FQHE and Hitchin systems on modular curves

S.C. [work in progress]

Hitchin Systems in Mathematics and Physics
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• The $tt^*$ equations (in 2d for one coupling) are the Hitchin equations

$$F + [\Phi, \bar{\Phi}] = 0, \quad D_A \Phi = \bar{D}_A \Phi = 0$$

or, more generally, the condition that a connection on a hyperKähler manifold is hyperholomorphic and invariant under translation in some number of directions [SC, D. Gaiotto, C. Vafa, JHEP 1405 (2014) 055]

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Cumrun Vafa [arXiv:1511.03372] has suggested that the

**Fractional Quantum Hall Effect (FQHE),**

as actually observed in the laboratory, may be modeled by the $tt^*$ geometry of some complicated $\mathcal{N} = 4$ SQM systems.

Object of main interest: the **Berry holonomy** of the vacuum bundle

$$\mathcal{V} \to \mathcal{U} \equiv \text{(space of universal parameters)}$$

- $\mathcal{U}$ a complex manifold
- $\mathcal{V}$ a holomorphic Hermitian bundle whose fiber $\mathcal{F}$ is the vector space of vacua (zero energy states) of the SQM model specified by parameters
- the associated TFT defines a holomorphic $\Phi \in \Omega^1(\text{End } \mathcal{V})$ given by the action of the chiral fields on the vacua: $\mathcal{R} \subset \text{End } \mathcal{V}$ and $\Phi \in \Omega^1(\mathcal{R})$
- the Berry connection $A$ satisfies

$$[D_A, \overline{D}_A] + [\Phi, \overline{\Phi}] = D_A \overline{\Phi} = \overline{D}_A \Phi = D_A \Phi = \overline{D}_A \Phi = 0$$

- **These are Hitchin systems with actual technological implications**

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One of his motivations:
in FQHE phenomenology central role amplitudes of the form

$$\int_{\gamma} e^{-\sum_{i=1}^{N} V(x_i)} \prod_{1 \leq i < j \leq N} (x_i - x_j)^{1/\nu} \, dx_1 \cdots dx_N$$

$V(z)$ one-particle potential

$$V(z) = \left( \sum_{y \in \Lambda} \log(z - y) + \sum_{s \in S} e(s) \log(z - s) \right)$$

$\Lambda \subset \mathbb{C}$ a lattice, $S \subset \mathbb{C}$ a discrete set where defects (quasi-holes) of various charges $e(s) \in \mathbb{Z}$ are placed. $0 < \nu \leq 1$ is the filling fraction.

**Proper definition:** finite volume and then thermodynamical limit
Amplitudes of the form

\[ \int e^{-\sum_{i=1}^{N} V(x_i)} \prod_{1 \leq i < j \leq N} (x_i - x_j)^{1/\nu} \, dx_1 \cdots dx_n \]  

(\ast)

arise in (2,2) systems as BPS brane amplitudes in a double scaling limit \( \Lambda \to 0, \zeta \to 0 \) with \( \Lambda / \zeta \) fixed (\( \Lambda \) mass scale, \( \zeta \) \( \mathbb{P}^1 \) twistor parameter)

**Idea:** take seriously the SQM and its \( tt^* \) which give (\ast) in the limit

**Many possibilities for the \( N = 4 \) SQM which yields** (\ast): Basically, possible \( N = 4 \) models classified by their Witten index as function of \( N \)

\[ w(1) = \# \text{(one-particle low-lying states)} \]

\[ w_B(N) = \binom{N + w(1) - 1}{N} \quad \text{“Bose statistics”} \]

\[ w_F(N) = \binom{w(1)}{N} \quad \text{“Fermi statistics”} \]

We focus on the “fermionic” version: much simpler! (but still quite hard)

Defined if \( \nu > 0 \), natural when \( 0 < \nu \leq 1 \) (physical range)
In the “fermionic” model we are effectively reduced to study the one-particle $tt^*$ geometry, i.e. the LG model with superpotential

$$V(z) = \left( \sum_{y \in \Lambda} \log(z - y) + \sum_{s \in S} e(s) \log(z - s) \right)$$

