Title: Frobenius algebras, Hopf algebras and 3-categories

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Abstract: It is well known that commutative Frobenius algebras can be represented as topological surfaces, using the graphical calculus of dualizable objects in monoidal 2-categories. We build on related ideas to show that the interacting Frobenius algebras of Duncan and Dunne, which have a Hopf algebra structure, arise naturally in a similar way, by requiring a single 3-morphism in a 3-category to be invertible. We show that this gives a purely geometrical proof of Mueger's version of Tannakian reconstruction of Hopf algebras from fusion categories equipped with a fibre functor. We also relate our results to the theory of lattice code surgery.
Frobenius algebras, Hopf algebras and 3-categories

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Hopf algebras in Kitaev’s quantum double models
Perimeter Institute, Canada

August 3, 2017
The plan

- **Part 1.** Motivation
- **Part 2.** 2-categories
- **Part 3.** 3-categories
- **Part 4.** Hopf algebras
- **Part 5.** Higher linear algebra
- **Part 6.** Lattice models
The plan

- **Part 1.** Motivation
- **Part 2.** 2-categories
- **Part 3.** 3-categories
- **Part 4.** Hopf algebras
- **Part 5.** Higher linear algebra
- **Part 6.** Lattice models
Part 1
Motivation
What is higher algebra?

- Ordinary algebra lets us compose along a line:

  \[ xy^2 zyx^3 \]
What is higher algebra?

- Ordinary algebra lets us compose along a line: \( xy^2 zyx^3 \)

- *Higher algebra* lets us compose in higher dimensions:
A tradeoff between algebra and topology

Frobenius algebras

\[ \begin{array}{c}
\text{Diagram 1} \quad = \quad \text{Diagram 2} \\
\text{Diagram 3} \end{array} \]
A tradeoff between algebra and topology

Frobenius algebras as a ‘shadow’ of a two dimensional theory.
A tradeoff between algebra and topology

Frobenius algebras as a ‘shadow’ of a two dimensional theory.

\[
\begin{align*}
&\text{simple} \quad \text{topology} \quad + \quad \text{harder} \quad \text{algebra} \\
&\text{harder} \quad \text{topology} \quad + \quad \text{simple} \quad \text{algebra}
\end{align*}
\]
A tradeoff between algebra and topology

Frobenius algebras as a ‘shadow’ of a two dimensional theory.

study topology in terms of algebra

lower dimensional topology + harder algebra  

higher dimensional topology + simpler algebra
A tradeoff between algebra and topology

Frobenius algebras as a ‘shadow’ of a two dimensional theory.

study topology in terms of algebra

lower dimensional topology + harder algebra = higher dimensional topology + simpler algebra

‘outsource’ algebra to topology
A tradeoff between algebra and topology

Frobenius algebras as a ‘shadow’ of a two dimensional theory.

study topology in terms of algebra

lower dimensional topology + harder algebra = higher dimensional topology + simpler algebra

‘outsource’ algebra to topology

Next hour: Hopf algebras as a ‘shadow’ of a three dimensional theory.
Part 2
2-categories
Algebra in the plane = 2-category theory

The language describing algebra in the plane is *2-category theory*:

\[ \begin{align*}
A & \quad \xrightarrow{f} \quad B \\
\text{object} & \quad 1\text{-morphism} & \quad 2\text{-morphism}
\end{align*} \]
Algebra in the plane = 2-category theory

The language describing algebra in the plane is 2-category theory:

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\uparrow g & & \uparrow f \\
A & \xrightarrow{\eta} & B
\end{array}
\]

object          1-morphism          2-morphism

We can compose 2-morphisms like this:

\[
\begin{array}{ccc}
A & \xrightarrow{\epsilon} & B \\
\uparrow \eta & & \uparrow \epsilon \\
A & \xrightarrow{\eta} & B \\
\end{array}
\]

vertical composition          horizontal composition

These are pasting diagrams.
Algebra in the plane = 2-category theory

The language describing algebra in the plane is 2-category theory:

- Object: $A$
- 1-morphism: $f: A \rightarrow B$
- 2-morphism: $g: f \rightarrow g$

We can compose 2-morphisms like this:

- Vertical composition:
  - $A \xrightarrow{\epsilon} B$
  - $A \xrightarrow{\eta} B$

- Horizontal composition:
  - $A \xrightarrow{\eta} B \xrightarrow{\epsilon} C$

These are pasting diagrams. The dual diagrams are the graphical calculus.
Algebra in the plane = 2-category theory

The language describing algebra in the plane is 2-category theory:

- **Object**: $A$
- **1-morphism**: $f : A \to B$
- **2-morphism**: $g : f \Rightarrow f$

We can compose 2-morphisms like this:

- **Vertical composition**: $A \xrightarrow{\eta} B$
- **Horizontal composition**: $A \xrightarrow{\eta} B \xleftarrow{\epsilon} C$

These are pasting diagrams. The dual diagrams are the graphical calculus.
Algebra in the plane = 2-category theory

The language describing algebra in the plane is 2-category theory:

\[
\begin{align*}
A & \xrightarrow{f} B \\
\end{align*}
\]

object 1-morphism 2-morphism

We can compose 2-morphisms like this:

\[
\begin{align*}
A & \xrightarrow{\eta} B \\
\end{align*}
\]

vertical composition

\[
\begin{align*}
A & \xrightarrow{\eta} B \xrightarrow{\epsilon} C \\
\end{align*}
\]

horizontal composition

These are pasting diagrams.
Algebra in the plane = 2-category theory

The language describing algebra in the plane is 2-category theory:

- Object: $A$
- 1-morphism: $f: A \to B$
- 2-morphism: $g: f \to g$

We can compose 2-morphisms like this:

- Vertical composition: $\eta: A \to B$
- Horizontal composition: $\epsilon: B \to C$

These are pasting diagrams. The dual diagrams are the graphical calculus.
Algebra in the plane = 2-category theory

The language describing algebra in the plane is 2-category theory:

object

1-morphism

2-morphism

We can compose 2-morphisms like this:

vertical composition

horizontal composition

These are pasting diagrams. The dual diagrams are the graphical calculus. A 2-category with one object (the 'empty region') is a monoidal category.
Dualizable 1-morphisms

A 1-morphism $A \xrightarrow{f} B$ has a dual $B \xrightarrow{f^*} A$ if there are 2-morphisms:
Dualizable 1-morphisms

A 1-morphism $A \xrightarrow{f} B$ has a dual $B \xrightarrow{f^*} A$ if there are 2-morphisms:

such that the following hold:
Dualizable 1-morphisms

A 1-morphism $A \xrightarrow{f} B$ has a dual $B \xrightarrow{f^*} A$ if there are 2-morphisms:

such that the following hold:

**Theorem.** Graphical calculus for duals $\leftrightarrow$ oriented wires in the plane
Dualizable 1-morphisms

A 1-morphism $A \xrightarrow{f} B$ has a dual $B \xrightarrow{f^*} A$ if there are 2-morphisms:

such that the following hold:

**Theorem.** Graphical calculus for duals $\leftrightarrow$ oriented wires in the plane

| Tangle hypothesis. |
| Bord$_{1,0}^{2D}$ $\cong$ free monoidal category on a dualizable object |
Dualizable 1-morphisms

A 1-morphism $A \xrightarrow{f} B$ has a dual $B \xrightarrow{f^*} A$ if there are 2-morphisms:

such that the following hold:

**Theorem.** Graphical calculus for duals $\leftrightarrow$ oriented wires in the plane

**Definition.** $G$ directed graph $\Rightarrow \mathcal{F}_2(G) :=$ free 2-category with duals on $G.$
Dualizable 1-morphisms

