BPS states, torus links, and wild character varieties

• String theoretic approach to the cohomology of wild character varieties; with Ron Donagi and Tony Pantev.

• Generalizes previous work with Wu-yen Chuang and Guang Pan, and Wu-yen Chuang, Ron Donagi and Tony Pantev.

• Physical derivation and generalization of conjectures of Hausel, Mereb, Wong and Shende, Treumann, Zaslow.
1. Wild Character varieties
[Witten, Boalch]

- $(C,p)$ smooth projective curve with marked point
- $Q$ irregular type at $p$ with values in $t \subset gl(r, \mathbb{C})$ i.e. $t$-valued meromorphic function germ at $p$ modulo holomorphic terms
  \[ Q = \sum_{k=1}^{n-1} \frac{A_k}{z^k}, \quad A_k \in t \]
for some $n \in \mathbb{Z}$, $n \geq 2$.
- $R \in t$ diagonal matrix
- Flat $gl(r, \mathbb{C})$-connections on $C \setminus \{p\}$ locally gauge
equivalent to

$$dQ + \frac{R}{z} + \text{holomorphic terms}$$

at $p$.

- $S_{Q,R}$ the variety of Stokes data [Witten, Boalch].
  Rigorous construction [Boalch] by quasi-hamiltonian reduction.

- $S_{Q,R}$ smooth quasi-projective for generic $R$. Dimension

$$d_{\mu,n} = 2r^2(g - 1) + n(r^2 - \sum_{i=1}^{\ell} m_i^2) + 2$$

where $\mu$ is the partition of $r$ determined by the eigenvalues of $R$. 
• \[ WP(Q, R; u, v) \] the mixed Poincaré polynomial:

\[
WP(Q, R; u, v) = \sum_{k,j} \dim Gr_k^W H^j(S_{Q,R}) u^{k/2} v^j
\]

where \( W_k H^j \) weight filtration.

• Wild non-abelian Hodge correspondence
  [Boalch,Biquard], [Mochizuki]

  Irregular flat connections ↔ Irregular Higgs bundles

Moduli spaces related by hyper-Kähler rotation.

• \( P = W \) correspondence
  [de Cataldo, Hausel, Migliorini]

\[
W_{2k} H^j(\text{character variety}) = P_k H^j(\text{Higgs bundle moduli space})
\]

where \( P_k H^j \) perverse Leray filtration from Hitchin map.
2. The conjecture of Hausel, Mereb, and Wong

- $R$ regular and sufficiently generic

$$Z_{HMW}(z, w) = \sum_{\lambda} \Omega_{\lambda}^{g,n}(z, w) \tilde{H}(x; z^2, w^2)$$

where:

- the sum in the right hand side is over all Young diagrams $\lambda$,
- for each $\lambda$

$$\Omega_{\lambda}^{g,n} = \prod_{\Box \in \lambda} \frac{(-z^{2a(\Box)}w^{2l(\Box)})^{r(2a(\Box)+1)} - w^{2l(\Box)+1})^{2g}}{(z^{2a(\Box)+2} - w^{2l(\Box)})(z^{2a(\Box)} - w^{2l(\Box)+2})},$$

and
• $x = (x_1, x_2, \ldots)$ is an infinite set of formal variables and $\tilde{H}_\lambda(x; z^2, w^2)$ are the modified Macdonald polynomials.

• Define $\mathbb{H}_{\mu, n}(z, w)$ by

$$Z_{HMW}(z, w) = \sum_{k \geq 1} \sum_{\mu} \frac{(-1)^n|\mu|w^{k\mu_n}z^{n-k\mu_n}}{1 - z^{2k}(w^{2k} - 1)} m_\mu(x^k)$$

where $m_\mu(x)$ are the monomial symmetric functions and $x^k = (x_1^k, x_2^k, \ldots)$. Then one has the following conjectural formula

$$WP(Q, R; u, v) = \mathbb{H}_{(1^r)}(u^{1/2}, -u^{-1/2}v^{-1})$$

for any $r, n \geq 1$ and any $Q$ and any generic regular $R$. 