Solving $tt^*$ (≡ Hitchin eqns.) is simpler when the SQM model has:

- Abelian symmetry group $\mathcal{A}$ acting freely and transitively on the vacua
- Fiber $\mathcal{F}$ of vacuum bundle $\mathcal{V}$ regular representation of $\mathcal{A}$

$$\mathcal{F} = L^2(\mathcal{A}) \cong L^2(\text{Hom}(\mathcal{A}, U(1)))$$

- $\mathcal{A}$ centralizes $tt^*$ metric and Berry holonomy
- both are diagonal in the character basis
- infinite number of vacua (in the thermodynamic limit)

$\Rightarrow$ $\mathcal{A}$ should also get infinite: a group of translations:

$$\Lambda \cup S = L \subset \mathbb{C} \text{ a lattice,} \quad e : L/\Lambda \to U(1) \text{ an additive character}$$

$$e(s + s') = e(s) e(s'), \quad e(s + \Lambda) = e(s)$$

- position of quasi-holes well-defined on the elliptic curve $E(\Lambda) \equiv \mathbb{C}/\Lambda$

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\[ V(z, \tau) = \sum_{y \in L/\Lambda} e(y) \log \theta_1((z - y)/2, \tau), \quad \Lambda = 2\pi \mathbb{Z} \oplus 2\pi \tau \mathbb{Z}, \quad \tau \in \mathbb{H} \]

\[ V'(z, \tau) = \sum_{y \in L/\Lambda} e(y) \left( \zeta(z - y, \tau) - \frac{\eta}{\pi} (z - y) \right), \]

\( \zeta(z, \tau) \) Weierstrass \( \zeta \)-function, \( \zeta(z + 2\pi, \tau) = \zeta(z) + 2\eta_1. \)

**To preserve invariance under translation by the lattice \( \Lambda \), \( V'(z, \tau) \) should be an elliptic function \( \iff \) \( e(s) \) not the trivial character**

Unfortunately \( e(s) \) **trivial** is the most interesting case for FQHE: \( e(s) \) roots of 1: OK for SQM. **Quadratic characters**: real charges

Superpotential \( V(z, \tau) \) still multi-valued on \( E(\Lambda) \) for 2 reasons:

i) \( \theta_1 \) just quasi-periodic for \( \Lambda \),

ii) branches of log

\( \Rightarrow \) SQM model has the symmetry \( L \) but in a very subtle way

the group \( L/\Lambda \) acts as discrete \( R \)-symmetry
Classification: We may assume $e$ to be faithful (otherwise $\Lambda \to \ker e$).

$$L/\Lambda \simeq \mathbb{Z}/M_1\mathbb{Z} \oplus \mathbb{Z}/M_2\mathbb{Z} \quad \text{with} \quad M_1 | M_2.$$  

$L/\Lambda$ has a faithful character iff $\gcd(M_1, M_2) = 1$ so 

$$L/\Lambda \simeq \mathbb{Z}/M\mathbb{Z}, \quad M \geq 2$$  

Models with the required symmetry are parametrized by 

1) an elliptic curve $E \equiv E(\Lambda)$  
2) a torsion subgroup $T \equiv L/\Lambda \subset E$, $T \simeq \mathbb{Z}/M\mathbb{Z}$, $(M \geq 2)$  
3) a faithful character $e: T \to U(1)$ up to equivalence $e \sim e^{-1}$

Equivalently by the pairs $(E, p)$ with $p \in E$ the unique point of order strictly $M$ such that $e(p) = e^{2\pi i/M}$ (a fixed primitive $M$-root)

Pairs $(E, p)$: elliptic curve with a level $M$ structure of type $\Gamma_1(M)$
Space of models of given level $M \equiv$ \textbf{moduli space of elliptic curves with structure} $\Gamma_1(M)$ (mod isomorphism)

$$\Gamma_1(M) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mod M \right\} \subset SL(2, \mathbb{Z})$$

