Abstract: In this talk, I would like to discuss how we can realize the correspondence between AdS/CFT and tensor network in quantum field theories (i.e. the continuous limit). As the first approach I will discuss a possible connection between continuous MERA and AdS/CFT. Next I will introduce the second approach based on the optimization of Euclidean path-integral, where the structures of hyperbolic spaces and entanglement wedges emerge naturally. This second approach is closely related to the idea of tensor network renormalization.
Contents

① Introduction
② cMERA and AdS/CFT [refer also to Guifre’s, Sully’s talk]
③ Optimization of Path-Integral and AdS/CFT
④ Conclusions

Path-integral on flat space

Optimization of Path-integral

Path-integral on Hyperbolic Space
[Caputa-Kundu-Miyaji-Watanabe-TT 17]

Tensor Network Renormalization

[Evenbly-Vidal 14, 15]
[Verstraete’s talk]
Tensor Networks for Quantum Field Theories II
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Two Continuous Approaches to AdS/Tensor Network duality

Tadashi Takayanagi

Yukawa Institute for Theoretical Physics (YITP), Kyoto University

Overview of AdS/cMERA
+ 1703.00456 Caputa-Kundu-Miyaji-Watanabe-TT
Collaborators

Shinsei Ryu (Chicago)
Pawel Caputa (YITP, Kyoto, IFQ fellow)
Nilay Kundu (YITP, Kyoto)
Ali Mollabashi (IPM)
Masahiro Nozaki (Chicago)
Tokiro Numasawa (YITP/Osaka→McGill)
Noburo Shiba (Harvard)
Xueda Wen (UIUC)

Masamichi Miyaji (YITP, Kyoto)
Kento Watanabe (YITP, Kyoto)
References of Continuous Approach to TN I: cMERA

Original paper of cMERA:
Haegeman-Osborne-Verschelde-Verstraete [PRL 110(2013)100402]

Conformal symmetry in cMERA  Hu-Vidal 1703.04798 [Guifre’s talk]

Interactions in cMERA  Cotler-Molina-Vilaplana-Mueller 1612.02427

Information metric and Holography for cMERA:

Time evolution (Quantum quenches) and finite temp. in cMERA:

General Formulation (use of boundary state)

Miyaji-Numasawa-Shiba-Watanabe-TT  1506.01353
[PRL 115 (2015) 171602]
References of Continuous Approach to TN II:
Optimization of Path-integral

Qualitative Argument:

AdS from Optimization of path-integral:
1703.00456 Caputa-Kundu-Miyaji-Watanabe-TT
170?.?????? Work in progress

Refer also to Czech’s talk
① Introduction

**AdS/CFT** (or more generally holography) [Maldacena 20 yrs ago]
⇒ “Geometrization” of Dynamics in QFTs

One important fact behind holography is **Quantum entanglement**
⇒ “Geometry” of Quantum States in many-body system

**Emergent spacetime from Entanglement**

Quantum States

\[
|\Psi(t)\rangle = \sum_{\{i_k\}} c_{\{i_k\}}(t) |i_1\rangle \otimes |i_2\rangle \ldots \otimes |i_N\rangle
\]

Algebraically complicated, but geometrically look nicer!

Tensor Network [Guifre,..] = AdS [Swingle,..]

*Pirs: 17040042*
In holography, the entanglement is computed as the area of minimal surface \[ [Ryu-TT 06, Hybeny-Rangamani-TT 07]\]

\[
S_A = \frac{\text{Area}(\gamma_A)}{4G_N} \approx \frac{\text{Area}(\gamma_A)}{l_{pl}^2}.
\]

\(\gamma_A\): Minimal Area surface

Planck length \(\sim 1\) qubit

Spacetime in gravity = Collections of bits of entanglement

\(\Rightarrow\) Emergent space via tensor network?
Coarse-graining = Isometry

\[ [T]_{abc}^\dagger [T]_{bcd} = \delta_{ad} \]

Disentangler = Unitary trf.

\[ S_A \leq \text{Min[\#links]} \]
\[ \propto \log L \]
\[ \implies \text{agrees with results in 2d CFT!} \]
The original idea: Tensor Network of MERA (∃ scale inv.)
= a time slice of AdS space

Questions [see e.g. Beny 2011, Bao et.al. 2015, Czech et.al. 2015]
(a) Special Conformal invariance?
(b) Non-isotropic tensor \(\rightarrow\) ∃ causal structure in MERA
(c) Why the EE bound is saturated?
(d) Sub AdS scale Locality?