2. The conjecture of Shende, Treumann and Zaslow

- $\Sigma \subset \mathbb{A}^2$ is a reduced rational plane curve with one singular point $\nu$.
- $\pi : \mathbb{A}^2 \rightarrow \mathbb{A}^1$ projection onto one of the coordinate axes.
- $\Sigma \simeq$ affne part of a spectral curve for a meromorphic Hitchin system on $\mathbb{P}^1$ with a pole at $\infty$.
- Wild non-abelian Hodge correspondence $\Rightarrow$ character variety $S_\Sigma$.
- $L$ link of singular point $\nu \in \Sigma$.
- $P_{L}^{(0)}(u)$ leading term in Homfly polynomial of $L$. 
Then
\[ P_L^{(0)}(u) = WP(S_\Sigma, u, -1) \]
for a specific normalization of \( P_L^{(0)}(u) \)

- Non-abelian Hodge ‘mirror’ of the conjecture of Oblomkov and Shende
  \[ P_L^{(0)}(u) = \sum_{n \in \mathbb{Z}} u^n \chi(Hilb_n(S, \nu)). \]

- Refined generalization [Oblomkov, Rassmusen, Shende]
- Refined colored generalization conjectured by [D, Hua, Soibelman] using the stable pair theory of the conifold. Will be used later in the talk.
- Unrefined colored generalization proven by [Maulik]
2. Irregular parabolic Higgs bundles

- \((C, p)\) curve with marked point, \(D = np\), \(n \geq 1\), \(M = K_C(D)\).
- \(\xi = (\xi_1, \ldots, \xi_\ell)\) \(\ell \geq 1\) sections of \(K_C(D)|_D\).
- Irregular Higgs \(\xi\)-parabolic Higgs bundle \((E, \Phi, E^*_D, \alpha)\)

\[ E : \text{vector bundle on } C, \quad E_D = E \otimes \mathcal{O}_D \]
\[ \Phi : E \rightarrow E \otimes M, \quad \Phi_D = \Phi|_D \]
\[ 0 \subset E_D^1 \subset \cdots \subset E_D^\ell = E_D \quad E_D^i/E_D^{i-1} \text{ loc. free } \mathcal{O}_D\text{-modules} \]
\[ \Phi_D(E_D^i) \subseteq E_D^i, \quad \text{gr}^i(\Phi_D) = \xi_i \otimes 1 \]
\[ \alpha = (\alpha_1, \ldots, \alpha_\ell) \quad \text{realparabolic weights} \]
\[ 0 < \alpha_\ell < \cdots < \alpha_1 < 1. \]
• Parabolic stability condition [Maruyama, Yokogawa] 
  \[ \Rightarrow \text{moduli stack } \mathcal{H}^{ss}_{\xi}(C, D; \alpha, \underline{m}, d) \text{ where} \]
  \[ \underline{m} = (m_1, \ldots, m_\ell), \quad m_i = \text{length}_{O_D}(E_D^i/E_D^{i-1}) \]
  \[ d = \deg(E), \quad \sum_{i=1}^\ell m_i = \text{rk}(E) \]

• for generic weights \(C^x\)-gerbe over smooth coarse moduli space

• similar objects introduced by [Inaba, Saito]
2. Irregular parabolic Higgs bundles

- $(C, p)$ curve with marked point, $D = np, n \geq 1$, $M = K_C(D)$.
- $\xi = (\xi_1, \ldots, \xi_\ell) \ \ell \geq 1$ sections of $K_C(D)|_D$.
- Irregular Higgs $\xi$-parabolic Higgs bundle $(E, \Phi, E^*_D, \alpha)$

$E$: vector bundle on $C$, $E_D = E \otimes \mathcal{O}_D$
$\Phi: E \rightarrow E \otimes M$, $\Phi_D = \Phi|_D$
$0 \subset E^1_D \subset \cdots \subset E^\ell_D = E_D \quad E^i_D/E^{i-1}_D$ loc. free $\mathcal{O}_D$-modules
$\Phi_D(E^i_D) \subseteq E^i_D$, $\text{gr}^i(\Phi_D) = \xi_i \otimes 1$
$\alpha = (\alpha_1, \ldots, \alpha_\ell)$ real parabolic weights
$0 < \alpha_\ell < \cdots < \alpha_1 < 1.$
3. Spectral correspondence