A 1-morphism $A \xrightarrow{f} B$ has a dual $B \xrightarrow{f^*} A$ if there are 2-morphisms:

\[
\begin{array}{cccc}
  f^* & f & f^* & f \\
  \text{diag} & \text{non-diag} & \text{diag} & \text{non-diag}
\end{array}
\]

such that the following hold:

\[
\begin{array}{cccc}
  \text{diag} & \text{non-diag} & \text{diag} & \text{non-diag}
\end{array}
\]

**Theorem.** Graphical calculus for duals $\leftrightarrow$ oriented wires in the plane

**Definition.** $G$ directed graph $\Rightarrow \mathcal{F}_2(G) :=$ free 2-category with duals on $G$.

**Example.** $\mathcal{F}_2\left(\begin{array}{c} 1 \\ \rightarrow \end{array}\right):$ free 2-category on dualizable 1-morphism
Dualizable 1-morphisms

A 1-morphism $A \xrightarrow{f} B$ has a dual $B \xrightarrow{f^*} A$ if there are 2-morphisms:

such that the following hold:

**Theorem.** Graphical calculus for duals ↔ oriented wires in the plane

**Definition.** $G$ directed graph $\Rightarrow \mathcal{F}_2(G) :=$ free 2-category with duals on $G$.

**Example.** $\mathcal{F}_2\left(\begin{array}{c}
\text{defect data} \\
\text{2-1-0}
\end{array}\right)$: free 2-category on dualizable 1-morphism
Frobenius algebras and dualizable 1-morphisms

A *Frobenius algebra* in a monoidal category is an object with morphisms:

such that:

```
=  
```

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Frobenius algebras and dualizable 1-morphisms

A *Frobenius algebra* in a monoidal category is an object with morphisms:

such that:

\[
\begin{align*}
\begin{array}{cccc}
\ triangledown & = & \ triangledown & = \\
\ triangle & = & \ y & = \\
\ y & = & \ y & = \\
\ y & = & \ y & = \\
\ u & = & \ u & = \\
\ u & = & \ u & = \\
\ end{array}
\end{align*}
\]
Frobenius algebras and dualizable 1-morphisms

A *Frobenius algebra* in a monoidal category is an object with morphisms:

such that:

*Frob*: free monoidal category on a Frobenius algebra
Frobenius algebras and dualizable 1-morphisms

A Frobenius algebra in a monoidal category is an object with morphisms:

\[ \begin{array}{ccc}
\circlearrowright & \xrightarrow{} & \circlearrowright \\
\circlearrowright & \xrightarrow{} & \circlearrowright \\
\end{array} \]

such that:

\[ \begin{array}{ccc}
\circlearrowright & \xrightarrow{} & \circlearrowright \\
\circlearrowright & \xrightarrow{} & \circlearrowright \\
\end{array} \]

\textbf{Frob}: free monoidal category on a Frobenius algebra

There is a 2-functor \( \text{Frob} \) 'thickening' \( F_2 := \mathcal{F}_2 \left( \begin{array}{c}
\circlearrowright \\
\circlearrowright \\
\end{array} \right) \).
Frobenius algebras and dualizable 1-morphisms

A Frobenius algebra in a monoidal category is an object with morphisms:

\[
\begin{align*}
& 
\begin{array}{cccc}
\implies & \implies & \implies & \implies \\
\implies & \implies & \implies & \implies \\
\implies & \implies & \implies & \implies \\
\implies & \implies & \implies & \implies \\
\end{array}
\end{align*}
\]

such that:

\[
\begin{align*}
\begin{array}{cccc}
\implies & \implies & \implies & \implies \\
\implies & \implies & \implies & \implies \\
\implies & \implies & \implies & \implies \\
\implies & \implies & \implies & \implies \\
\end{array}
\end{align*}
\]

Frob: free monoidal category on a Frobenius algebra

There is a 2-functor \(\text{Frob} \xrightarrow{\text{thickening}} \mathcal{F}_2 := \mathcal{F}_2 \left( \begin{array}{c} \square \implies \square \end{array} \right)\).
Frobenius algebras and dualizable 1-morphisms

A *Frobenius algebra* in a monoidal category is an object with morphisms:

![Diagram](image)

such that:

![Diagram](image)

**Frob**: free monoidal category on a Frobenius algebra

There is a 2-functor \( \text{Frob} \overset{\text{thickening}}{\rightarrow} F_2 := \mathcal{F}_2 \left( \begin{array}{c} \square \rightarrow \square \end{array} \right) \).

**Theorem.** This induces a monoidal equivalence \( \text{Frob} \cong F_2 \left( \begin{array}{c} \square, \square \end{array} \right) \).
Frobenius algebras and dualizable 1-morphisms

A *Frobenius algebra* in a monoidal category is an object with morphisms:

such that:

\[ \begin{align*}
&\begin{array}{c}
\xymatrix{
\triangle & \triangle \\
\downarrow & \downarrow \\
\square & \square \\
\end{array}
& \begin{array}{c}
\xymatrix{
\square & \square \\
\downarrow & \downarrow \\
\triangle & \triangle \\
\end{array}
& \begin{array}{c}
\xymatrix{
\triangle & \triangle \\
\downarrow & \downarrow \\
\square & \square \\
\end{array}
& \begin{array}{c}
\xymatrix{
\triangle & \triangle \\
\downarrow & \downarrow \\
\square & \square \\
\end{array}
& \begin{array}{c}
\xymatrix{
\triangle & \triangle \\
\downarrow & \downarrow \\
\square & \square \\
\end{array}
& \begin{array}{c}
\xymatrix{
\triangle & \triangle \\
\downarrow & \downarrow \\
\square & \square \\
\end{array}
& \begin{array}{c}
\xymatrix{
\triangle & \triangle \\
\downarrow & \downarrow \\
\square & \square \\
\end{array}
& \begin{array}{c}
\xymatrix{
\triangle & \triangle \\
\downarrow & \downarrow \\
\square & \square \\
\end{array}
\end{align*}
\]

\[\textbf{Frob}: \text{free monoidal category on a Frobenius algebra}\]

There is a 2-functor \(\text{Frob} \xrightarrow{\text{thickening}} F_2 := F_2 \left( \begin{array}{c}
\begin{array}{c}
\square & \square \\
\downarrow & \downarrow \\
\triangle & \triangle \\
\end{array}
\end{array} \right)\).

\textbf{Theorem}. This induces a monoidal equivalence \(\text{Frob} \cong F_2 \left( \begin{array}{c}
\begin{array}{c}
\square & \square \\
\downarrow & \downarrow \\
\triangle & \triangle \\
\end{array}
\end{array} \right)\).
Frobenius algebras and dualizable 1-morphisms

A *Frobenius algebra* in a monoidal category is an object with morphisms:

such that:

\[
\begin{array}{cccc}
\begin{array}{c}
\alpha \\
\beta \\
\gamma \\
\delta \\
\end{array}
& \to &
\begin{array}{c}
\alpha \\
\beta \\
\gamma \\
\delta \\
\end{array}
& = &
\begin{array}{c}
\alpha \\
\beta \\
\gamma \\
\delta \\
\end{array}
\end{array}
\]

\[
\begin{array}{cccc}
\begin{array}{c}
\epsilon \\
\zeta \\
\eta \\
\iota \\
\end{array}
& \Rightarrow &
\begin{array}{c}
\epsilon \\
\zeta \\
\eta \\
\iota \\
\end{array}
& = &
\begin{array}{c}
\epsilon \\
\zeta \\
\eta \\
\iota \\
\end{array}
\end{array}
\]

**Frob:** free monoidal category on a Frobenius algebra

There is a 2-functor \( \text{Frob} \) `thickening` \( \text{F}^2 := \mathcal{F}_2 (\begin{array}{c}
\begin{array}{c}
\Box \\
\Box \\
\end{array}
\end{array}) \).

**Theorem.** This induces a monoidal equivalence \( \text{Frob} \cong \text{F}^2 (\begin{array}{c}
\begin{array}{c}
\Box \\
\Box \\
\end{array}
\end{array}) \).

