Title: Revisiting quantum incompatibility
Date: Jul 26, 2017  02:00 PM
URL: http://pirsa.org/17070048

Abstract:
Revisiting Quantum Incompatibility

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quantum contextuality

incompatibility

nondisturbance
commutativity

compatibility
Incompatibility

- is a unifying concept
- has many manifestations
- covers several features of quantum theory
this talk

1) simultaneous measurement of quantum observables
2) incompatibility of quantum channels
3) incompatibility of a quantum channel and observable
4) special nature of quantum incompatibility?
quantum observable
quantum observable

POVM

\[ A(x) \geq 0 \]

\[ \sum_x A(x) = I \]
quantum observable

\[ \mathcal{O} \quad A \]

\[ \text{POVM} \]

\[ A(x) \geq 0 \]

\[ \sum_x A(x) = I \]

\[ \text{tr}[\mathcal{O}A(x)] \]
quantum channel

input port  output port

input port  1. output port

input port  2. output port
measuring two observables simultaneously
Definition

and are broadcastable

if there exists such that

for all .
Remark

means that

and
Remark

means that

and
Remark

means that

$$\text{tr} [\Lambda(\varrho) A(x) \otimes I] = \text{tr} [\varrho A(x)]$$

and

$$\text{tr} [\Lambda(\varrho) I \otimes B(y)] = \text{tr} [\varrho B(y)]$$
Remark

means that

and
Example:

\[ A \quad \text{and} \quad B \quad \text{are broadcastable} \]

- Fix an orthonormal basis \( \{\varphi_j\} \)

- Use channel \( \Lambda(\varrho) = \sum_{j=1}^{d} \langle \varphi_j | \varrho \varphi_j \rangle | \varphi_j \otimes \varphi_j \rangle \langle \varphi_j \otimes \varphi_j | \)

- \( A \) and \( B \) are broadcastable if all operators \( A(x), B(y) \) are diagonal in the fixed basis
Therefore,

implies that is measured without disturbing
Proposition

can be measured without disturbing if and only if

there exist and such that

for all
Example:

$A$ can be measured without disturbing $B$ when

- $[A(x), B(y)] = 0$ for all outcomes $x, y$

Lüders instrument of $A$

$\varrho \mapsto \sqrt{A(x)} \varrho \sqrt{A(x)}$
Example: $A$ **cannot** be measured without disturbing $B$ when

- is any nontrivial observable

and

- is an informationally complete observable

Why?
Example:

\[
\text{cannot be measured without disturbing } \quad A \\
\text{when}
\]

- is any nontrivial observable

and

- is an informationally complete observable

Why?

*No information without disturbance*
Example: 

\[ A \text{ cannot be measured without disturbing } B \]

when

- \([A(x), B(y)] \neq 0 \) for some \(x, y\)

- \( \dim [\text{span}\{B(1), \ldots, B(n)\}] \geq (d-1)^2 + 1 \)

(B is informationally complete if \( \dim [\text{span}\{B(1), \ldots, B(n)\}] = d^2 \))
**Proposition**

For two qubit observables $A$ and $B$, the following are equivalent:

1. $A$ and $B$ are broadcastable
2. $A$ can be measured without disturbing $B$
3. $B$ can be measured without disturbing $A$
4. $[A(x), B(y)] = 0$ for all outcomes $x,y$
Remarks

For $d \geq 3$, the nondisturbance relation is

- not equivalent to commutativity
- not symmetric
Definition

and are compatible

if there exists such that

for all ; otherwise incompatible.
Definition

$A$ and $B$ are compatible

if there exists $G$ such that

$$\sum_y G(x, y) = A(x)$$

$$\sum_x G(x, y) = B(y)$$
Proposition

and are compatible

if and only if

there exist , , such that

for all .
Proof

• Assume that $A$ and $B$ are compatible, hence there exists $G$ such that

$$A(x) = \sum_y G(x, y), \quad B(y) = \sum_x G(x, y)$$

• Fix two sets of orthonormal unit vectors:

$$\{\varphi_x\} \text{ and } \{\eta_y\}$$
**Proof**

- Assume that \( A \) and \( B \) are compatible, hence there exists \( G \) such that

\[
A(x) = \sum_y G(x, y), \quad B(y) = \sum_x G(x, y)
\]