Recent developments in lattice models [Sully’s talk]
• Improved TN models: [Perfect TN: Pastawski-Yoshida-Harlow-Preskill 15]
  ⇒ (a),(b),(c) [Random TN: Hayden-Nezami-Qi-Thomas-Walter-Yang 16]
  [Hyper inv. TN: Evenbly 17]

• Another Interpretation:
  ⇒ (a),(b) [MERA as Kinematic Space (dS): Czech, Lamprou, McCandlish, Sully 15]
We expect some of these problems are due to lattice artifacts. Moreover, we would like to eventually understand the genuine AdS/CFT based on QFTs.

Therefore, in this talk we would like to focus on the continuous approaches to TNs.

Below we will discuss two different approaches.
Contents

1. Introduction
2. cMERA and AdS/CFT  [refer also to Guifre’s talk]
3. Optimization of Path-Integral and AdS/CFT
4. Conclusions

Path-integral on flat space

Optimization of Path-integral

Path-integral on Hyperbolic Space

Tensor Network Renormalization

[Evenbly-Vidal 14, 15]
Comment: irrespective of details, we can find a basic principle expected to be satisfied in AdS/TN:

**Surface/State correspondence** [Miyaji-TT 15]

A $d$ dim. convex space-like surface in $M$ (closed and homologically trivial)

$$|\Phi(\Sigma)\rangle \in H^M$$

If $\Sigma$ is open or topologically non-trivial, it corresponds to a mixed state.
The surface/state correspondence is realized in "nice" tensor networks description of holography (e.g. perfect TNs).

\[ |\Phi(\Sigma)\rangle \quad \text{AdS Bdy} \quad |\Phi_{\text{UV}}\rangle \]
(2-1) cMERA formulation [Haegeman-Osborne-Verschelde-Verstraete 11]
[Vidal’s talk]

The continuous MERA is defined as follows:

\[ \left| \Psi(u) \right\rangle = P \cdot \exp \left( -i \int_{-\infty}^{u} ds \left( K(s) + L \right) \right) \cdot \left| \Omega \right\rangle. \]

\[ \text{State at scale } \approx e^{-u} \]

\( K(u) \) : (dis)entangler,
\( UV : u = 0, \ IR : u = -\infty \)

\( L \) : coarse-graining (space-like rescaling).

\( \left| \Omega \right\rangle \) : unentangled state in real space

\( \rightarrow S_A = 0 \) for any \( A \).

This is identified with so called a boundary state.
[Miyaji-Ryu-Wen-TT 14]
IR State as Boundary State

\[
\frac{\langle \Omega | O(x_1)O(x_2) \cdots O(x_n) | \Omega \rangle}{\langle \Omega | \Omega \rangle} \approx \prod_{i=1}^{n} \langle O(x_i) \rangle.
\]

\( \Rightarrow \) When \((x_i-x_j) \gg \delta\), there are no correlations!

\( \Rightarrow \) Disentangled!
General Structures of cMERA for 2d CFTs (below UV cutoff) [Miyaji-Watanabe-TT 16]

Conformal Sym. (Virasoro): \( L_n, \tilde{L}_n \ (n \in \mathbb{Z}) \)

Space-like rescaling: \( L \Rightarrow \) Defined as radius quantum quench.

IR state: \( |\Omega\rangle \approx |B_0\rangle \) boundary states s.t. \( (L_n - \tilde{L}_{-n})|\Omega\rangle = 0 \).

Moreover, we have \([L_n - \tilde{L}_{-n}, L] = [L_n - \tilde{L}_{-n}, K] = 0\).

\( \Rightarrow \) Consistent with the condition \( L |\Omega\rangle = 0 \).

\( K(u) + L = D \) (i.e. dilatation of CFT)

\( \Rightarrow \) We can determine \( K(u) \).