\( \text{Frob} \) as a 'shadow' of the theory of dualizable 1-morphisms in 2-categories.
Other algebraic theories?

What about *commutative* Frobenius algebras

\[
\begin{align*}
\begin{tikzpicture}[scale=0.5]
\draw[thick] (0,0) -- (1,0) -- (1,2) -- (0,2) -- cycle;
\end{tikzpicture}
\end{align*}
\]
Other algebraic theories?

What about *commutative* Frobenius algebras or *bialgebras*?

\[
\begin{align*}
&\quad = \\
\text{Diagram 1} &\quad = \\
\text{Diagram 2} &\quad = \\
\end{align*}
\]
Other algebraic theories?

What about *commutative* Frobenius algebras or *bialgebras*?

Only make sense in (at least) three dimensional space.
Other algebraic theories?

What about *commutative* Frobenius algebras or *bialgebras*?

Only make sense in (at least) three dimensional space.

\[ \Downarrow \]

Shadows of 3D structures?
Part 3
3-categories
Algebra in three dimensions = 3-category theory

The language describing algebra in three dimensions is 3-category theory:
Algebra in three dimensions = 3-category theory

The language describing algebra in three dimensions is **3-category theory**:
Algebra in three dimensions = 3-category theory

The language describing algebra in three dimensions is \textit{3-category theory}:

object

1-morphism
Algebra in three dimensions = 3-category theory

The language describing algebra in three dimensions is 3-category theory:

object

1-morphism

2-morphism
Algebra in three dimensions = 3-category theory

The language describing algebra in three dimensions is 3-category theory:

- object
- 1-morphism
- 2-morphism
- 3-morphism
Algebra in three dimensions = 3-category theory

The language describing algebra in three dimensions is 3-category theory:

We can compose 3-morphisms like this:
Algebra in three dimensions = 3-category theory

The language describing algebra in three dimensions is 3-category theory:

object

1-morphism

2-morphism

3-morphism

We can compose 3-morphisms like this:

vertical composition
Algebra in three dimensions = 3-category theory

The language describing algebra in three dimensions is 3-category theory:

object
1-morphism
2-morphism
3-morphism

We can compose 3-morphisms like this:

vertical composition
horizontal composition
Algebra in three dimensions = 3-category theory

The language describing algebra in three dimensions is 3-category theory:

object 1-morphism 2-morphism 3-morphism

We can compose 3-morphisms like this:

vertical composition horizontal composition layered composition
Algebra in three dimensions = 3-category theory

The language describing algebra in three dimensions is 3-category theory:

- object
- 1-morphism
- 2-morphism
- 3-morphism

We can compose 3-morphisms like this:

- vertical composition
- horizontal composition
- layered composition

A one object (the ‘empty region’) 3-category is a monoidal 2-category.
Algebra in three dimensions = 3-category theory

The language describing algebra in three dimensions is 3-category theory:

object

1-morphism

2-morphism

3-morphism

We can compose 3-morphisms like this:

vertical composition

horizontal composition

layered composition

A one object (the ‘empty region’) 3-category is a monoidal 2-category.

A one object and one 1-morphism 3-category is a braided monoidal category.
Duals in 3-categories

A 1-morphism $A$ has an oriented dual $A^*$ if there are 2-morphisms (folds):
**Duals in 3-categories**

A 1-morphism $A$ has an *oriented dual* $A^*$ if there are 2-morphisms (*folds*):

![Diagram showing 1-morphisms and oriented duals](image)

and 3-morphisms (*cusps, saddles and births/deaths of the circle*):

![Diagram showing 3-morphisms](image)

+ horizontal and vertical reflections and opposite orientation
Duals in 3-categories

A 1-morphism $A$ has an oriented dual $A^*$ if there are 2-morphisms (folds):

and 3-morphisms (cusps, saddles and births/deaths of the circle):

such that the following hold (and reflections and opposite orientation):

Ref: Section 162
Duals in 3-categories

**Theorem.** Graphical calculus for duals $\leftrightarrow$ oriented surfaces in 3D space
Duals in 3-categories

**Theorem.** Graphical calculus for duals $\leftrightarrow$ oriented surfaces in 3D space

**Tangle hypothesis.**
$\text{Bord}^{3D}_{2,1,0} \cong$ free monoidal 2-category on a dualizable object
Duals in 3-categories

**Theorem.** Graphical calculus for duals $\leftrightarrow$ oriented surfaces in 3D space

Let $G$ be a 2-globular set $G = \left(\begin{array}{c}
\text{2-Edges} \leftrightarrow \text{Edges} \leftrightarrow \text{Vertices}\end{array}\right)$
Duals in 3-categories

**Theorem.** graphical calculus for duals $\leftrightarrow$ oriented surfaces in 3D space

Let $G$ be a 2-globular set $G = \left( \begin{array}{c} \text{2-Edges} & \text{Edges} & \text{Vertices} \end{array} \right)$
**Theorem.** Graphical calculus for duals ↔ oriented surfaces in 3D space

Let $\mathcal{G}$ be a 2-globular set $\mathcal{G} = \left(\begin{array}{c}
\text{2-Edges} \\
\text{Edges} \\
\text{Vertices}
\end{array}\right)$
**Duals in 3-categories**

**Theorem.** Graphical calculus for duals $\leftrightarrow$ oriented surfaces in 3D space

Let $\mathcal{G}$ be a 2-globular set $\mathcal{G} = \left(\begin{array}{c} \text{2-Edges} \leftrightarrow \text{Edges} \leftrightarrow \text{Vertices} \end{array}\right)$

**Def.** $\mathcal{F}_3(\mathcal{G})$: free 3-category with duals for 2- and 1-morphisms given in $\mathcal{G}$.

**Examples.**

$\mathcal{F}_3\left(\begin{array}{c} \text{square} \end{array}\right)$: free 3-category on a dualizable 1-morphism

$\mathcal{F}_3\left(\begin{array}{c} \text{triangle} \end{array}\right)$: free 3-category on \{two dualizable 1-morphisms
\{one dualizable 2-morphism\}
**Duals in 3-categories**

**Theorem.** Graphical calculus for duals $\leftrightarrow$ oriented surfaces in 3D space

Let $\mathcal{G}$ be a 2-globular set $\mathcal{G} = \left( \begin{array}{c} \text{2-Edges} \leftrightarrow \text{Edges} \leftrightarrow \text{Vertices} \end{array} \right)$

**Def.** $\mathcal{F}_3(\mathcal{G})$: free 3-category with duals for 2- and 1-morphisms given in $\mathcal{G}$.

**Examples.**

$\mathcal{F}_3\left( \begin{array}{c} \text{\includegraphics[width=1cm]{example1}} \end{array} \right)$: free 3-category on a dualizable 1-morphism

$\mathcal{F}_3\left( \begin{array}{c} \text{\includegraphics[width=1cm]{example2}} \end{array} \right)$: free 3-category on \{two dualizable 1-morphisms, one dualizable 2-morphism\}

**Summary.** The graphical calculus of $\mathcal{F}_3(\mathcal{G})$ is given by regions, surfaces and wires in three dimensional space.
**Duals in 3-categories**

**Theorem.** Calculus for duals $\leftrightarrow$ oriented surfaces in 3D space

Let $\mathcal{G}$ be a 2-globular set $\mathcal{G} = \left( 2\text{-Edges} \leftrightarrow \text{Edges} \leftrightarrow \text{Vertices} \right)$

**Def.** $\mathcal{F}_3(\mathcal{G})$: free 3-category with duals for 2- and 1-morphisms given in $\mathcal{G}$.

**Examples.**

$\mathcal{F}_3(\mathcal{G})$ for defect bordisms embedded in $\mathbb{R}^3$

$\mathcal{F}_3$ (defect bordisms)