- Goal: construct holomorphic symplectic surface $S_\xi$ such that
  \[ \mathcal{H}^{ss}_\xi(C, D; \alpha, m, d) \]
  \[ \mid \ell \]
  Moduli stack Bridgeland stable dim 1 sheaves on $S_\xi$

- [Kontsevich, Soibelman] construct $S_\xi$ such that
  Hitchin base
  \[ \mid \ell \]
  Linear system on $S_\xi$

- [Szabo] proves (*) for certain open subsets
$T_\xi = \text{blow-up of the images of sections}$

$\xi_1, \ldots, \xi_\ell : D \to M_D \text{ on } M$

(assumed pairwise distinct and nonzero)
• $S_\xi = T_\xi \backslash$ anti-canonical divisor

• For any $\underline{m} = (m_1, \ldots, m_\ell)$ there is a linear system on $S_\xi$ consisting of compact divisors which are finite covers of $C$.

• There is a compactly supported $B$-field $\beta \in H_c^2(S_\xi, \mathbb{R})$

\[
\beta(\Sigma_m) = n \sum_{i=1}^{\ell} m_i \beta_i
\]

for any divisor $\Sigma_m$ in this linear system, where $\beta_i \in \mathbb{R}, 1 \leq i \leq \ell$.

• $\beta$-stability for dimension one sheaves $F$ on $S_\xi$ with

$$\text{ch}_1(F) = \Sigma_m, \quad \chi(F) = c.$$
\( \mathcal{M}^{ss}_\beta(S_\xi; m, c) \) moduli stack of \( \beta \)-semistable sheaves

**Spectral correspondence**

\[
\mathcal{M}^{ss}_\beta(S_\xi; m, c) \simeq \mathcal{H}_\xi(C, D; \alpha, m, c + r(g - 1))
\]

\[
\beta(\Sigma_m) = n \sum_{i=1}^{\ell} m_i \alpha_i,
\]

where \( g \) is the genus of \( C \) and \( r = \sum_{i=1}^{\ell} m_i \).

- holds for any \( g \geq 0 \) and any number of singular points.
4. Calabi-Yau threefold and refined invariants

- \( Y_\xi = S_\xi \times \mathbb{A}^1 \).
  \[ m_\beta^s(S_\xi; m, c) \simeq S_\xi^s(C, D; \alpha, m, c + r(g - 1)) \times \mathbb{A}^1 \]

- Stable pair theory [Pandharipande, Thomas]
  \[ PT_{Y_\xi}(m, c) = \text{virtual number of pairs} \]
  \[ \mathcal{O}_{Y_\xi} \to F, \quad s \text{ generically surjective} \]
  \[ \chi(F) = c \]
  \[ F \text{ compact support}. \]
- Refined stable pair theory
  [Kontsevich and Soibleman]

  \[ PT_{Y_{\xi}}(m, c; y) = \text{virtual Poincare polynomial of moduli space of such pairs.} \]

- Generating function

  \[ Z_{Y_{\xi}}(q, Q_1, \ldots, Q_{\ell}, y) = \]
  \[ 1 + \sum_{c,m \atop m \neq (0,\ldots,0)} q^c \prod_{i=1}^{\ell} Q_i^{m_i} PT_{Y_{\xi}}(m_1, \ldots, m_{\ell}, c; y). \]

- Goal: compute the generating function.
5. Refined invariants via refined Chern-Simons theory

- From now on $C = \mathbb{P}^1$, one marked point $p \in C$.
- Torus action $\mathbb{C}^\times \times S_\xi \to S_\xi$
- Refined virtual localization [Nekrasov, Okonkov], [Maulik]
- Stable pair theory localizes on a finite collection of rational curves $\Sigma_1, \ldots, \Sigma_\ell$ in $S_\xi$
- Direct localization computations **hard!**
• local equation at \( \mathbf{o} \)

\[
\prod_{i=1}^{\ell} (y - \lambda_i x^{n-2}) = 0.
\]
• Oblomkov-Shende framework:

Refined invariants of singular plane curve

\[ \Sigma_1 + \cdots + \Sigma_\ell \]

\[ \uparrow \]

Refined invariants of \((\ell, (n-2)\ell)\)-torus links

• One needs the colored refined variant of the conjecture formulated by [D,Hua,Soibelman].