In this way, we can obtain \( K(u) \) and \( L \) for general 2d CFTs.

\[\text{Tensor Networks have Conformal Symmetry} \]
(2-2) AdS/CFT and cMERA

[Miyaji-Numasawa-Shiba-Watanabe-TT 15, Miyaji-Watanabe-TT 16]

**AdS3/CFT2 setup**

Consider the global AdS3 space:

\[ ds^2 = R^2 (-\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\phi^2) , \]

whose isometry \( SL(2,R) \times SL(2,R) \) is generated by

\[
L_0 = i \partial_+ , \quad \bar{L}_0 = \partial_- , \\
L_{\pm 1} = i e^{\pm i x^+} \left[ \frac{\cosh 2\rho}{\sinh 2\rho} \partial_+ - \frac{1}{2} \sinh \frac{2\rho}{\sinh 2\rho} \partial_\mp \pm \frac{i}{2} \partial_\rho \right] , \\
\bar{L}_{\pm 1} = i e^{\pm i x^-} \left[ \frac{\cosh 2\rho}{\sinh 2\rho} \partial_- - \frac{1}{2} \sinh \frac{2\rho}{\sinh 2\rho} \partial_+ \mp \frac{i}{2} \partial_\rho \right] .
\]

On a time slice, the isometry is generated by \( l_n = L_n - \bar{L}_{-n} \quad (n = 0, \pm 1) \).
Derivation of cTN from AdS3/CFT2

\[ |\Psi(\rho)\rangle = P \exp\left[ -i \int_0^{\rho} M(\tilde{\rho})d\tilde{\rho} \right] |\Psi(0)\rangle, \]

\[ M(\rho) = \int_0^{2\pi} d\phi M(\rho, \phi). \]

⇒ Surface/state correspondence: Miyaji-TT 15

Since we have \( 2\partial_\rho \rho = e^{i\phi} l_{-1} - e^{-i\phi} l_1 \) in AdS3/CFT2, we can identify \( M \) as follows:

\[ M(\rho, \phi) \approx \lim_{\delta \to 0} \frac{1}{\delta} \int_{-\delta/2}^{\delta/2} d\tilde{\phi} \cdot \tilde{T}_\phi(\phi + \tilde{\phi}) = D(\phi). \]

In this way, \( M \) is identified with the dilatation \( D \).

⇒ This reproduces the cMERA network (below UV cut off).

\[ |\Psi(\rho)\rangle = |0\rangle_{R(\rho)}, \quad R(\rho) = \frac{\sinh \rho}{\sinh \rho_{\infty}} = e^u. \]
(2-2) AdS/CFT and cMERA

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\bar{L}_{\pm 1} &= i e^{\pm i x^-} \left[ \frac{\cosh 2\rho}{\sinh 2\rho} \partial_- - \frac{1}{\sinh 2\rho} \partial_+ \mp i \frac{\rho}{2} \right].
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\[ |\Psi(\rho)\rangle = |0\rangle_{R(\rho)}, \quad R(\rho) = \frac{\sinh \rho}{\sinh \rho_{\infty}} = e^u. \]
Comments

We find that cMERa satisfies the condition for the time slice of AdS3 (=2d hyperbolic space H2) below UV cut off. However, the same symmetry argument can be applied to 2d de-Sitter slices (dS2) in AdS3.

Hyperbolic slice

Corresponds to different choices of K(u) above the UV cut off?

de Sitter slice
(2-3) cMERA for a Massless Free Scalar Theory

[Haegeman-Osborne-Verschelde-Verstraete 11]

Hamiltonian: \[ H = \frac{1}{2} \int d^d k \left[ \pi(k) \pi(-k) + k^2 \phi(k) \phi(-k) \right]. \]

Ground state \[ |0\rangle : a_k |0\rangle = 0. \]

IR state: \[ a_x |\Omega\rangle = 0, \quad \left( a_x \equiv \sqrt{\Lambda} \phi(x) + \frac{i}{\sqrt{\Lambda}} \pi(x) \right), \]

i.e. \[ |\Omega\rangle = \prod_x |0\rangle_x \quad \Rightarrow \quad S_A = 0. \]

**Strength of disentanglers**

\[ cMERA: \quad K(u) = \frac{i}{2} \int d^d k \left[ \chi(u) \cdot \Gamma(k/\Lambda) a_k^+ a_{-k}^+ + (h.c.) \right], \]

where \[ \Gamma(x) \] is a cut off function: \[ \Gamma(x) = \theta(1-|x|). \]

For massless scalar, \[ \chi(u) = \frac{1}{2}. \]
(2-4) Information Metric from cMERA [Nozaki-Ryu-TT 12]

