_j) - 1 \)
Fission graph + legs
Wild Character Varieties

In this example \((p, 0, q)\) \(Q = A / B^b, \; \text{GL}_2(C)\)

\[
\mathcal{M}_B = \tilde{\mathcal{M}}_B \parallel H \frac{N_1, N_2}{(q_1, q_2)}
\]

\[
= \text{Rep}^* (\Gamma, V) \parallel H \frac{N_1, N_2}{(q_1, q_2)}
\]

"multiplicative quiver variety"

\[
\Gamma = \begin{array}{c}
\hdots \\
\vdots \\
\hdots
\end{array}, \; V = C \oplus C
\]

E.g. \(k=3\) (Painlevé II Betti space)

\[
\mathcal{M}_B \cong \left\{ xyz + x + y + z = b - b^{-1} \right\} \quad b \in \mathbb{C}^+ \text{ constant}
\]

(Flaschka-Newell Surface)
Wild Character Varieties

In this example \((p^1, 0, Q)\) \(Q = A/\mathbb{Z}^k, \quad \mathcal{G}_2(C)\)

\[ \mathcal{M}_B = \text{Rep}^*(\Pi, V) \backslash \! \! \! \backslash \mathcal{H} \]

"multiplicative quiver variety"

Also \(\mathcal{M}^* \cong \text{Rep}(\Pi, V) \backslash \! \! \! \backslash \mathcal{H} \)

"Nakajima/additive quiver variety"

(Re 2008, Hiroe-Yamakawa 2013)

E.g. \(k=3\) (Painlevé 2 Betti space)

\[ \mathcal{M}_B \cong \left\{ xyz + x + y + z = b - b^{-1} \right\} \quad b \in \mathbb{C}^* \text{ constant} \]

(Flaschka-Newell Surface)
Wild Character Varieties

In this example \((q^1, 0, Q)\) \(Q = A/\mathbb{Z}^k, \ \text{GL}_2(C)\)

\[ M_B = \text{Rep}^*(\Pi, V) \big/ \mathbb{H} \]

"multiplicative quiver variety"

Also \(M^* \cong \text{Rep}(\Pi, V) \big/ \mathbb{H} \)

"Nakajima/additive quiver variety"

(PB 2008, Hiss--Tymekawa 2013)
Conjectural classification (of $M_3$) in $dim_a = 2$:

(Nonabelian Hodge Surfaces) (1203.6607) "K2 surfaces"

Affine Weyl group
Minimal rank of bundles
Pole orders
Conjectural classification (of $U_3$) in $dim = 2$:

(Non-abelian Hodge Surfaces) (1203.6607) "K2 surfaces"

$E_8$, $E_7$, $E_6$

$P_4$, $A_3 = P_3$

$P_6$, $P_5$

$P_4$, $P_2$, $P_1$

Phase spaces for Painlevé differential equations
Conjectural classification (of $M_3$) in $dim_0 = 2$:

(Non-abelian Hodge surfaces) \((1203 \cdot 6607)\) “K2 surfaces”

$E_8 \quad E_7 \quad E_6$ \hfill $P_1 \quad A_3 = P_3 \quad P_2 \quad P_1$

Phase spaces for Painlevé differential equations
Conjectural classification (of $M^3$) in $dim_a = 2$:

(Non-abelian Hodge Surfaces) $(1203, 660?)$ "K3 surfaces"

$M^* \cong$ ALE

$M^*_1 \cong$ ALF

---

$E_8, E_7, E_6, D_4, A_3 = D_3, \{D_2, D_1, D_0\}$

---

$[M^* = M \text{ open piece where bundle holomorphic trivial}]$
Conjectural classification (of $U_3$) in $dim = 2$:

(Nonabelian Hodge Surfaces) $\quad (1203 \cdot 6607) \quad \text{"K2 surfaces"}$

$U_1^* \cong \text{ALE}$

$U_2^* \cong \text{ALF}$

$E_8 \quad E_7 \quad E_6$

$A_3 = A_3$

$D_2 \quad D_1 \quad D_0$

$A_2 \quad A_1 \quad A_0$

$[U^*_1 = U \text{ open piece where bundle holom. trivial}]$
Summary

$$B_2 = \mathbb{B}(v_1, v_2)$$
$$\mu \sim (a, b) = ab + 1$$

$$\mathbb{F}_2 \otimes \mathbb{F}_2$$
$$\mu \sim (a, b)(c, d)$$

Continuants factorise:

$$(a, b, c, d) = (a, b)(c', d)$$

$$= (a, b')(c, d)$$

$$c' = (a, b)^{-1}(a, b, c)$$

$$b' = (b, c, d)(c, d)^{-1}$$
Summary

\[ B_2 = B(u, v) \]
\[ \mu \sim (a, b) = ab + 1 \]
\[ R_2 \oplus B_2 \xrightarrow{L} \xrightarrow{R} B_4 \]
\[ \mu \sim (a, b)(c, d) \]
\[ \mu \sim (a, b, c, d) \]

Continuants factorise: \( (a, b, c, d) = (a, b)(c', d) = (a, b')(c, d) \)
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\[ b' = (b, c, d)(c, d)^{-1} \]

Thm (B.- Paluba - Yamakawa): All such factorisation maps relate the quasi-Hamiltonian structures
Summary

\[ B_2 = B(\nu, \kappa) \]
\[ \mu \sim (a, b) = ab + 1 \]

Continuants factorise:
\[ (a, b, c, d) = (a, b)(c', d) = (a, b')(c, d) \]
\[ c' = (a, b)^{-1}(a, b, c) \]
\[ b' = (b, c, d)(c, d)^{-1} \]

Thm (B.-Paluba-Yamakawa)
All such factorisation maps relate the quasi-Hamiltonian structures
- Count all factorisations (into linear factors) \( \sim 14 \)
- Similarly \( B_n \) has \( C_n = \frac{1}{n+1} \left( \frac{2n}{n} \right) \) factorisations (Catalan no.)
- Follows since \( L \& R \) form "free duplicial algebra" (Loday)
Summary

\[ B_2 = B(u_1, u_2) \]

\[ \mu \sim (a, b) = ab + 1 \]

Continuants factorise:

\[ \begin{align*}
(a, b, c, d) &= (a, b)(c', d) \\
&= (a, b')(c, d)
\end{align*} \]

\[ c' = (a, b)^{-1}(a, b, c) \]

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Summary

\[ \mathcal{B}_2 = \mathcal{B}(V_1, V_2) \]

\[ \mu \sim (a, b) = ab + 1 \]

\[ \mathcal{B}_2 \otimes \mathcal{B}_2 \xrightarrow{L} \mathcal{B}_4 \]

\[ \mu \sim (a, b)(c, d) \]

\[ \mu \sim (a, b, c, d) \]

Continuants factorise: \((a, b, c, d) = (a, b)(c', d) = (a, b')(c, d)\)

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\[ b' = (b, c, d)(c, d)^{-1} \]

**Thm (B. - Paluba - Yamakawa)**

All such factorisation maps relate the quasi-Hamiltonian structures.

- Count all factorisations (into linear factors) \(\sim 14\)
- Similarly \(\mathcal{B}_n\) has \(C_n = \frac{1}{n+1} \left( \frac{2^n}{n} \right)\) factorisations (Catalan no.)
- Follows since \(L \& R\) form "free duplicial algebra" (Loday)
- \(C_n\) also counts triangulations of \((n+2)\)-gon (Euler 1751, Segner 1758)
Summary

\[ B_2 = B(v_1, v_2) \]

\[ \mu \sim (a, b) = ab + 1 \]

\[ B_2 \otimes B_2 \xrightarrow{L} \quad B_4 \]

\[ \mu \sim (a, b)(c, d) \]

Continuants factorise:

\[ (a, b, c, d) = (a, b)(c', d) = (a, b')(c, d) \]

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\( B \) similarly \( B_n \) has \( C_n = \frac{1}{n+1} \binom{2n}{n} \) factorisations (Catalan no.)