\textit{moduli space of elliptic curve with} $\Gamma_1(M)$ \textit{structure} $= Y_1(M) \equiv \mathbb{H}/\Gamma_1(M)$

better compactify the space: $\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$

compactified moduli space $X_1(M) \equiv \overline{\mathbb{H}}/\Gamma_1(M)$

added points: \textbf{cusps} $\mathbb{P}^1(\mathbb{Q})/\Gamma_1(M)$

**Modular curve** $X_1(M)$ \textit{is the space of models (coupling constant space)}

$X_1(M)$ a Riemann surface of genus

$$g(X_1(M)) = 1 + \frac{M^2}{24} \prod_{p|M} (1 - p^{-2}) - \frac{1}{4} \sum_{d|M} \phi(d) \phi(M/d),$$

$(\phi$ Euler totient function $\phi(n) = |(\mathbb{Z}/n\mathbb{Z})^\times|)$
To compactify the coupling constant space $Y_1(M)$ we added the **cusps**

$$\#\text{cusps}(X_1(M)) = \frac{1}{2} \sum_{d|M} \phi(d) \phi(M/d).$$

**Cusps**: points at infinite distance in the natural hyperbolic metric from any regular theory. They are singular limits. Various kinds:

- **U type cusps**: a BPS particle gets zero mass closing the mass-gap
- **I type cusps**: a BPS particle gets infinite mass and decouples
- **I/U type cusps**: both mechanisms. **Not possible for** $M$ **prime**

**Other “bad” points in** $Y_1(M) = X_1(M) \setminus \{\text{cusps}\}$ **where the mass gap closes or states decouple**?

- Not expected since they are at finite distance from regular models
- For $M = 2$ one checks that all non-cusp points are regular
- Likely to be true for general $M$
Additional structures

The modular curve $X_1(M)$ has an important group of automorphisms

$$(\mathbb{Z}/M\mathbb{Z})^\times / \{\pm 1\} \cong \text{Gal}(\mathbb{Q}[\cos(2\pi/M)]/\mathbb{Q})$$

given by the \textit{diamond automorphisms}

$$\langle m \rangle : X_1(M) \to X_1(M), \quad m \in (\mathbb{Z}/M\mathbb{Z})^\times, \quad \langle m \rangle (E, p) = (E, mp)$$

Since the curve $X_1(M)$ is the space of theories, $\langle m \rangle$ send one theory to another: \textit{it is a duality}. Its effect is to change the character (charge assignment of ‘quasi-holes’) $e \mapsto e^m$. All models with a given $E$ obtained from any one by acting with $\text{Gal}(\mathbb{Q}[\cos(2\pi/M)]/\mathbb{Q})$: weakly-coupled model with charges $e$ is a strongly-coupled limit of the model with charges $e^m$ for all choices of $m \neq 1$

\textit{Solutions to tt* should be Gal}(\mathbb{Q}[\cos(2\pi/M)]/\mathbb{Q})-covariant

Similar story with \textit{Hecke correspondences $T_m$ ($m \in \mathbb{N}$)} (subtler dualities)

$$\begin{array}{c}
X_1(M) \leftarrow X_1^1(M, m) \rightarrow X_1(M)
\end{array}$$
$tt^*$ equations same as Hitchin equations in coupling space

The $tt^*$ equations for the level $M$ models: a family of Hitchin systems parametrized by the characters of $\Lambda$,

$$\text{Hom}(\Lambda, U(1)) \simeq S^1 \times S^1,$$

over the modular curve $X_1(M)$ with prescribed singularities at the cusps and covariant under the diamond automorphisms $\langle m \rangle$ (Hecke ?)

- $\text{Hom}(\Lambda, U(1)) \simeq S^1 \times S^1$ depends on a choice of generators for $\Lambda$ (or $L$)

$$\begin{pmatrix} \tilde{\phi} \\ \tilde{\theta} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \phi \\ \theta \end{pmatrix} \Rightarrow \text{action of } \Gamma_1(M)$$

on the family of Hitchin systems:

an invariance
Using the symmetry $L,$

$$\text{End}(\mathcal{F}) \cong \text{End}(L^2(S^1 \times S^1)) \otimes \text{End}(L^2(\text{Hom}(L/\Lambda, U(1))))$$

In the TFT trivialization, both $A$ and $\Phi$ act as multiplicative operators on $\text{End}(L^2(S^1 \times S^1))$ and $\Phi$ is proportional to $\text{Id} \in \text{End}(L^2(S^1 \times S^1))$