In cMERA, it is not straightforward to calculate EE analytically. Instead, there is another quantity which is more tractable: information metric.

\[
1 - \left| \left\langle \Psi(u) \right| e^{iLdu} \left| \Psi(u + du) \right\rangle \right|^2 = (du)^2 \cdot G_{uu} \cdot \left[ \int dx^d \cdot \int_0^{\Lambda e^u} dk^d \right] .
\]

The total volume of phase space at scale \( u \)

\[
= V_d \cdot \Lambda^d e^{du}.
\]

Note: The operation \( e^{iLdu} \) removes the coarse-graining procedure to extract the strength of disentanglers.
(2-3) cMERA for a Massless Free Scalar Theory

$\text{Hamiltonian: } H = \frac{1}{2} \int dk^d \left[ \pi(k) \pi(-k) + k^2 \phi(k) \phi(-k) \right].$

**Ground state** $|0\rangle : a_k |0\rangle = 0.$

**IR state:** $a_x |\Omega\rangle = 0,$

$$a_x = \sqrt{\Lambda} \phi(x) + \frac{i}{\sqrt{\Lambda}} \pi(x),$$

i.e. $|\Omega\rangle = \prod_x |0\rangle_x \Rightarrow S_A = 0.$

**cMERA:**

$$K(u) = \frac{i}{2} \int dk^d \left[ \chi(u) \cdot \Gamma(k / \Lambda) a_k^+ a_{-k}^+ + (h.c.) \right],$$

where $\Gamma(x)$ is a cut off function: $\Gamma(x) = \theta(1 - |x|).$

For massless scalar, $\chi(u) = \frac{1}{2}.$
Heuristic and Phenomenological Interpretation of Guu

Since \( \text{Guu} \propto \text{density}^2 \) of disentanglers, we expect

\[
S_A \sim \int_{u_{IR}}^{0} du \sqrt{G_{uu}} \cdot e^{(d-1)u} \sim \text{Hol.EE}
\]

\[\Rightarrow \text{Guu} \sim \text{guu} \quad \text{in} \quad ds_{\text{Gravity}}^2 = g_{uu} du^2 + \frac{e^{2u}}{\varepsilon^2} \cdot d\vec{x}^2 - g_{tt} dt^2.\]

Example: Free Scalar

\[K(u) = \frac{i}{2} \int dk^d \left[ \chi(u) \Gamma(k / \Lambda) a_k^+ a_{-k}^+ + (h.c.) \right], \quad \Rightarrow \quad g_{uu} = \chi(u)^2.\]

\[\chi(u) = \frac{1}{2} \cdot \frac{e^{2u}}{e^{2u} + m^2 / M^2}, \quad \text{for } m = 0, \quad \chi(u) = 1/2.\]
(2-5) Excited States after Quantum Quenches

\[ (A_k a_k + B_k a_k^+ ) |\Psi\rangle = 0, \quad (|A_k|^2 - |B_k|^2 = 1). \]

\[ A_k = \frac{1}{2} \left( \left( \frac{k^2 + m\theta^2}{k^2} \right)^{1/4} + \left( \frac{k^2}{k^2 + m\theta^2} \right)^{1/4} \right) e^{ikt}, \]

\[ B_k = \frac{1}{2} \left( \left( \frac{k^2 + m\theta^2}{k^2} \right)^{1/4} - \left( \frac{k^2}{k^2 + m\theta^2} \right)^{1/4} \right) e^{-ikt}. \]

To realize these states, we extend the ansatz as

\[ K(u) = \frac{i}{2} \int dk^d \Gamma(k / \Lambda) \left[ \chi(u) a_k^+ a_k^+ + \chi^*(u) a_k a_k \right]. \]
Time evolution (2d Scalar) under Quantum Quenches

\[ \sqrt{G_{uu}} = |\chi(u)| \]

Looks like a propagation of gravitational waves.

We can also (analytically) confirm \(\chi(u) \propto t\) and \(SA \propto t\) at late time. The same is true in higher dim.

This is consistent with the known CFT (2d) [Calabrese-Cardy 05]. and with the holographic result (any d).