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- \( C_n \) also counts triangulations of \((n+2)\)-gon (Euler 1751, Segner 1758)
Summary

\[ B_2 = B(v_1, v_2) \]

\[ B_2 \otimes B_2 \xrightarrow{L} B_4 \]

\[ \mu \sim (a, b) = ab + 1 \]

\[ \mu \sim (a, b)(c, d) \]

\[ \mu \sim (a, b, c, d) \]

Continuants factorise:

\[ (a, b, c, d) = (a, b)(c', d) = (a, b')(c, d) \]

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Continuants factorise:
\[ (a, b, c, d) = (a, b)(c', d) \]
\[ = (a, b')(c, d) \]

Thm (B.-Paluba-Yamakawa)

All such factorisation maps relate the quasi-Hamiltonian structures
- Count all factorisations (into linear factors) \( \sim 14 \)
- Similarly \( B_n \) has \( C_n = \frac{1}{n+1} \frac{(2n)!}{n!} \) factorisations (Catalan no.)
- Follows since \( L \& R \) form "free duplicial algebra" (Loday)
- \( C_n \) also counts triangulations of \( (n+2) \)-gon (Euler 1751, Segner 1758)
Summary

\[ B_2 = B(U_1, U_2) \]

\[ L(B_2 \Theta B_2) = \mathfrak{S}_4 \]

\[ \mu \sim (a,b) = ab+1 \]

\[ \mu \sim (a,b)(c,d) \]

\[ \mu \sim (a,b,c,d) \]

Continuants factorise:

\[ (a,b,c,d) = (a,b)(c',d) \]

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\[ c' = (a,b)^{-1}(a,b,c) \]

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- \( C_n \) also counts triangulations of \((n+2)\)-gon (Euler 1751, Segner 1758)
Summary

\[ B_2 = B(v_1, v_2) \]
\[ L(B_2 \otimes B_2) = B_4 \]
\[ \mu \sim (a, b) = ab + 1 \quad \mu \sim (a, b)(c, d) \quad \mu \sim (a, b, c, d) \]

Continuants factorise:
\[ (a, b, c, d) = (a, b)(c', d) \]
\[ = (a, b')(c, d) \]

\[ c' = (a, b)^{-1}(a, b, c) \]
\[ b' = (b, c, d)(c, d)^{-1} \]

Thm. \((B.-Pakula-Yamakawa)\)

All such factorisation maps relate the quasi-Hamiltonian structures

- Count all factorisations (into linear factors) \(\sim 14\)
- Similarly \(B_n\) has \(C_n = \frac{1}{n+1} \binom{2n}{n}\) factorisations (Catalan no.)
- Follows since \(L \& R\) form "free duplicial algebra" (Loday)
- \(C_n\) also counts triangulations of \((n+2)\)-gon \((Euler 1751,\; Segner 1758)\)
other the inferior (Stokes 1857) Fig. 1. Stokes diagram of Airy equation

\[ q = \pm 2 \omega^{3/2} \]

The curve will evidently have the form represented...
Thm (B. Yamakawa, arXiv:1512)

- Can define twisted Stokes local systems (any reductive $G$)
- (Stokes structures already known $G \times \mathbb{N}$)
- Moduli spaces of framed twisted Stokes local systems are (twisted) quasi-Hamiltonian
- Completes project of understanding "symplectic nature of wild $\Omega$"

\[ B_i \cong GL(n_i) \quad \mu \sim (a) \]
\[ B_3 \cong \{ a, b, c \in GL(n) \mid \det(a, b, c) \neq 0 \} \]
\[ \mu \sim (a, b, c) \]

Can now glue these Airy triangles ($B_i$) as before, so clearly

factorisations $\Rightarrow$ triangulations

\[ B^h_i \rightarrow B^h \]

If $\dim(k) = 1$ this is familiar from complex WKB, but now see how to glue the triangles via GRT fusion