$A$ is a $S^1 \times S^1$ family of connections in *the Cartan of $\mathfrak{sl}(M)$*

$$A(\phi, \theta) = \text{diag}(A(\phi, \theta)_1, A(\phi, \theta)_2, \cdots, A(\phi, \theta)_M),$$

$$\Phi \in \Omega^1(\mathfrak{sl}(M))$$

The spectral curve in $K_{X_1(M)}$ has the form

$$\det[\lambda - \Phi] = \lambda^M - \rho$$

for a meromorphic $M$-differential $\rho \in \Gamma\left(X_1(M), K_{X_1(M)}^M\right)$ which has an arithmetic construction:

**Topological side is “arithmetic”**
Arithmetic construction of spectral cover

**Notation:** \( e(k) = e^{2\pi i \ell k/M} \) with \( \ell \in (\mathbb{Z}/M\mathbb{Z})^\times \), \( \ell \sim (M - \ell) \)

By a **spectral cover** \( \tilde{X} \xrightarrow{\pi} X_1(M) \) I mean a cover over which the eigenvalue of \( \Phi_\ell \) is a globally defined holomorphic one-form \( \mu_\ell \)

\[
\pi^* \det[\lambda - \Phi_\ell] = \prod_{k=0}^{M-1} (\pi^* \lambda - e^{2\pi i k/M} \mu_\ell)
\]

In physical terms the spectral cover is a curve parametrizing pairs

\((\text{SQM model, vacuum mod } \Lambda) \equiv (\text{point in } X_1(M), \text{eigenvalue of } \Phi)\)

\(V'(z, \tau)\) is an elliptic function with simple poles at \( z = 2\pi k/M \), \( k \in \mathbb{Z}/M\mathbb{Z} \), such that \( V'(z + 2\pi/M, \tau) = e(1) V'(z, \tau) \). Thus vacua \( z_0 + 2\pi k/M, \ k \in \mathbb{Z}/M\mathbb{Z} \), and \( Mz_0 = 0 \) (Abel’s thm)

classical vacua have a simple characterization in terms of the Weil pairing
\[ E[M] \subset E \text{ the group of } M\text{-torsion points,} \]

\[ \langle -, - \rangle_{\text{Weil}} : E[M] \times E[M] \to \mathbb{Z} / M\mathbb{Z} \text{ is the Weil pairing} \]

SQM model parametrized by \((E, p) \in T \subset E[M], e(p) = e^{2\pi i / M}, \)

\[
q \in E \text{ is a classical vacuum } \iff q \in E[M] \text{ and } \langle p, q \rangle_{\text{Weil}} = 1
\]

\[
(\text{model, vacuum}) \equiv (E, p, q : p, q \in E[M], \langle p, q \rangle_{\text{Weil}} = 1)
\]

**Spectral cover:** moduli space of such triples \((E, p, q)\) which are called

*Elliptic curves with a level } M\text{ structure of type } \Gamma(M)*

moduli space of such triples is yet another modular curve
Principal congruence subgroup $\Gamma(M) \subset SL(2, \mathbb{Z})$

$$\Gamma(M) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod M \right\} \subset SL(2, \mathbb{Z})$$

1 → $\Gamma(M) \to \Gamma_1(M)$ → $\mathbb{Z}/M\mathbb{Z}$ → 0,

$[\Gamma_1(M) : \Gamma(M)] = M$,

spectral cover $\equiv Y(M) \equiv \mathbb{H}/\Gamma(M) \overset{\text{comp.}}{\longrightarrow} X(M) \equiv \overline{\mathbb{H}}/\Gamma(M)$

The (compactified) spectral cover is the principal modular curve of level $M$, $X(M)$. The $M$-fold spectral cover

$X(M) \overset{\pi}{\longrightarrow} X_1(M)$ is the canonical projection

$\overline{\mathbb{H}}/\Gamma(M) \overset{\pi}{\longrightarrow} \overline{\mathbb{H}}/\Gamma_1(M)$, $M$-fold cover