- Fix two sets of orthonormal unit vectors:

\[
\{ \varphi_x \} \quad \text{and} \quad \{ \eta_y \}
\]

- Observables:

\[
A'(x) = |\varphi_x\rangle\langle\varphi_x| \quad \text{and} \quad B'(y) = |\eta_y\rangle\langle\eta_y|
\]
\textbf{Proof}

- Assume that $A$ and $B$ are compatible, hence there exists $G$ such that

$$A(x) = \sum_y G(x, y), \quad B(y) = \sum_x G(x, y)$$

- Fix two sets of orthonormal unit vectors:

$$\{\varphi_x\} \quad \text{and} \quad \{\eta_y\}$$

- Observables:

$$A'(x) = |\varphi_x\rangle \langle \varphi_x| \quad \text{and} \quad B'(y) = |\eta_y\rangle \langle \eta_y|$$

- Channel:

$$\Lambda(\rho) = \sum_{x, y} \text{tr}[\rho G(x, y)] \ |\varphi_x\rangle \langle \varphi_x| \otimes |\eta_y\rangle \langle \eta_y|$$
If $A$ and $B$ are compatible, do they have a joint observable $G$ of some specific “canonical” form?
simple sufficient condition for compatibility

\[ G(x, y) = \frac{1}{2} \{ A(x), B(y) \} \]

\[ = \frac{1}{2} (A(x)B(y) + B(y)A(x)) \]

is a valid joint observable of \( A \) and \( B \) if

\[ \{ A(x), B(y) \} \geq 0 \]

for all outcomes \( x, y \).
simple sufficient condition for compatibility

\[ G(x, y) = \frac{1}{2} \{ A(x), B(y) \} \]
\[ = \frac{1}{2} (A(x)B(y) + B(y)A(x)) \]

is a valid joint observable of \( A \) and \( B \) if

\[ \{ A(x), B(y) \} \geq 0 \]

for all outcomes \( x, y \).

Remark:

\[ [A(x), B(y)] = 0 \Rightarrow \{ A(x), B(y) \} \geq 0 \]
**Proposition**

If $A$ or $B$ is sharp, then the following are equivalent:

1. $A$ and $B$ are compatible
2. $A$ be measured without disturbing $B$
3. $B$ can be measured without disturbing $A$
4. $[A(x), B(y)] = 0$ for all outcomes $x, y$
ignoring one output port
joint channel

is a joint channel of

and

if

and

=
incompatibility

Two channels

and

are **compatible** if they have a joint channel.

Otherwise they are **incompatible**.
incompatibility

Two channels

and

are **compatible** if they have a joint channel.

Otherwise they are **incompatible**.

• For this definition to make sense, channels must have the same input space. However, they can have different output spaces.
concatenation

• If \( \text{blue} \) and \( \text{red} \) are compatible,

then also \( \text{blue} \) and \( \text{red} \) and \( \text{green} \) are compatible.
concatenation

- If $\text{blue}$ and $\text{red}$ are compatible,

then also $\text{blue}$ and $\text{green}$ and $\text{red}$ are compatible.

**Proof:**

\[
\text{green} \quad \text{blue} \quad \text{green} = \quad \text{blue} \quad \text{green}
\]
observables as channels

\[ A \leftrightarrow \Gamma_A \]

orthogonal pure states

\[ \Gamma_A(\varrho) = \sum_x \text{tr}[\varrho A(x)]\xi_x \]
Proposition
Two observables (POVMs) $A$ and $B$ are compatible
if and only if
the corresponding channels $\Gamma_A$ and $\Gamma_B$ are compatible.
universal broadcasting

is a (perfect and universal) broadcasting channel if

\[= \text{identity channel}\]

and

\[= \text{identity channel}\]
no-broadcasting theorem

The following are equivalent:

1. Universal broadcasting is possible.
2. Two identity channels are compatible.
3. Any pair of channels are compatible.

No-broadcasting theorem in incompatibility flavor:
*There exists a pair of incompatible channels.*
self-compatible channels

is **self-compatible** if there exists such that

\[\text{and} \quad = \]

\[=\]
self-compatible channels

is self-compatible if there exists such that

and

**Proposition**

self-compatible channel = anti-degradable channel
\( n \)-self-compatible channels

is 3-\textit{self-compatible} if there exists such that

\[ \begin{align*}
\text{and} & \\
\text{and} &
\end{align*} \]
Every observable is $n$-self-compatible for every $n$:

$\rho$

\[2,1,1,3,4,2,1,3,4,..\]

\[2,1,1,3,4,2,1,3,4,..\]

\[2,1,1,3,4,2,1,3,4,..\]

\[2,1,1,3,4,2,1,3,4,..\]
Theorem

is \( n \)-self-compatible for every \( n=2,3,\ldots \)

if and only if

any collection of states
instrument
instrument
instrument
**Definition**

and are compatible if there exists such that

= and

=
two questions

1) Given 🎥, what are all possible channels compatible with it?

2) How does incompatibility relate to noise and disturbance?
post-processing

\[
\begin{pmatrix}
0.4 & 0.2 & 0.4 \\
0.3 & 0.1 & 0.6 \\
0.5 & 0.4 & 0.1
\end{pmatrix}
\]
Example: binary qubit observables

\[ A_1(\pm 1) = \frac{1}{2} (I \pm \sigma_1) \]

\[ A_t(\pm 1) = \frac{1}{2} (I \pm t\sigma_1) \]

sharp observable

post-processing

coin tossing observable
noise - disturbance trade-off
part 1

If $B$ is a post-processing of $A$, then any channel compatible with $A$ is also compatible with $B$. 
noise - disturbance trade-off
part 2

If all channels compatible with $A$ are compatible with $B$, then $B$ is a post-processing of $A$. 
noise - disturbance trade-off
part 3

\[ A \quad B \]

- \( B \) is a post-processing of \( A \) if and only if the least disturbing channel of \( A \) is compatible with \( B \).

- any channel compatible with \( A \) can be obtained from its least disturbing channel by concatenating it with some other channel.
mathematical construction of the least disturbing channel

- think $A$ as channel $\Gamma_A$
- take a minimal Stinespring dilation of $\Gamma_A$
- form the conjugate channel $\bar{\Gamma}_A$ of $\Gamma_A$
- $\bar{\Gamma}_A$ is the least disturbing channel of $A$ in the sense that any channel $\Lambda$ compatible with $A$ can be written as $\Lambda = \Lambda' \circ \bar{\Gamma}_A$
Example: Pauli channels

- two one-parametric families of binary qubit observables

\[ A_t(\pm 1) = \frac{1}{2} (I \pm t\sigma_1) \]
\[ B_s(\pm 1) = \frac{1}{2} (I \pm s\sigma_2) \]

- Pauli channels

\[ \Lambda_p(\rho) = \sum_{j=0}^{3} p_j \sigma_j \rho \sigma_j \]
\[ p = (p_0, p_1, p_2, p_3) \]
Example: Pauli channels

\[ s^2 + t^2 \leq 1 \]
Example: Pauli channels

\[ A_t \quad \quad s^2 + t^2 \leq 1 \quad \quad B_s \]

\[ t \leq 2(\sqrt{p_0 p_1} + \sqrt{p_2 p_3}) \]

\[ \Lambda_p \]
Example: Pauli channels

\[ s^2 + t^2 \leq 1 \]

completely depolarizing channel

\[ p_0 = p_1 = p_2 = p_3 = \frac{1}{4} \]

\[ t \leq 2(\sqrt{p_0p_1} + \sqrt{p_2p_3}) \]

\[ s \leq 2(\sqrt{p_0p_2} + \sqrt{p_1p_3}) \]

\[ \Lambda_p \]
Is there anything special about *quantum incompatibility*?
general probabilistic theory