$\mathcal{F}_3$ (3-2-1-0)

$\mathcal{F}_3$ (dualizable 1-morphism)

$\mathcal{F}_3$ (free 3-category on)

**Summary.** The graphical calculus of $\mathcal{F}_3(\mathcal{G})$ is given by regions, surfaces and wires in three dimensional space.
Commutative Frobenius algebras

A *commutative* Frobenius algebra is a Frobenius algebra such that:

\[ \text{Diagram:} \]

David Reutter

Hopf algebras and 3-categories

August 3, 2017

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Commutative Frobenius algebras

A *commutative* Frobenius algebra is a Frobenius algebra such that:

\[
\begin{align*}
\text{cFrob}: & \quad \text{free braided monoidal category on a commutative Frobenius algebra}
\end{align*}
\]
Commutative Frobenius algebras

A *commutative* Frobenius algebra is a Frobenius algebra such that:

\[
\begin{array}{ccc}
\begin{tikzpicture}
  \draw (0,0) -- (0,1);
  \draw (0,1) -- (0.5,1.5);
\end{tikzpicture}
& = &
\begin{tikzpicture}
  \draw (0,0) -- (0,1);
  \draw (0,0) arc (180:0:0.5);
  \draw (0,1) arc (90:0:0.5);
\end{tikzpicture}
\end{array}
\]

\[\begin{tikzpicture}
  \draw (0,0) -- (0,1);
  \draw (0,1) -- (0.5,1.5);
\end{tikzpicture}\]

\[\begin{tikzpicture}
  \draw (0,0) -- (0,1);
\end{tikzpicture}\]

**cFrob**: free *braided* monoidal category on a commutative Frobenius algebra
Commutative Frobenius algebras

A *commutative* Frobenius algebra is a Frobenius algebra such that:

\[ \begin{array}{c}
\quad = \\
\quad = \\
\end{array} \]

\( \textbf{cFrob} \): free *braided* monoidal category on a commutative Frobenius algebra

There is a 3-functor \( \textbf{cFrob} \rightarrow F_3 := \mathcal{F}_3 \left( \begin{array}{c}
\quad \\
\quad \\
\end{array} \right) \).
Commutative Frobenius algebras

A *commutative* Frobenius algebra is a Frobenius algebra such that:

\[ \begin{array}{ccc}
\text{commutative} & \Rightarrow & \text{Frobenius} \\
\end{array} \]

\[ \begin{array}{ccc}
\text{cFrob} : & \text{free braided} & \text{monoidal category on a commutative Frobenius algebra} \\
\end{array} \]

There is a 3-functor \( \text{cFrob} \xrightarrow{\text{'thickening'}} F_3 \) := \( F_3 \left( \begin{array}{c}
\text{\includegraphics[width=1cm]{example1.png}}
\end{array} \right) \).

**Theorem.** This induces a braided monoidal equivalence \( \text{cFrob} \cong F_3 \left( \begin{array}{c}
\text{\includegraphics[width=1cm]{example2.png}}
\end{array} \right) \).
Commutative Frobenius algebras

A *commutative* Frobenius algebra is a Frobenius algebra such that:

\[ \begin{array}{c}
\includegraphics[width=2cm]{diagram1.png} \\
= \\
\includegraphics[width=2cm]{diagram2.png} \\
= \\
\includegraphics[width=2cm]{diagram3.png}
\end{array} \]

**cFrob**: free *braided* monoidal category on a commutative Frobenius algebra

There is a 3-functor \( \text{cFrob} \xrightarrow{\text{thickening}} F_3 := \mathcal{F}_3 \left( \begin{array}{c}
\includegraphics[width=2cm]{diagram4.png}
\end{array} \right) \).

**Theorem.** This induces a braided monoidal equivalence \( \text{cFrob} \cong F_3 \left( \begin{array}{c}
\includegraphics[width=2cm]{diagram5.png}
\end{array} \right) \).

\( \text{cFrob} \) as a *shadow* of the theory of dualizable 1-morphisms in 3-categories.
Part 4
Hopf algebras
Hopf algebras

A Hopf algebra in a braided monoidal category is a pair of

an algebra \( \begin{array}{c} \text{a } \text{coalgebra} \end{array} \)

that form a bialgebra

\[
\begin{array}{c}
\quad =
\quad =
\quad =
\quad =
\end{array}
\]

and have an antipode; an endomorphism \( S \) fulfilling

\[
\begin{array}{c}
\quad =
\quad =
\quad =
\end{array}
\]
Hopf algebras

A *Hopf algebra* in a braided monoidal category is a pair of

an algebra \( \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array} \) a coalgebra \( \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array} \)

that form a *bialgebra*

\begin{align*}
\begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array} &= \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array} \\
\begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array} &= \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array} \\
\begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array} &= \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array} \\
\begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array} &= \Box
\end{align*}

and have an *antipode*; an endomorphism \( S \) fulfilling

\begin{align*}
\begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array} &= \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array} \\
\begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array} &= \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array} \\
\begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array} &= \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array}
\end{align*}

Here, we consider more restrictive algebras.
Unimodular Hopf algebras

A *unimodular Hopf algebra* is a pair of Frobenius algebras

\[
\left( \begin{array}{c}
\uparrow \quad \uparrow \quad \downarrow \\
\quad \quad \quad \quad \\
\downarrow \quad \downarrow \quad \uparrow
\end{array} \right) \quad \left( \begin{array}{c}
\uparrow \quad \uparrow \quad \downarrow \\
\quad \quad \quad \quad \\
\downarrow \quad \downarrow \quad \uparrow
\end{array} \right)
\]
Unimodular Hopf algebras

A unimodular Hopf algebra is a pair of Frobenius algebras

that form a bialgebra
Unimodular Hopf algebras

A unimodular Hopf algebra is a pair of Frobenius algebras

\[
\left( \begin{array}{cc}
\text{\tiny \includegraphics[width=0.3\textwidth]{unimodular-hopf-algebra-1}} \\
\text{\tiny \includegraphics[width=0.3\textwidth]{unimodular-hopf-algebra-2}}
\end{array} \right)
\]

that form a bialgebra

\[
\begin{array}{c}
\text{\tiny \includegraphics[width=0.3\textwidth]{bialgebra-1}} \\
\text{\tiny \includegraphics[width=0.3\textwidth]{bialgebra-2}} \\
\text{\tiny \includegraphics[width=0.3\textwidth]{bialgebra-3}} \\
\text{\tiny \includegraphics[width=0.3\textwidth]{bialgebra-4}}
\end{array}
\]

and such that

\[
\begin{array}{cc}
\text{\tiny \includegraphics[width=0.3\textwidth]{unimodular-hopf-algebra-5}} & \text{\tiny \includegraphics[width=0.3\textwidth]{unimodular-hopf-algebra-6}} \\
\text{\tiny \includegraphics[width=0.3\textwidth]{unimodular-hopf-algebra-7}} & \text{\tiny \includegraphics[width=0.3\textwidth]{unimodular-hopf-algebra-8}}
\end{array}
\]
Unimodular Hopf algebras