On $X(M)$ the eigenvalue $\mu_\ell$ of $\Phi_\ell$ is well defined $\mu_\ell \in \Omega^1_{X(M)}(\log D_1)$, $D_1$ $\equiv$ divisor of type I cusps
Explicit form of eigenvalue one-form $\mu_\ell$ on $X(M) \equiv \overline{\mathbb{H}}/\Gamma(M)$

$$
\mu_\ell = \sum_{k=0}^{M-1} e^{2\pi i \ell k/M} \frac{d Z_{\ell,k}}{Z_{\ell,k}}
$$

where $Z_{\ell,k}(\tau)$ is the partition function of a complex free chiral fermion on a torus of periods $(2\pi, 2\pi \tau)$ subjected to the boundary conditions

$$
\psi(z + 2\pi) = e^{2\pi i \ell/M} \psi(z), \quad \psi(z + 2\pi \tau) = -e^{2\pi i k/M} \psi(z),
$$

$$
Z_{\ell,k} = q^{B_2(\ell/M)/2} \prod_{m=1}^{\infty} \left( 1 - e^{2\pi i k/M} q^{m-\ell/M} \right) \left( 1 - e^{-2\pi i k/M} q^{m-(M-\ell)/M} \right)
$$

$$
\frac{d Z_{\ell,k}}{Z_{\ell,k}} \quad \text{meromorphic one-form on } X(M)
$$
Modular properties

\[ \overline{H} \overset{\alpha}{\to} X(M) \equiv \overline{H}/\Gamma(M) \]

\[ \omega^* \mu_\ell = F_\ell(\tau) \ d\tau \]

\( F_\ell(\tau) \) is a meromorphic (poles at cusps) modular function with character for the congruence subgroup \( \Gamma_1(M) \) and good properties under \( \Gamma_0(M) \)

\[ \Gamma_0(M) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mod M \right\} \subset SL(2, \mathbb{Z}) \]

\[ 1 \to \Gamma_1(M) \to \Gamma_0(M) \overset{\alpha}{\to} (\mathbb{Z}/M\mathbb{Z})^\times \to 1 \]

\[ F_\ell\left( \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^2 e^{2\pi i a b (M-\ell)/M} F_{a\ell}(\tau), \]
Behavior at cusps

At cusps on $X(M)$ ($A(\phi, \theta), \Phi$) have regular singularities

$$A(\phi, \theta) = \frac{1}{2} q(\phi, \theta) \left( \frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right) + \text{regular},$$

$$\Phi = C \frac{dz}{z} + \text{regular}$$

$U$ cusp

$$\begin{cases} 
q(\phi, \theta) \text{ non-trivial} \\
C \text{ nilpotent}
\end{cases}$$

$I$ cusp

$$\begin{cases} 
q(\phi, \theta) = 0 \\
C \text{ semi-simple}
\end{cases}$$

- Eigenvalues of $q(\phi, \theta)$: the states which become massless at a $U$ cusp are described by a SCFT, the $q(\theta, \varphi)$ are the $U(1)_R$ charges of the susy vacua in this SCFT
- $C$: action on chiral ring $R_{SCFT}$ of operator $O$ perturbing away from cusp point
U vs. I cusps: Example

$X(5)$ has 12 cusps which map into 4 inequivalent cusps for the physical coupling curve $X_1(5)$. Which ones are U respectively I type?

Schur (1917) considered the infinite product:

$$K(q) = q^{-1/5} \prod_{n \geq 0} \frac{(1 - q^{5n+1})(1 - q^{5n+3})}{(1 - q^{5n+2})(1 - q^{5n+3})} = \frac{G(q)}{q^{1/5} H(q)} = \left( \frac{q^{1/5}}{1 + \frac{q^2}{1+q^2}} \right)^{-1}$$

$G(q), H(q)$ Rogers-Ramanujan funct. $\equiv$ characters of (2,5) minimal CFT

and asked for which roots of unity it converges. Answer:

it converges at $q = e^{2\pi i a/b}$ ($a/b \in \mathbb{Q}$) $\iff a/b$ a I cusp for the $M = 5$ model

$K(q)$ Hauptmodul of $X(5)$. Eigenvalue $\mu_\ell$ rational differential in $K(q)$. Icosahedral group $SL(2,\mathbb{Z})/\Gamma(5)$ acts on $K(q)$ by Möbius maps. Its action determines $\mu_\ell$