- set of states $S = \text{convex closed subset of a real vector space } V$

- observable = affine map from $S$ to a set of probability distributions (no-restriction hypothesis)
valid in all non-classical theories

No-broadcasting theorem:

*There exists an incompatible pair of channels.*

[Barnum et al, PRL 99, 240501 (2007)]
valid in all non-classical theories

No-broadcasting theorem:
There exists an incompatible pair of channels.
[Barnum et al, PRL 99, 240501 (2007)]

There exists an incompatible pair of observables.
[Plávala, PRA 94, 042108 (2016)]
valid in all non-classical theories

No-broadcasting theorem:
There exists an incompatible pair of channels.
[Barnum et al, PRL 99, 240501 (2007)]

No-measurement-without disturbance:
There exists an incompatible pair of a channel and observable.

There exists an incompatible pair of observables.
[Plávala, PRA 94, 042108 (2016)]
Observable $A$ can discriminate two states $s_1$ and $s_2$ if there are subsets of outcomes $X_1$ and $X_2$ such that

$$prob(A \in X_1 | s_1) = 1 \quad \text{and} \quad prob(A \in X_2 | s_2) = 1$$
Observable \( B \) is **pure state informationally complete** if for any two different pure states \( s_1 \) and \( s_2 \),

\[
prob(B|s_1) \neq prob(B|s_2)
\]
In quantum theory, two observables $A$ and $B$ are incompatible if $A$ can discriminate (at least) two states and $B$ is pure state informationally complete.
In quantum theory, two observables $A$ and $B$ are incompatible if $A$ can discriminate (at least) two states and $B$ is pure state informationally complete.

In the square state space, there exist compatible observables $A$ and $B$ such that $A$ can discriminate two states and $B$ is pure state informationally complete.
valid in all non-classical theories

No-broadcasting theorem:
There exists an incompatible pair of channels.
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No-measurement-without disturbance:
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Something (related to incompatibility) that is valid in quantum theory but not in all general probabilistic theories?
In quantum theory, two observables $A$ and $B$ are incompatible if $A$ can discriminate (at least) two states and $B$ is pure state informationally complete.

In the square state space, there exist compatible observables $A$ and $B$ such that $A$ can discriminate two states and $B$ is pure state informationally complete.
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\[ a + c = b + d \]
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In the square state space, there exist compatible observables $A$ and $B$ such that $A$ can discriminate two states and $B$ is pure state informationally complete.

\[
\begin{align*}
G(1,1) &= B(1) & G(2,1) &= 0 \\
G(1,2) &= B(2) & G(2,2) &= 0 \\
G(1,3) &= 0 & G(2,3) &= B(3) \\
G(1,4) &= 0 & G(2,2) &= B(4)
\end{align*}
\]
In the square state space, there exist compatible observables $A$ and $B$ such that $A$ can discriminate two states and $B$ is pure state informationally complete.

<table>
<thead>
<tr>
<th></th>
<th>$G(1,1) = B(1)$</th>
<th>$G(2,1) = 0$</th>
<th>$B(1)$</th>
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<tbody>
<tr>
<td>$G(1,2) = B(2)$</td>
<td>$G(2,2) = 0$</td>
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<tr>
<td>$G(1,4) = 0$</td>
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<tr>
<td>$A(1)$</td>
<td>$A(2)$</td>
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</tbody>
</table>
1) **simultaneous measurement of quantum observables**
   - TH, Simultaneous measurement of two quantum observables: Compatibility, broadcasting, and in-between, PRA 93, 042118 (2016)

2) **incompatibility of quantum channels**
   - TH and T. Miyadera, Incompatibility of quantum channels, JPA 50, 135302 (2017)

3) **incompatibility of a quantum channel and observable**
   - TH and T. Miyadera, Qualitative noise-disturbance relation for quantum measurements, PRA 88, 042117 (2013)

4) **incompatibility in GPTs**
   - S. Filippov, TH and L. Leppäräri, Necessary condition for incompatibility of observables in general probabilistic theories, PRA 95, 032127 (2017)