A *unimodular Hopf algebra* is a pair of Frobenius algebras

\[
\left( \begin{array}{c}
\scriptstyle * \vspace{-3pt} \\
\scriptstyle * \vspace{-3pt} \\
\scriptstyle * \vspace{-3pt} \\
\scriptstyle * \vspace{-3pt}
\end{array} \right) \quad \left( \begin{array}{c}
\scriptstyle * \vspace{-3pt}
\scriptstyle * \vspace{-3pt} \\
\scriptstyle * \vspace{-3pt} \\
\scriptstyle * \vspace{-3pt} \\
\scriptstyle * \vspace{-3pt}
\end{array} \right)
\]

that form a *bialgebra*

\[
\begin{align*}
\begin{array}{c}
\scriptstyle * \\
\scriptstyle * \\
\scriptstyle * \\
\scriptstyle *
\end{array} &= \begin{array}{c}
\scriptstyle * \\
\scriptstyle * \\
\scriptstyle * \\
\scriptstyle *
\end{array} \\
\begin{array}{c}
\scriptstyle * \\
\scriptstyle * \\
\scriptstyle * \\
\scriptstyle *
\end{array} &= \begin{array}{c}
\scriptstyle * \\
\scriptstyle * \\
\scriptstyle * \\
\scriptstyle *
\end{array} \\
\begin{array}{c}
\scriptstyle * \\
\scriptstyle * \\
\scriptstyle * \\
\scriptstyle *
\end{array} &= \begin{array}{c}
\scriptstyle * \\
\scriptstyle * \\
\scriptstyle * \\
\scriptstyle *
\end{array} \\
\begin{array}{c}
\scriptstyle * \\
\scriptstyle * \\
\scriptstyle * \\
\scriptstyle *
\end{array} &= \begin{array}{c}
\scriptstyle * \\
\scriptstyle * \\
\scriptstyle * \\
\scriptstyle *
\end{array}
\end{align*}
\]

and such that

\[
\begin{align*}
\begin{array}{c}
\scriptstyle * \\
\scriptstyle * \\
\scriptstyle * \\
\scriptstyle *
\end{array} &= \begin{array}{c}
\scriptstyle * \\
\scriptstyle * \\
\scriptstyle * \\
\scriptstyle *
\end{array} \\
\begin{array}{c}
\scriptstyle * \\
\scriptstyle * \\
\scriptstyle * \\
\scriptstyle *
\end{array} &= \begin{array}{c}
\scriptstyle * \\
\scriptstyle * \\
\scriptstyle * \\
\scriptstyle *
\end{array} \\
\begin{array}{c}
\scriptstyle * \\
\scriptstyle * \\
\scriptstyle * \\
\scriptstyle *
\end{array} &= \begin{array}{c}
\scriptstyle * \\
\scriptstyle * \\
\scriptstyle * \\
\scriptstyle *
\end{array} \\
\begin{array}{c}
\scriptstyle * \\
\scriptstyle * \\
\scriptstyle * \\
\scriptstyle *
\end{array} &= \begin{array}{c}
\scriptstyle * \\
\scriptstyle * \\
\scriptstyle * \\
\scriptstyle *
\end{array}
\end{align*}
\]

\textbf{uHopf}: free *braided* monoidal category on a unimodular Hopf algebra
Unimodular Hopf algebras

A *unimodular Hopf algebra* is a pair of Frobenius algebras

\[
\left( \begin{array}{c}
\begin{array}{c}
\text{Diagram 1}
\end{array}
\end{array}\right) \quad \left( \begin{array}{c}
\begin{array}{c}
\text{Diagram 2}
\end{array}\end{array}\right)
\]

that form a *bialgebra*

\[
\begin{array}{c}
\begin{array}{c}
\text{Equation 1}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\text{Equation 2}
\end{array}\end{array} \quad \begin{array}{c}
\begin{array}{c}
\text{Equation 3}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\text{Equation 4}
\end{array}\end{array} \quad \begin{array}{c}
\begin{array}{c}
\text{Equation 5}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\text{Equation 6}
\end{array}\end{array}
\end{array}
\]

and such that

\[
\begin{array}{c}
\begin{array}{c}
\text{Equation 7}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\text{Equation 8}
\end{array}\end{array} \quad \begin{array}{c}
\begin{array}{c}
\text{Equation 9}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\text{Equation 10}
\end{array}\end{array} \quad \begin{array}{c}
\begin{array}{c}
\text{Equation 11}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\text{Equation 12}
\end{array}\end{array}
\end{array}
\]

\textbf{uHopf}: free *braided* monoidal category on a unimodular Hopf algebra

**Theorem.** The antipode of a unimodular Hopf algebra is \( \mathcal{O} = \left( \begin{array}{c}
\begin{array}{c}
\text{Diagram 3}
\end{array}\end{array}\right) \).
Unimodular Hopf algebras

A \textit{unimodular Hopf algebra} is a pair of Frobenius algebras
\[
\left( \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\end{array} \right), \left( \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\end{array} \right)
\]
that form a \textit{bialgebra}
\[
\left( \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\end{array} \right) = \left( \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\end{array} \right) = \left( \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\end{array} \right)
\]
and such that
\[
\left( \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\end{array} \right) = \left( \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\end{array} \right) = \left( \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\end{array} \right)
\]

\textbf{uHopf}: free \textit{braided} monoidal category on a unimodular Hopf algebra

\textbf{Theorem}. The antipode of a unimodular Hopf algebra is
\[
\left( \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\end{array} \right) = \left( \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\end{array} \right) = \left( \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\end{array} \right)
\]

\textbf{U. Hopf algebras in }\textbf{Vect}_k\text{ are }\textit{finite dimensional unimodular Hopf algebras}.
Unimodular Hopf algebras

A *unimodular Hopf algebra* is a pair of Frobenius algebras

\[
\left( \begin{array}{ccc}
\text{\includegraphics[width=1cm]{unimodular_hopf1.png}} & \text{\includegraphics[width=1cm]{unimodular_hopf2.png}} & \text{\includegraphics[width=1cm]{unimodular_hopf3.png}} \\
\end{array} \right)
\]

that form a *bialgebra*

\[
\begin{array}{ccc}
\text{\includegraphics[width=2cm]{bialgebra1.png}} & = & \text{\includegraphics[width=2cm]{bialgebra2.png}} \\
\text{\includegraphics[width=2cm]{bialgebra3.png}} & = & \text{\includegraphics[width=2cm]{bialgebra4.png}} \\
\end{array}
\]

and such that

\[
\begin{array}{ccc}
\text{\includegraphics[width=2cm]{bialgebra5.png}} & = & \text{\includegraphics[width=2cm]{bialgebra6.png}} \\
\text{\includegraphics[width=2cm]{bialgebra7.png}} & = & \text{\includegraphics[width=2cm]{bialgebra8.png}} \\
\end{array}
\]

\textbf{uHopf}: free *braided* monoidal category on a unimodular Hopf algebra

\textbf{Theorem}. The antipode of a unimodular Hopf algebra is

\[
\text{\includegraphics[width=2cm]{unimodular_hopf4.png}} = \text{\includegraphics[width=2cm]{unimodular_hopf5.png}} = \text{\includegraphics[width=2cm]{unimodular_hopf6.png}}.
\]

\textbf{U. Hopf algebras in Vect}_k \textit{are finite dimensional unimodular Hopf algebras.}

\textbf{Example}. Any finite dimensional semisimple and cosemisimple Hopf algebra.
A topological 3-category

Start with $\mathcal{F}_3$
A topological 3-category

Start with $\mathcal{F}_3$:

- free 'topological' 3-category on
  - a blue surface
  - a red surface
  - a blue-red boundary wire
A topological 3-category

Start with $\mathcal{F}_3$:

free ‘topological’ 3-category on
- a blue surface
- a red surface
- a blue-red boundary wire

**Definition.** $\mathbb{H}$ is the free 3-category with duals on two surfaces and a boundary wire, such that the following hold:

is inverse to $\quad$ and $\quad$ =
A topological 3-category

Start with $\mathcal{F}_3$:

```
  \begin{array}{c}
    \texttt{a blue surface} \\
    \texttt{a red surface} \\
    \texttt{a blue-red boundary wire}
  \end{array}
```

**Definition.** $\mathcal{H}$ is the free 3-category with duals on two surfaces and a boundary wire, such that the following hold:

- is inverse to
- and

Explicitly, invertibility of the saddles means:

```
  \begin{array}{c}
    \texttt{Seite 85 von 182}
  \end{array}
```
A Hopf algebra in $\mathbb{H}$

There is a Hopf algebra in $\mathbb{H}$. 
A Hopf algebra in $\mathbb{H}$

There is a Hopf algebra in $\mathbb{H}$.
It lives on the following ‘thickened’ wire:
A Hopf algebra in $\mathbb{H}$

There is a Hopf algebra in $\mathbb{H}$.