I cusps (width 5) $= \{0, \frac{1}{2}\}$, U cusps (width 1) $= \{\frac{2}{5}, \infty\}$

$M$ odd prime: $\frac{1}{2}\phi(M)$ U cusps (width 1), $\frac{1}{2}\phi(M)$ I cusps (width $M$)

width of cusp related to emergent $U(1)_R$ symmetry in the limit theory
Behavior at cusps

Residue of \( \mu_\ell \) at cusp \((a:c) \in \mathbb{P}^1(\mathbb{Q}), \gcd(a, c) = 1\)

\[
\kappa_{M,\ell}(a:c) \overset{\text{def}}{=} \frac{1}{2} M \sum_{k=0}^{M-1} e^{2\pi i \ell k/M} \tilde{B}_2 \left( \frac{a \ell + ck}{M} \right) \in \frac{1}{2M} \mathbb{Z}[e^{2\pi i/M}],
\]

\[
\tilde{B}_2(x) = \{x\}^2 - \{x\} + \frac{1}{6}, \quad \text{with} \quad \{x\} \equiv x - [x].
\]

**The cusp** \((a:c)\) **is U type iff** \(\kappa_{M,\ell}(a:c) = 0\)

\[
\kappa_{M,\ell}(a + sM; c + tM) = \kappa_{M,\ell}(a:c) \quad \forall s, t \in \mathbb{Z},
\]

\[
\kappa_{M,\ell}(a:c) = 0 \quad \text{if and only if} \quad \gcd(M, c) > 1,
\]

\[
\gcd(M, c) = 1 \Rightarrow \kappa_{M,\ell}(a + a'; c) = \varrho(a'c) \kappa_{M,\ell}(a;c),
\]

\[
\gcd(M, b) = 1 \Rightarrow \kappa_{M,\ell}(a;c) = \kappa_{M,\ell}(\bar{a}b; bc)
\]

\((\bar{a} \equiv \text{inverse in } \mathbb{Z}/M\mathbb{Z}, \bar{a}a = 1 \mod M, \varrho(k) = e^{2\pi i k(\ell-\ell)/M})\)

\(\tau = i\infty\) always a U cusp, \(\tau = 0\) always a l cusp
\( \mu_\ell: \text{ expressions are much simpler when } M \leq 5 \)

Genus covering curve \( X(M) \)

\[
g(X(M)) = \begin{cases} 
1 + \frac{M-6}{24} M^2 \prod_{p|\text{M}} \left( 1 - p^{-2} \right) & M > 2 \\
0 & M = 2,
\end{cases}
\]

For \( M \leq 5 \), \( g = 0 \) and \( X(M) \sim \mathbb{P}^1 \), isomorphism given by \textbf{Hauptmodul} \( z = z(\tau) \). There exists a Hauptmodul such that

\[
z(\tau + 1) = e^{2\pi i/M} z(\tau)
\]

\( \mu_\ell \) rational differential of the form

\[
\mu_\ell = \sum_{(a: c) \text{ type cusp of } X_1(M)} \kappa_\ell(a: c) \sum_{k=0}^{M-1} \frac{e^{2\pi i k \ell (M-\ell)}}{z - e^{2\pi i k / M} z(a : c)}
\]

\textbf{Example:} \( M = 5 \), \( z(\tau) = K(e^{2\pi i \tau})^{-1}; \ z(a : c) = (\frac{5}{c}) \left[ e^{2\pi i ac / 5} \frac{\sqrt{5}-1}{2} \right]^{\frac{5}{c}} \)
Example $M = 2$: $z(\tau) = 1 - \lambda(\tau)$

where $\lambda(\tau)$ Legendre modular function

$$z(0) = 0, \quad z(1) = \infty, \quad z(i\infty) = 1.$$ 

$z(\tau)$ modular invariant for $\Gamma(2)$ and in fact a $\text{Hauptmodul}$

$$z : X(2) \xrightarrow{\sim} \mathbb{P}^1,$$

$$z(T \tau) = z(\tau + 1) = \frac{1}{z(\tau)}.$$

In terms of the $\mathbb{P}^1$ coordinate $z = z(\tau)$, projection $X(2) \to X_1(2)$ is

$$z \sim z^{-1}.$$

- $z = 0$ and $z = \infty$ map to the unique $I$ type cusp on $X_1(2) \equiv X_0(2)$
- $z = 1$ is a $U$ type cusp