• Colored refined invariants of torus links can be obtained from refined Chern-Simons theory [Aganagic, Shakirov], [Shakirov], also using some large \(N\) duality tricks.
Conjecture 1

The refined stable pair theory of $Y$ is given by

$$Z_{Y^t}(q, Q_1, \ldots, Q_\ell, y) = \sum_{\mu_1, \ldots, \mu_\ell} \left( \widetilde{W}_{\mu_1, \ldots, \mu_\ell}^{(n-2)}(s, t) \right) \prod_{i=1}^{\ell} \left[ (ts^{-1}Q_i)^{|\mu_i|/2} f_{\mu_i}(s, t)^{n-1} P_{\mu_i}(t, s; \overline{s}) \right] \bigg|_{s=qy, \ t=qy^{-1}}$$

where

$$\widetilde{W}_{\mu_1, \ldots, \mu_\ell}^{(n-2)}(s, t) = \sum_{\lambda_1, \ldots, \lambda_{\ell-1}} N_{\mu_1, \lambda_{\ell-1}}^{\lambda_{\ell-1}} N_{\mu_2, \lambda_{\ell-2}}^{\lambda_{\ell-2}} \cdots N_{\mu_3, \lambda_{\ell-3}}^{\lambda_2} N_{\mu_4, \lambda_1}^{\lambda_1} f_{\lambda_{\ell-1}}(s, t)^2 P_{\lambda_{\ell-1}}(s, t; t).$$
• $P_\lambda(t, s; x)$, $x = (x_1, x_2, \ldots)$, are the $(t, s)$-Macdonald polynomials

• $N^\sigma_{\nu, \lambda}$ are the $(s, t)$-Littlewood-Richardson coefficients

$$P_\nu(t, s; x) P_\lambda(t, s; x) = \sum_\sigma N^\sigma_{\nu, \lambda} P_\sigma(t, s; x).$$

• $f_\lambda(s, t)$ are refined framing factors,

$$f_\lambda(s, t) = \prod_{\square \in \lambda} s^{a(\square)} t^{-l(\square)},$$

• $t = (t^{1/2}, t^{3/2}, \ldots)$, $s = (s^{1/2}, s^{3/2}, \ldots)$. 
Conjecture 2: refined Gopakumar-Vafa expansion

\[ Z_{Y_\xi}(q, Q_1, \ldots, Q_\ell, y) = \exp \left( - \sum_{k \geq 1} \sum_{\mu} m_\mu(Q_1^k, \ldots, Q_\ell^k, 0, \ldots) \frac{y^{-kr}(qy^{-1})^{kd_\mu,n/2} P_{\mu,n}((qy)^{-k}, -(qy)^k)}{(1 - (qy)^{-k})(1 - (qy^{-1})^k)} \right) \]

- \( m_\mu(x_1, \ldots) \) monomial symmetric functions.
- \( P_{\mu,n}(u, v) \) perverse Poincaré polynomial of \( \mathcal{H}_\xi^*(C, D; \alpha, m, d) \) for generic \( \alpha, \mu \) partition

\[ r = m_1 + \cdots + m_\ell. \]
Conjecture 3

For $C = \mathbb{P}^1$ with one marked point,

$$WP(Q, R; u, v) = P_{\mu,n}(u, v)$$

where $\mu$ is the partition of multiplicities of eigenvalues of $R$ (assumed generic).
• agrees with HMW in many rank 2 and 3 examples with R regular

• For $\mu = (2,1)$, $n = \{5,6\}$, $P_{\mu,n}(1,v)$ agrees with Poincare polynomial of Higgs bundle moduli space computed by localization.

• Can one prove the $v = 1$ specialization by counting rational points on wild character varieties?