③ Optimization of Path-Integral and AdS/CFT

(3-1) Motivation

Remember that the MERA can be obtained from the `optimization’ of tensor networks

⇒ Tensor network renormalization  [Evenbly-Vidal 14, 15]

Euclidean Path-Integral ⇒

MERA ⇐
Optimization of Path-Integral and AdS/CFT

(3-1) Motivation

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⇒ Tensor network renormalization  [Evenbly-Vidal 14, 15]

Euclidean Path-Integral ⇒

MERA ⇔
Optimization of Path-Integral

Euclidean Time (-z)

Optimization of Path-integral

Space (x)

TNR: Optimization of TN

Hyperbolic Space = Time slice of AdS
Optimization of Path-Integral

Euclidean Time (-z)

Optimization of Path-integral

Space (x)

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Hyperbolic Space = Time slice of AdS
(3-2) Formulating Optimization of Path-integral

**A Basic Rule:** Simplify the path-integral such that it produces the correct UV wave functional $\Psi_{UV}^{\text{Flat}}[\Phi(x)]$.

Below we focus on 2d CFTs for simplicity.

**Modification of discretizations in path-integral**

= Curved space metric s.t. one cell (bit) = unit length:

$$ds^2 = e^{2\varphi(x,z)}(dx^2 + dz^2).$$

[cf. Original flat metric:

$$ds^2 = \varepsilon^{-2} \cdot (dx^2 + dz^2),$$

where $\varepsilon$ is the UV cutoff.]

**Ground state UV wave function in curved space**

$$\Psi_{UV}^g[\Phi(x)] = \int \prod_{0<z<\infty} \prod_{-\infty<x<\infty} D\Phi(x, z) e^{-S_{\text{CFT}}(\Phi)} \cdot \delta(\Phi(x) - \Phi(x, z = 0))$$
In CFTs, owing to conformal sym., we have

\[ \Psi_{UV}^{g_{ab}=e^{2\varphi}\delta_{ab}}[\Phi(x)] = N[\varphi(x, z)] \cdot \Psi_{UV}^{\text{Flat}}[\Phi(x)]. \]

**Our Proposal** (Optimization of Path-integral for CFTs):

Minimize \( N[\varphi(x, z)] \) w.r.t \( \varphi(x, z) \)
with the boundary condition \( e^{2\varphi} \big|_{z=\varepsilon} = \varepsilon^{-2} \).

**Motivation**

The normalization \( N \) estimates repetitions of same
operations of path-integration. \( \rightarrow \) Minimize this!

\( \Rightarrow \) Our conjecture:

\[ N[\varphi(x, z)] \approx \exp[C[\varphi]] \]

\[ C[\varphi] \equiv \text{complexity of TN}[\varphi] \]

[Refer to a nice explanation based on TN : Czech’s talk afternoon]
In CFTs, owing to conformal sym., we have
\[ \Psi_{UV}^{g_{ab}=e^{2\phi}\delta_{ab}}[\Phi(x)] = N[\phi(x, z)] \cdot \Psi_{UV}^{{\text{Flat}}}[\Phi(x)]. \]

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The normalization \( N \) estimates repetitions of same operations of path-integration. \( \rightarrow \) Minimize this!

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where \( \varepsilon \) is the UV cutoff.]

Ground state UV wave function in curved space

\[
\Psi_{UV}^g[\Phi(x)] = \int \prod_{0<z<\infty} D\Phi(x, z) e^{-S_{\text{CFT}}(\Phi)} \cdot \delta(\Phi(x) - \Phi(x, z = 0))
\]
**A Sketch: Optimization of Path-Integral**

\[
\begin{align*}
\Psi_{UV}^{\text{Flat}}(\varphi(x)) & \quad \iff \quad \Psi_{UV}^{g}(\varphi(x)) \\
\text{z dependent state} & \\
\Psi[z, \varphi(x)] & \\
\text{[Cf. Leigh’s talk]} & \end{align*}
\]

\[ds^2 = \frac{dx^2 + dz^2}{z^2}\]

Time

- \(z = 0\)
- \(z = \infty\)

Space \(X\)

Optimize

High energy modes \(k > 1/z\) are not important!
(3-3) 2d CFT Vacuum

\[ \Psi_g[\Phi(x)] \] depends on metric due to Weyl anomaly and UV regularization.

\[ \frac{\Psi_\varphi}{\Psi_0} \propto e^{\frac{c}{24\pi} S_L} \]

Conformal Anomaly

\[ S_L = \int dx dz \left[ (\partial_x \varphi)^2 + (\partial_z \varphi)^2 + \frac{\mu}{4} e^{2\varphi} \right] \]

\[ = \int dx dz \left[ (\partial_x \phi)^2 + (\partial_z \phi + \sqrt{\mu} \cdot e^\varphi / 2)^2 \right] \]

\[ \Rightarrow \text{Minimum: } e^{2\varphi} = \frac{4}{\mu} \cdot z^{-2}. \]

Reproduces \( H_2 \) !