The eigenvalue 1-form $\mu$ on $\mathbb{P}^1 \cong X(2)$ has simple poles at $z = 0$ and $z = \infty$, this fixed it up to overall coefficient

$$\mu = -\frac{1}{4z} \frac{dz}{z}.$$
**tt* equations/Hitchin equations for \( M = 2 \)**

**tt* metric along the fiber**

\[
G(\phi, \theta, z) = \left| \varphi(\pi \tau) - \varphi(\pi (\tau + 1)) \right| \exp\left( \sigma_3 L(\phi, \theta, z) \right)
\]

\[
\partial_z \partial_{\bar{z}} L(\phi, \theta, z) = \frac{1}{16|z|^2} \sinh\left( 2L(\phi, \theta, z) \right)
\]

\[
L(\phi, \theta, z) = -L(-\phi, -\theta, z)
\]

Setting \( x = -\frac{1}{4} \log z \) we reduce to the well known \( \hat{A}_1 \) Toda equation

\[
\partial_x \partial_{\bar{x}} L(\phi, \theta, x) = \sinh\left( 2L(\phi, \theta, x) \right)
\]

many special solutions are known (often reduces to Painlevé III)

\( x \) not univalued on \( X(2) \), but univalued when pulled back to \( \mathbb{H} \)

We need an automorphic family of solutions to \( \hat{A}_1 \) Toda

\[
L\left( \phi, \theta, x(\tau) \right) = L\left( a\phi + b\theta, c\phi + d\theta, \frac{a\tau + b}{c\tau + d} \right), \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(2)
\]
No solution to Toda (known to me) has the right automorphic properties

**We can solve the equations near the cusps:**
- $z = 1$ is a U type cusp. The model is asymptotic to the LG model

$$W(X) = -2q^{1/2}(e^X - e^{-X}), \quad q = e^{2\pi i \tau} \to 0, \quad (\text{related to the } \mathbb{P}^1 \sigma\text{-model})$$

whose $tt^*$ equations are also $\tilde{A}_1\text{Toda}$

$$\partial_x \partial_x L(\theta, x) = \sinh(2L(\theta, x))$$

where $L(\theta, x), 0 \leq \theta < 2\pi,$ is the family of **all** solutions which vanish at infinity and are regular for $x \neq 0$ (they are Painlevé transcendentals)

setting $L(\phi, \theta, x) = L(\theta, x)$ solves the equations, reality constraints, regularity conditions, has the right asymptotics as $z \to 1,$ and passes other consistency checks

**yet it cannot be the correct solution since it is not automorphic**

- $\Rightarrow$ the solution cannot become trivial at the I cusp $z = 0$
- $tt^*$ eqns. linearize. Their solutions very reminiscent of Maass form
as before

\[ x = x + \pi i \]
No solution to Toda (known to me) has the right automorphic properties

**We can solve the equations near the cusps:**

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whose $tt^*$ equations are also $\hat{A}_1$ Toda

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- $\Rightarrow$ the solution cannot become trivial at the l cusp $z = 0$
- $tt^*$ eqns. linearize. Their solutions very reminiscent of Maass form
The case of general $M$ similar: $tt^*$ eqns. are $\hat{A}_{M-1}$ Toda equations

$$G(\phi, \theta) = \left| \sum_{k=0}^{M-1} e^{2\pi i\ell k/M} \varphi\left(\frac{2\pi (\ell \tau - k)}{M}\right) \right| \exp\left[\text{diag}(L_k(\phi, \theta, x))\right]$$

$$\sum_{k=1}^{M} L_k(\phi, \theta, x) = 0, \quad L_k(\phi, \theta; \tau) + L_{M-\ell-k}(-\phi, -\theta; \tau) = 0$$