It lives on the following ‘thickened’ wire:
A Hopf algebra in $\mathbb{H}$

There is a Hopf algebra in $\mathbb{H}$. It lives on the following ‘thickened’ wire:
A Hopf algebra in $\mathbb{H}$

There is a Hopf algebra in $\mathbb{H}$.
It lives on the following ‘thickened’ wire:
A Hopf algebra in $\mathbb{H}$

There is a Hopf algebra in $\mathbb{H}$.
It lives on the following ‘thickened’ wire:
A Hopf algebra in $\mathbb{H}$

There is a Hopf algebra in $\mathbb{H}$.
It lives on the following ‘thickened’ wire:
A Hopf algebra in $\mathbb{H}$

There is a Hopf algebra in $\mathbb{H}$.
It lives on the following ‘thickened’ wire:

The two interacting Frobenius structures are:

\[
\left( \begin{array}{c}
\begin{array}{c}
\text{,}
\end{array}
\end{array}\right) \quad \left( \begin{array}{c}
\begin{array}{c}
\text{,}
\end{array}\right)
\right)
\]
A Hopf algebra in $\mathbb{H}$

There is a Hopf algebra in $\mathbb{H}$. It lives on the following ‘thickened’ wire:

The two interacting Frobenius structures are:
A Hopf algebra in $\mathbb{H}$

There is a Hopf algebra in $\mathbb{H}$. It lives on the following ‘thickened’ wire:

The two interacting Frobenius structures are:

\[
\left( \begin{array}{c}
\mathcal{A}, \mathcal{Y}, \mathcal{X} \\
\end{array} \right) \quad \left( \begin{array}{c}
\mathcal{A}, \mathcal{Y}, \mathcal{X} \\
\end{array} \right)
\]
A Hopf algebra in $\mathbb{H}$

Let's check (some of) the axioms of unimodular Hopf algebras:

\[
\begin{align*}
&\begin{tikzpicture}
  \draw[thick] (0,0) -- (0,-1);
  \draw[thick] (0,-2) -- (0,-3);
  \draw[fill=green] (0,-1) circle (0.1);
  \draw[fill=red] (0,-2) circle (0.1);
\end{tikzpicture} =
  \begin{tikzpicture}
  \draw[thick] (0,0) -- (0,-1);
\end{tikzpicture} \\
&\begin{tikzpicture}
  \draw[thick] (0,0) .. controls (0,-1) and (0,-2) .. (0,-3);
  \draw[fill=green] (0,-1) circle (0.1);
\end{tikzpicture} =
  \begin{tikzpicture}
  \draw[thick] (0,0) .. controls (0,-1) and (0,-2) .. (0,-3);
  \draw[fill=green] (0,-1) circle (0.1);
\end{tikzpicture} \\
&\begin{tikzpicture}
  \draw[thick] (0,0) .. controls (0,-1) and (0,-2) .. (0,-3);
  \draw[fill=red] (0,-1) circle (0.1);
\end{tikzpicture} =
  \begin{tikzpicture}
  \draw[thick] (0,0) .. controls (0,-1) and (0,-2) .. (0,-3);
  \draw[fill=green] (0,-1) circle (0.1);
\end{tikzpicture}
\end{align*}
\]
A Hopf algebra in $\mathbb{H}$

Let's check (some of) the axioms of unimodular Hopf algebras:

$\begin{align*}
\text{Diagram 1} & = \text{Diagram 2} = \text{Diagram 3} \\
\text{Diagram 4} & = \text{Diagram 5}
\end{align*}$
A Hopf algebra in $\mathbb{H}$

Let’s check (some of) the axioms of unimodular Hopf algebras:

\[
\begin{align*}
\begin{array}{ccc}
\uparrow & = & \downarrow \\
\text{Fig. 1} & = & \text{Fig. 2}
\end{array}
\end{align*}
\]
A Hopf algebra in $\mathbb{H}$

Let's check (some of) the axioms of unimodular Hopf algebras:

\[
\begin{align*}
\Delta(1) &= 1 \\
\epsilon(\Delta(1)) &= \epsilon(1) \\
\Delta(\epsilon(\Delta(1))) &= \epsilon(1) \\
\epsilon(\Delta(1)) &= 1
\end{align*}
\]
A Hopf algebra in $\mathbb{H}$

The bialgebra laws correspond to the invertibility of the saddle:

\[
\begin{array}{c}
\ \\
\end{array}
\]
A Hopf algebra in $\mathbb{H}$

The bialgebra laws correspond to the invertibility of the saddle:
A Hopf algebra in $\mathbb{H}$

The bialgebra laws correspond to the invertiblity of the saddle:
A Hopf algebra in $\mathbb{H}$

The bialgebra laws correspond to the invertibility of the saddle:

\[
\begin{align*}
\text{Diagram representation of the bialgebra laws}
\end{align*}
\]
A Hopf algebra in $\mathbb{H}$

The bialgebra laws correspond to the invertiblity of the saddle:
A Hopf algebra in \( \mathbb{H} \)

The bialgebra laws correspond to the invertibility of the saddle:
A Hopf algebra in $\mathbb{H}$

The bialgebra laws correspond to the invertibility of the saddle:
A Hopf algebra in $\mathbb{H}$

The bialgebra laws correspond to the invertibility of the saddle:
A Hopf algebra in $\mathbb{H}$

The bialgebra laws correspond to the invertibility of the saddle:
Summary. $u\text{Hopf}$ is a shadow of a simpler 3-category.
A Hopf algebra in $\mathbb{H}$

**Summary.** $\mathbf{uHopf}$ is a shadow of a simpler 3-category.

A *unimodular Hopf algebra* is a pair of Frobenius algebras
\[
\left( \begin{array}{c} \Upsilon, \; \Upsilon, \; \Upsilon, \; \Upsilon \end{array} \right)
\quad \left( \begin{array}{c} \Upsilon, \; \Upsilon, \; \Upsilon, \; \Upsilon \end{array} \right)
\]
that form a *bialgebra*
\[
= \begin{array}{c} \Upsilon \end{array} \quad = \begin{array}{c} \Upsilon \end{array} \quad = \begin{array}{c} \Upsilon \end{array}
\]
and such that
\[
\begin{array}{c} \Upsilon \end{array} = \begin{array}{c} \Upsilon \end{array} = \begin{array}{c} \Upsilon \end{array} \quad \begin{array}{c} \Upsilon \end{array} = \begin{array}{c} \Upsilon \end{array} = \begin{array}{c} \Upsilon \end{array}
\]
A Hopf algebra in $\mathbb{H}$

**Summary.** $u\text{Hopf}$ is a shadow of a simpler 3-category.

$\mathbb{H}$ is the free 3-category with duals on two surfaces and a boundary wire, such that the following hold:

- Is inverse to
- And

[Diagram showing the relationships between the elements]
A Hopf algebra in $\mathbb{H}$

**Summary.** $u\text{Hopf}$ is a shadow of a simpler 3-category.

A *unimodular Hopf algebra* is a pair of Frobenius algebras

\[
\left( \begin{array}{c}
\begin{array}{c}
\text{diagram 1}
\end{array}
\end{array}
\right)
\quad \left( \begin{array}{c}
\begin{array}{c}
\text{diagram 2}
\end{array}
\end{array}
\right)
\]

that form a *bialgebra*:

\[
\begin{array}{c}
\begin{array}{c}
\text{diagram 3}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\text{diagram 4}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\text{diagram 5}
\end{array}
\end{array}
\]

and such that

\[
\begin{array}{c}
\begin{array}{c}
\text{diagram 6}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\text{diagram 7}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\text{diagram 8}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{diagram 9}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\text{diagram 10}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\text{diagram 11}
\end{array}
\end{array}
\]


A Hopf algebra in $\mathcal{H}$

**Summary.** $u\text{Hopf}$ is a shadow of a simpler 3-category.

$\mathcal{H}$ is the free 3-category with duals on two surfaces and a boundary wire, such that the following hold:

![Diagram showing two surfaces and a boundary wire with inverse relationship and equality symbol]
A Hopf algebra in $\mathbb{H}$

Summary. $u\text{Hopf}$ is a shadow of a simpler 3-category. Formally, we have defined a 3-functor $u\text{Hopf} \rightarrow \mathbb{H}$. 
A Hopf algebra in $\mathbb{H}$

**Summary.** $\text{uHopf}$ is a shadow of a simpler 3-category. Formally, we have defined a 3-functor $\text{uHopf} \to \mathbb{H}$.