(we set \( \mu = 4 \))

UV div. \( \sim \varepsilon^{-2} \)

Liouville Theory [Polyakov 1981, ...]
(3-4) BTZ and Thermo Field Double

TFD state is described as

\[ \Psi_g[\Phi_1(x), \Phi_2(x)] = \int D\Phi(x, z) e^{-S_{\text{CFT}}(\Phi)} \cdot \delta(\Phi_1(x) - \Phi(x, z = \beta/4)) \delta(\Phi_2(x) - \Phi(x, z = -\beta/4)) \]

\[ \Rightarrow e^{2\varphi} = \frac{4\pi^2}{\beta^2} \cdot \frac{1}{\cos^2(2\pi z / \beta)}. \]

This agrees with the time slice of BTZ black hole!
(i.e. Einstein-Rosen Bridge).
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A Sketch: Optimization of Path-Integral

\[ \Psi_{\text{Flat}}^{UV}[\phi(x)] \rightarrow \infty \rightarrow \Psi_{\text{Flat}}^{g}[\phi(x)] \rightarrow \Psi[z, \phi(x)] \]

- **z dependent state**
- **Optimize**
- **Space x**
- **High energy modes**
  \( k > 1/z \) are not important!

\[ ds^2 = \frac{dx^2 + dz^2}{z^2} \]

[Cf. Leigh’s talk]
(3-4) BTZ and Thermo Field Double

TFD state is described as

\[ \Psi_g[\Phi_1(x), \Phi_2(x)] = \int \prod_{-\beta/4 < z < \beta/4} D\Phi(x, z) e^{-S_{CFT}(\Phi)} \cdot \delta(\Phi_1(x) - \Phi(x, z = \beta/4)) \delta(\Phi_2(x) - \Phi(x, z = -\beta/4)) \]

\[ \implies e^{2\phi}_{opt} = \frac{4\pi^2}{\beta^2} \cdot \frac{1}{\cos^2 \left( \frac{2\pi z}{\beta} \right)} . \]

This agrees with the time slice of BTZ black hole!
(i.e. Einstein-Rosen Bridge).
(3-5) Global AdS3 and Excitation

We insert an operator $O(x)$ in the center of disk. $O(x)$: conformal dim. = $(h, h) \Rightarrow O(x) \sim e^{-2h\cdot \phi}$.

Thus we minimize $\frac{\Psi^\phi}{\Psi_0} \propto e^{\frac{c}{24\pi} S_L} \cdot e^{-2h\phi(x_0)}$.

\[ \partial_w \partial_{w^*} \phi - \frac{\mu}{16} e^{2\phi} + \frac{6\pi h}{c} \delta^2(w) = 0. \]
Solution: \[ A(w) = w^a, \quad B(\bar{w}) = \bar{w}^a, \quad (a \equiv 1 - 12h / c). \]

Metric: \[ ds^2 = \frac{4d\zeta d\bar{\zeta}}{(1 - |\zeta|^2)^2}, \quad \zeta \equiv w^a = re^{i\theta} \]

⇒ Deficit angle geometry \[ \theta \sim \theta + 2\pi a. \]

This agrees with the gravity dual if \( h/c \ll 1 \).

Note: the AdS/CFT predicts \( a = \sqrt{1 - 24h / c} \).

Interestingly, if we consider the quantum Liouville CFT, then \[ h = \frac{\gamma \alpha}{4}(Q - \alpha \gamma / 2), \quad c = 1 + 3Q^2, \quad (Q \equiv 2 / \gamma + \gamma). \]

⇒ We get \[ a = \sqrt{1 - 24h / c}. \]
**Heuristic Summary**

Time

Local excitation (energy source)

Optimize

We locally need a fine graining
⇒ The metric gets larger!

