Abstract: This talk will be about constraints on any model which reproduces the qubit stabilizer sub-theory. We show that the minimum number of classical bits required to specify the state of an n-qubit system must scale as $\sim n(n-3)/2$ in any model that does not contradict the predictions of the quantum stabilizer sub-theory. The Gottesman-Knill algorithm, which is a strong simulation algorithm is in fact, very close to this bound as it scales at $\sim n(2n+1)$. This is a result of state-independent contextuality which puts a lower bound on the minimum number of states a model requires in order to reproduce the statistics of the qubit stabilizer sub-theory.
Contextuality, PBR and their effect on the simulation of quantum systems

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The main result

The minimum number of classical bits required to specify the state of an $n$-qubit system in any model that reproduces stabilizer statistics is

$$\frac{n}{2} (n - 1)$$
Overview

• Why should you care?

• How is it related to contextuality?

• How did we do it?

• What now?
What does it mean to simulate quantum statistics?

\[
\begin{array}{c}
P \rightarrow M \rightarrow k \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|c}
P_1 & P_2 & \ldots & P_n \\
\hline
M_1 & & & \\
M_2 & & & \\
\vdots & & & \\
M_m & & & \\
\end{array}
\]

\[\Pr(k \mid P, M)\]
Why should one care about simulation of Quantum systems?

Quantum Computation  
Foundations of QM
Context

- Stabilizer sub-theory: Fault tolerant quantum computation
- Universal quantum computation: injecting “magic” states into stabilizer circuits

Qudits:
- magic states $\leftrightarrow$ Contextuality
- Non-negative Wigner functions $\rightarrow$ efficient classical sampling

Qubits: simulability
- state-independent contextuality
- Contextuality a computational resource?
- No efficient classical sampling
What we show

Qubits:

• The explicit effect of state-independent contextuality on size of the state-space of model
• Qubit stabilizer sub-theory is efficiently simulatable because the number of quantum states grows nicely
• A sampling algorithm cannot do much better than Gottesman-Knill
n-Qubit Stabilizer sub-theory

• Measurements: n-qubit Pauli Observables

• Preparations: eigenstates of n-qubit Pauli operators

• Transformations: Clifford Unitaries
Ontological Models

- State of the system $\lambda \in \Lambda$
- $\Pr(\lambda|P) = \mu_P(\lambda)$
- $\Pr(k|M, \lambda) = \xi_{k,M}(\lambda)$

Reproduce quantum predictions:

$$
\Pr(k|M, P) = \sum_{\Lambda} \mu_P(\lambda) \xi_{k,M}(\lambda) = Tr(\Pi_k \rho)
$$
Perfectly distinguishable preparation procedures cannot have ontic overlap

$$\text{Supp}(P_\rho) \cap \text{Supp}(P_\sigma) = \emptyset, \quad \text{Tr}(\rho \sigma) = 0$$
Perfectly distinguishable preparation procedures cannot have ontic overlap

\[ \text{Supp}(P_\rho) \cap \text{Supp}(P_\sigma) = \emptyset, \quad Tr(\rho \sigma) = 0 \]
Perfectly distinguishable preparation procedures cannot have ontic overlap

\[ \text{Supp}(P_\rho) \cap \text{Supp}(P_\sigma) = \emptyset, \quad \text{Tr}(\rho \sigma) = 0 \]
The state of the system can be described after a non-demolition measurement

\[
P \xrightarrow{\lambda} M_1 \xrightarrow{\lambda'} M_2 \rightarrow k'
\]

\[
\rho \rightarrow \rho'
\]

\[
\lambda \in \text{Supp}(P_\rho) \rightarrow \lambda' \in \text{Supp}(P_{\rho'})
\]
Two requirements:

1. Experimentally distinguishable states have disjoint support:
   \[ \text{Supp}(P_{\rho_i}) \cap \text{Supp}(P_{\rho_j}) = \emptyset, \quad Tr(\rho_i \rho_j) = 0 \]

2. The state of the system can be described even after a measurement:
   \[ \rho \rightarrow \rho' \]
   \[ \lambda \in \text{Supp}(\rho) \rightarrow \lambda' \in \text{Supp}(\rho') \]
PBR

\[ \bigcap_{PBR} \text{Supp}(\rho_i) = \emptyset \]

Proof:

\[
\begin{align*}
\rho_1 &= \{XI, IX, XX\} & \rho_1' &= \{YY, -ZZ, XX\} \\
\rho_2 &= \{ZI, IZ, ZZ\} & \rho_2' &= \{YY, ZZ, -XX\} \\
\rho_3 &= \{XI, IZ, XZ\} & \rho_3' &= \{YY, XZ, ZX\} \\
\rho_4 &= \{ZI, IX, ZX\} & \rho_4' &= \{YY, XZ, ZX\}
\end{align*}
\]
Contextuuality restricts overlap between states