Change of variable

$$\tau \mapsto x(\tau) = \int_{i\infty}^{\tau} \mu_\ell,$$

$$\partial_{\bar{\tau}} \partial_{x} L_k(\phi, \theta, x) =$$

$$= \exp\left[ L_k(\phi, \theta, x) - L_{k+1}(\phi, \theta, x) \right] - \exp\left[ L_{k-1}(\phi, \theta, x) - L_k(\phi, \theta, x) \right]$$

The solutions which are everywhere regular and vanish at $\infty$ are more or less known (and fully determined as a by-product of the present analysis) but they are not automorphic.
Asymptotics at U cusps more interesting:

**non-trivial action of the diamond duality group**

- it has important implications for general tt* geometry as well as for the theory of regular solutions to Toda eqns. (cfr. A. Its and co-worker)
- For $M$ an **odd prime** U cusps form a single orbit of $(\mathbb{Z}/M\mathbb{Z})^\times/\{\pm1\}$

The $\frac{1}{2}\phi(M) \Gamma_1(M)$-inequivalent U cusps are

$$\tau = i\infty, \text{ and } \tau = a/M \quad \text{with } 2 \leq a \leq (M - 1)/2 \equiv \phi(M)/2$$

The asymptotic behavior of model with character $e(k) = e^{2\pi i \ell k/M}$

$$q_\infty \equiv e^{2\pi i \tau} \sim 0 \quad W(X) \approx -M q_\infty^{(M-\ell)/M} \left[ \frac{e^{(M-\ell)X}}{M - \ell} + \frac{e^{-\ell X}}{\ell} \right].$$

$$q_{a/M} \equiv \exp\left[-2\pi i \frac{a\bar{\tau} - s}{MT - a}\right] \sim 0$$

$$W(X) \approx -M q_{a/M}^{[a\ell]/(M-|a\ell|)/M} \left[ \frac{e^{(M-|a\ell|)X}}{M - |a\ell|} + \frac{e^{-|a\ell|X}}{|a\ell|} \right],$$

$[n]$: the integer $n \mod M$ such that $0 \leq [n] \leq M - 1$. 
In other words: for $M$ odd prime we get at the several U type cusps all affine $\hat{A}(p, q)$ models with $p + q = M$ and $\text{gcd}(p, q) = 1$

**These models form an orbit of the duality** $(\mathbb{Z}/M\mathbb{Z})^\times / \{\pm 1\}$

They play a crucial role in classification of $N = 2$ susy in 2d and 4d:

- if $X \sim X + 2\pi i$ they are mirror to the $\sigma$-model with target the weighted projective line $\mathbb{P}(p, q)$
- if $X \sim X + 2\pi iK$ ($K$ an integer $K \to \infty$ in the thermodynamic limit) they are associated to the (mutation class of the) quiver obtained by orienting the affine Dynkin graph $\hat{A}_{MK-1}$ with $pK$ ($qK$) arrows in the positive (negative) direction
- the 2d BPS spectrum (in some chamber) is the Dynkin quiver
- the 2d quantum monodromy is minus the Coxeter of the affine quiver
- in 4d: $SU(2)$ SYM coupled to two Aargyres-Douglas of types $D_p$ and $D_q$
The $tt^*$ equations of all these models are the same $\widehat{A}_{M-1}$ Toda equations

$$\partial_x \partial_{\theta} L_k(\theta, x) = \exp \left[ L_k(\theta, x) - L_{k+1}(\theta, x) \right] - \exp \left[ L_{k-1}(\theta, x) - L_k(\theta, x) \right]$$

but with different reality constraints for different $p$

$$L_k(\theta, x) + L_{p-k}(-\theta, x) = 0$$

and different regularity conditions

It was a surprise that the regular solutions to these different PDE system are indeed related by the diamond duality $(\mathbb{Z}/M\mathbb{Z})^\times / \{ \pm 1 \}$

Regular solutions to the PDE recently described for $M = 5$ by A. Its et al.

Unexpected action of $(\mathbb{Z}/M\mathbb{Z})^\times / \{ \pm 1 \}$ explains their results and generalize them to arbitrary $M$

The **automorphic property** of solutions to $tt^*$ for the modular $\mathcal{N} = 4$ SQM models is actually useful (for a totally different problem)