This agrees with general relativity!
(3-6) Entanglement Wedge and Entropy

We optimize the reduced density matrix $\rho_A$

⇒ Geometry obtained by pasting two half disks:

$\rho_A = \begin{array}{c}
\begin{array}{c}
\delta = \pi(1-n)
\end{array}
\end{array}$

$\rho_A^n = \begin{array}{c}
\begin{array}{c}
\pi/2-\delta
\end{array}
\end{array}$

Optimize (Squeezing)

Boundary $\partial\Sigma$

Replica Method

Entanglement wedge!

$S_L^{(n)}[\varphi] = 2\int_{\Sigma} dx dz \left[ (\partial_x \varphi)^2 + (\partial_z \varphi)^2 + e^{2\varphi} \right] + 2\int_{\partial\Sigma} ds \left[ K \varphi + \mu_B e^{\varphi} \right]$ 

Reproduce the correct HEE!

$S_A = \frac{c}{6} \int_{\partial\Sigma} ds \ e^\varphi$

$\mu_B = \pi(1-n)$
Higher dimensional Generalization

For simplicity, we focus on the optimization of the Weyl rescaling mode:

$$ds^2 = e^{2\varphi(x,z)}(dz^2 + dx^2)$$

We argue that the function which we need to minimize for our optimization is given by

$$S_d = N \int dz dx^d \left[ e^{(d+1)\varphi} + e^{(d-1)\varphi} (\partial_z \varphi)^2 + e^{(d-1)\varphi} (\partial_x \varphi)^2 \right]$$

$$+ 2N \int_{bdy} dx^{d-1} \left[ \frac{e^{(d-1)\varphi} \cdot K_0}{d(d-1)} + \frac{\mu_B}{d} e^{(d-1)\varphi} \right].$$

This reproduces correct metrics, ent. wedge/entropy etc.
(3-8) Comments on Complexity

Recent proposals of holographic complexity:

(1) C=Vol \quad [\text{Susskind 2014,}...]

(2) C=Action \quad [\text{Brown-Roberts-Susskind-Swingle-Zhao 2015,}\
\quad \quad \text{Lehner-Myers-Poisson-Sorkin 2016,}\
\quad \quad \text{Chapman-Marrochio-Myers 2016,}...]

A Field theory definition of complexity is still missing.

[other approach: Myers’s talk]

One natural proposal based on our argument:

$$\text{Complexity of 2d CFT} = \min \left[ S_L [\varphi] \right]$$

[Refer to a nice explanation based on TN : Czech’s talk afternoon]
Evaluation of complexity (tentative results)

2d CFT  (1) Poincare AdS3: \[ S_L = \frac{c}{12\pi} \cdot \frac{L}{\varepsilon}. \]

(2) global AdS3: \[ S_L = \frac{c}{6} \cdot \left[ \frac{1}{\varepsilon} - 1 \right]. \]

(3) BTZ(TFD): \[ S_L = \frac{c}{3\varepsilon}. \]

3d CFT  global AdS4: \[ S_L = 4\pi N \left[ \frac{1}{\varepsilon^2} + \frac{1}{2} + \log\left( \frac{2}{\varepsilon} \right) \right]. \]

4d CFT  global AdS5: \[ S_L = 2\pi^2 N \left[ \frac{2}{3\varepsilon^3} + \frac{1}{\varepsilon} - \frac{5}{12} \right]. \]
4 Conclusions

In this talk we discussed two different approaches to AdS/TN duality.

cMERA

Our general symmetry argument supports the interpretation of a cMERA network as a slice of AdS.

Problems: (1) Explicit analysis can only be done for free CFTs. How about the holographic CFTs? (2) cMERA = H2 slice or dS2 slice of AdS? (3) AdS/cMERA above the UV cut off scale?
Optimization of Path-integral

We proposed:

Optimization of Euclidean path-integral of a CFT state
\[ \Leftrightarrow \] Time slice of its gravity dual in AdS/CFT

We explicitly studies this for 2d CFTs where the optimization can be done by minimizing the Liouville action.
\[ \Rightarrow \] Correct gravity dual metrics and entanglement wedge/entropy.

The Liouville action \~ a field theoretic counterpart of complexity. The minimizing complexity of TNs \[ \Rightarrow \] TN renormalization

Problems: Time-dependent dynamics ?
Correlation functions ?
Sub AdS locality ?