<table>
<thead>
<tr>
<th></th>
<th>$\rho_3$</th>
<th>$\rho_4$</th>
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<tbody>
<tr>
<td>$\rho_1$</td>
<td>$X_1$</td>
<td>$X_2$</td>
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<td>$\rho_2$</td>
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<td>$XZ$</td>
<td>$ZX$</td>
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Result applies to sets equivalent to PBR set

\[ \text{Def : } s = \{ \rho_i \}, h = \{ \sigma_i \}, s \sim h \iff \exists C \text{ s.t. } C^+ \rho_i C = \sigma_i \]

\[ \bigcap_{e(PBR)} \text{Supp}(\rho_i) = \emptyset \]

Proof:

\[ \rho_1 = \{ XI, IX, XX \} \]
\[ \rho_2 = \{ ZI, IZ, ZZ \} \]
\[ \rho_3 = \{ XI, IZ, XZ \} \]
\[ \rho_4 = \{ ZI, IX, ZX \} \]

\[ C^+ \rho_2 C = \rho_2' = \{ -YY, ZZ, XX \} \]
\[ C^+ \rho_3 C = \rho_3' = \{ -YY, XZ, ZX \} \]
\[ C^+ \rho_4 C = \rho_4' = \{ -YY, -XZ, ZX \} \]
Other PBR like sets with empty overlap

\[ e\{\langle ZI,IZ\rangle,\langle XI,IX\rangle,\langle XI,IY\rangle,\langle YI,IZ\rangle\} \]

\[ e\{\langle ZI,IZ\rangle,\langle XI,IX\rangle,\langle XI,IY\rangle,\langle YI,IY\rangle\} \]

\[ e\{\langle ZI,IZ\rangle,\langle XI,IX\rangle,\langle XI,IY\rangle,\langle XX,ZY\rangle\} \]

All sets can be used to construct proofs of contextuality
Other sets with empty overlap

For a system of 2 qubits,

$$\bigcap_{s} \text{Supp}(\rho_i) = \emptyset, \forall |s| > 5$$

Proof:
One cannot construct any set of states with more than 5 states, such that one of its subsets of 4 is not PBR like.
n-qubits

For a system of n qubits,

\[ \bigcap_{s} \text{Supp}(\rho_i) = \emptyset, \forall \mid s \mid > 3^{n-2}5 \]

Proof: On the board (If I have time)
n-qubits

$$\bigcap_{s} Supp(\rho_i) = \emptyset, \forall |s| > 3^{n-25}$$

This implies that any ontic state can be in support of at most $3^{n-25}$ stabilizer states (preparation procedures corresponding to $3^{n-25}$ stabilizer states).

Min no. ontic states required = (no.of stabilizer states) / (max no. of states the ontic state can be in the supp of)

$$\min |\Lambda| = \frac{|\text{stab}|}{\max |s|}$$
n-qubits

\[ \min |\Lambda| \approx 2^{\frac{n^2}{2} - \frac{1}{2}n} \]

Minimum number of classical bits required to specify ontic state:

\[ \approx \frac{1}{2} n(n - 1) \]

Gottesman-Knill simulation:

\[ n(2n + 1) \]
Answers to questions about contextuality and qubit stabilizers

Q: What is the effect of the presence of contextuality in the qubit sub-theory on simulation?
A: No model can do much better than Gottesman-Knill. The minimum information required for any model is asymptotically \( \sim n^2 \).

Q: How is it different from the qudit sub-theory?
A: The absence of contextuality allows a sampling algorithm to do better than Gottesman-Knill. Wigner function \( \sim n \).
Contextuality: an explicit link to classical simulation

• Can this approach be applied to other sub-theories?
• Can we develop a measure of contextuality that has a direct link to simulability?
Contextuality: an explicit link to classical simulation

Definition 3.1.1 A non-contextual value assignment for a set of observables \( O = \{O_i| i = 1, \ldots, n\} \) is a function \( \nu: O \to \mathbb{R} \) such that \( \nu(O_i) \) is an eigenvalue of the Hermitian operator describing \( O_j \) and \( \nu(O_iO_j) = \nu(O_i)\nu(O_j) \) if \( O_i \) and \( O_j \) commute.

Kochen-Specker proof ➔ No non-contextual value assignment possible
Contextuality: an explicit link to classical simulation

Theorem: The eigenstates of a set of observables that do not allow a non-contextual value assignment cannot have an ontic overlap

- The largest set of quantum states that can be simulated by a single ontic state is the largest set that does not allow a proof of contextuality

- Min. size of ontic space bounded by the size of the largest set of states that does not allow a proof of contextuality
Summary

• A link between contextuality in qubit stabilizer sub-theory

• A bound on the size of the state space of any model that reproduces qubit- stabilizer statistics

• Can this approach be applied to other quantum sub-theories?