**Conjecture.** This induces a braided equivalence $\text{uHopf} \cong \mathbb{H}\left(\begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array}, \begin{array}{c} \circlearrowright \\ \circlearrowleft \end{array}\right)$. 
A Hopf algebra in $\mathbb{H}$

**Summary.** $u\text{Hopf}$ is a shadow of a simpler 3-category. Formally, we have defined a 3-functor $u\text{Hopf} \rightarrow \mathbb{H}$.

**Conjecture.** This induces a braided equivalence $u\text{Hopf} \cong \mathbb{H}(\hat{\mathbb{H}}, \hat{\mathbb{H}})$. Several Hopf algebraic calculations simplify in this 3D model.
A Hopf algebra in $\mathbb{H}$

**Summary.** $u\text{Hopf}$ is a shadow of a simpler 3-category. Formally, we have defined a 3-functor $u\text{Hopf} \to \mathbb{H}$.

**Conjecture.** This induces a braided equivalence $u\text{Hopf} \cong \mathbb{H}(\overrightarrow{\cdot}, \overleftarrow{\cdot})$.

Several Hopf algebraic calculations simplify in this 3D model. For example, the antipode is the half twist:
A Hopf algebra in $\mathbb{H}$

**Summary.** $u\text{Hopf}$ is a shadow of a simpler 3-category. Formally, we have defined a 3-functor $u\text{Hopf} \to \mathbb{H}$.

**Conjecture.** This induces a braided equivalence $u\text{Hopf} \cong \mathbb{H}(\hfill, \hfill)$. Several Hopf algebraic calculations simplify in this 3D model. For example, the antipode is the half twist:

$\Rightarrow$ The antipode is an algebra antihomomorphism.
A Hopf algebra in $\mathbb{H}$

Summary. u
Formally, we have

Conjecture.
Several Hopf ... For example,

$\Rightarrow$ The antipode is an algebra antihomomorphism.
A Hopf algebra in $\mathbb{H}$

**Summary.** $u\text{Hopf}$ is a shadow of a simpler 3-category.

Formally, we have defined a 3-functor $u\text{Hopf} \to \mathbb{H}$.

**Conjecture.** This induces a braided equivalence $u\text{Hopf} \cong \mathbb{H}\left(\begin{array}{cc} \bullet & \bullet \\ \bullet & \bullet \end{array}\right)$.

Several Hopf algebraic calculations simplify in this 3D model. For example, the antipode is the half twist:

$\Rightarrow$ The antipode is an algebra antihomomorphism.
$\Rightarrow$ In a unimodular Hopf algebra, the antipode squares to the twist.

In particular, in a symmetric monoidal category, its 4th power is trivial.
Part 5

Higher linear algebra
Representations

So far: algebraic structures in terms of generators & relations
Representations

So far: algebraic structures in terms of generators & relations
Now: representations - instances of these structures in concrete categories
Representations

So far: algebraic structures in terms of generators & relations
Now: *representations* - instances of these structures in concrete categories
U. Hopf algebras in a BMC $C$ : braided monoidal functors $u\text{Hopf} \to C$
Representations

So far: algebraic structures in terms of generators & relations

Now: *representations* - instances of these structures in concrete categories

U. Hopf algebras in a BMC $\mathcal{C}$: braided monoidal functors $u\text{Hopf} \to \mathcal{C}$

*Linear representations* - representation functors with target $\textbf{Vect}$
Representations

So far: algebraic structures in terms of generators & relations

Now: representations - instances of these structures in concrete categories

U. Hopf algebras in a BMC $\mathcal{C}$: braided monoidal functors $u\text{Hopf} \rightarrow \mathcal{C}$

Linear representations - representation functors with target $\text{Vect}$

What are the appropriate linear targets for higher categorical theories?
Representations

So far: algebraic structures in terms of generators & relations

Now: representations - instances of these structures in concrete categories

U. Hopf algebras in a BMC $\mathcal{C}$: braided monoidal functors $u\text{Hopf} \rightarrow \mathcal{C}$

Linear representations - representation functors with target $\text{Vect}$

What are the appropriate linear targets for higher categorical theories?

Expectations:

- symmetric monoidal $n$-categories $n\text{Vect}$ categorifying $\text{Vect}$
Representations

So far: algebraic structures in terms of generators & relations

Now: representations - instances of these structures in concrete categories

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Representations

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What are the appropriate *linear* targets for higher categorical theories?

*Expectations:*
- symmetric monoidal $n$-categories $\text{nVect}$ categorifying $\text{Vect}$
- recover $\text{nVect}$ from $(n+1)\text{Vect}$: $(n+1)\text{Vect}(I, I) \cong \text{nVect}$

$n\text{Vect}$ a ‘shadow’ of $(n+1)\text{Vect} \leftrightarrow (n+1)\text{Vect}$ a ‘thickening’ of $n\text{Vect}$

\[ \xymatrix{ \text{Hopf algebras and 3-categories} & \text{August 3, 2017} & 25/34 } \]
Higher linear algebra

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David Reutter  Hopf algebras and 3-categories  August 3, 2017  26 / 34
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David Reutter

Hopf algebras and 3-categories

August 3, 2017

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## Higher linear algebra

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(*Diagram showing the relationship between objects 2Vect and Vect with Rep(-) as a transformation*)

---

*David Reutter*  
*Hopf algebras and 3-categories*  
*August 3, 2017*  
*26 / 34*
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They are **symmetric monoidal** 1-, 2- and 3-categories with duals.
## Higher linear algebra

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They are symmetric monoidal 1-, 2- and 3-categories with duals.

$$\textbf{3Vect}(I, I) \cong \textbf{2Vect} \quad \textbf{2Vect}(I, I) \cong \textbf{Vect} \quad \textbf{Vect}(I, I) = \mathbb{C}$$
## Higher linear algebra

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They are symmetric monoidal 1-, 2- and 3-categories with duals.

$$3\text{Vect}(I, I) \cong 2\text{Vect} \quad 2\text{Vect}(I, I) \cong \text{Vect} \quad \text{Vect}(I, I) = \mathbb{C}$$

Various generalizations are possible.
The 3-category $3\text{Vect}$

$C$

fusion category
The 3-category $\mathbf{3Vect}$

- fusion category
- bimodule category
The 3-category $\mathbf{3Vect}$

- Fusion category
- Bimodule category
- Intertwining functor
- Natural transformation
The 3-category $3\text{Vect}$

- fusion category
- bimodule category
- intertwining functor
- natural transformation

right $C$-module $\mathcal{M}$
The 3-category $\mathbf{3Vect}$

**Fusion category**

**Bimodule category**

**Intertwining functor**

**Natural transformation**

Right $\mathcal{M}$-module  $\mathcal{M}$  
Left $\mathcal{N}$-module  $\mathcal{N}$
The 3-category $\mathbf{3Vect}$

- fusion category
- bimodule category
- intertwining functor
- natural transformation

right $C$-module $\mathcal{M}$  left $C$-module $\mathcal{N}$  relative Deligne product $\mathcal{M} \boxtimes_C \mathcal{N}$
The 3-category \(3\text{Vect}\)

- **Fusion category**
- **Bimodule category**
- **Intertwining functor**
- **Natural transformation**

right \(C\)-module \(\mathcal{M}\)  
left \(C\)-module \(\mathcal{N}\)  
\textit{relative Deligne product} \(\mathcal{M} \boxtimes_C \mathcal{N}\)

\textbf{relative Deligne product: universal for} \(C\)-bilinear functors out of \(\mathcal{M} \times \mathcal{N}\)
The 3-category $\mathbf{3Vect}$

**fusion category**

**bimodule category**

**intertwining functor**

**natural transformation**

**right $C$-module $\mathcal{M}$**

**left $C$-module $\mathcal{N}$**

**relative Deligne product** $\mathcal{M} \boxtimes_C \mathcal{N}$

$\mathcal{F}_3 \rightarrow \mathbf{3Vect}$
The 3-category $3\text{Vect}$

- fusion category
- bimodule category
- intertwining functor
- natural transformation

right $C$-module $\mathcal{M}$  left $C$-module $\mathcal{N}$  relative Deligne product $\mathcal{M} \boxtimes_C \mathcal{N}$

$\mathcal{F}_3 \rightarrow 3\text{Vect} : \begin{cases} \text{a fusion category } \mathcal{C} \\ \text{two right } \mathcal{C}\text{-module categories } \mathcal{M}, \mathcal{N} \\ \text{an intertwining functor } \mathcal{M} \rightarrow \mathcal{N} \end{cases}$

David Reutter  Hopf algebras and 3-categories  August 3, 2017  27 / 34
Hopf algebras and fusion categories - a sketch

This 3-functor factors through $\mathbb{H}$ if the following hold:

- is inverse to

and

$\quad = \quad$
Hopf algebras and fusion categories - a sketch

This 3-functor factors through $\mathbb{H}$ if the following hold:

is inverse to $\Rightarrow$ is isomorphic to

and

$\Rightarrow$ is isomorphic to

and
Hopf algebras and fusion categories - a sketch

This 3-functor factors through $\mathbb{H}$ if the following hold:

is inverse to \hspace{1cm} and \hspace{1cm} =

⇒ is isomorphic to \hspace{1cm} and \hspace{1cm} is isomorphic to

David Reutter
Hopf algebras and fusion categories - a sketch

This 3-functor factors through $\mathbb{H}$ if the following hold:

```

is inverse to

and

\Rightarrow

is isomorphic to

and

is isomorphic to

In other words, $\mathcal{M} \boxtimes C \mathcal{N} \to \textbf{Vect}$ is an \textit{adjoint equivalence}!
Hopf algebras and fusion categories - a sketch

This 3-functor factors through $\mathbb{H}$ if the following hold:

\[
\begin{align*}
\text{is inverse to} & \quad \text{and} \quad \text{is isomorphic to} \\
\Rightarrow & \quad \text{is isomorphic to} & \quad \text{and} \quad \text{is isomorphic to}
\end{align*}
\]

In other words, $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \to \textbf{Vect}$ is an adjoint equivalence!

Data of a 3-functor $\mathbb{H} \to \textbf{3Vect}$:

\[
\begin{align*}
\text{a fusion category } \mathcal{C} \\
\text{a left and a right module category } \mathcal{M}, \mathcal{N} \\
an \text{adjoint equivalence } \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \to \textbf{Vect}
\end{align*}
\]
Hopf algebras and fusion categories - a sketch

This 3-functor factors through $\mathbb{H}$ if the following hold:

\[
\begin{array}{c}
\xymatrix{
\text{is inverse to} & \xymatrix{
\text{and} & \text{is isomorphic to} \\
\text{is isomorphic to} & \text{is isomorphic to} \\
\Rightarrow & \xymatrix{
\text{In other words, } & \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \to \textbf{Vect} \text{ is an adjoint equivalence!}
\end{array}
\end{array}
\]

Data of a 3-functor $\mathbb{H} \to \textbf{3Vect}$:

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\begin{array}{l}
\text{a fusion category } \mathcal{C} \\
\text{a left and a right module category } \mathcal{M}, \mathcal{N} \\
\text{an adjoint equivalence } \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \to \textbf{Vect}
\end{array}
\]

If $\mathcal{M}$ is the regular module $\mathcal{C}$, then $\mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{N} \cong \mathcal{N}$
Hopf algebras and fusion categories - a sketch

This 3-functor factors through $\mathcal{H}$ if the following hold:

\[ \begin{array}{c}
\text{is inverse to} \\
\Rightarrow \\
\text{is isomorphic to} \\
\text{and} \\
\text{is isomorphic to}
\end{array} \]

In other words, $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \to \textbf{Vect}$ is an adjoint equivalence!

Data of a 3-functor $\mathcal{H} \to \textbf{3Vect}$:

\[ \begin{cases} 
\text{a fusion category } \mathcal{C} \\
\text{a left and a right module category } \mathcal{M}, \mathcal{N} \\
\text{an adjoint equivalence } \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \to \textbf{Vect}
\end{cases} \]

If $\mathcal{M}$ is the regular module $\mathcal{C}$, then $\mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{N} \cong \mathcal{N} \cong \textbf{Vect}$. 
Hopf algebras and fusion categories - a sketch

This 3-functor factors through $\mathbb{H}$ if the following hold:

- $\Rightarrow$ is inverse to $\Rightarrow$ and $\Rightarrow = \Rightarrow$
- $\Rightarrow$ is isomorphic to $\Rightarrow$ and $\Rightarrow$ is isomorphic to $\Rightarrow$

In other words, $\Rightarrow : \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \to \textbf{Vect}$ is an adjoint equivalence!

Data of a 3-functor $\mathbb{H} \to 3\textbf{Vect}$:

- A fusion category $\mathcal{C}$
- A left and a right module category $\mathcal{M}, \mathcal{N}$
- An adjoint equivalence $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \to \textbf{Vect}$

If $\mathcal{M}$ is the regular module $\mathcal{C}$, then $\mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{N} \cong \mathcal{N} \cong \textbf{Vect}$.

A $\mathcal{C}$-module structure on $\textbf{Vect}$ is the same as a monoidal functor $\mathcal{C} \to \textbf{Vect}$. 
Hopf algebras and fusion categories - a sketch

This 3-functor factors through $\mathbb{H}$ if the following hold:

- is inverse to
- and

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$\Rightarrow$ | is isomorphic to |
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In other words, $\mathbb{M} \boxtimes_C \mathbb{N} \rightarrow \textbf{Vect}$ is an adjoint equivalence!

Data of a 3-functor:

- a fusion category $\mathcal{C}$
- a left and a right module category $\mathcal{M}, \mathcal{N}$
- an adjoint equivalence $\mathcal{M} \boxtimes_C \mathcal{N} \rightarrow \textbf{Vect}$

If $\mathcal{M}$ is the regular module $\mathcal{C}$, then $\mathcal{C} \boxtimes_C \mathcal{N} \cong \mathcal{N} \cong \textbf{Vect}$.

A $\mathcal{C}$-module structure on $\textbf{Vect}$ is the same as a monoidal functor $\mathcal{C} \rightarrow \textbf{Vect}$.

Data of a 3-functor with $\mathcal{M} = \mathcal{C}$:

- a fusion category $\mathcal{C}$
- a monoidal functor $\mathcal{C} \rightarrow \textbf{Vect}$
Tannaka reconstruction

Given a fusion category \( \mathcal{C} \) with a monoidal functor \( \mathcal{C} \xrightarrow{F} \text{Vect} \)

\( \Rightarrow \) The following vector space is a unimodular Hopf algebra:
Tannaka reconstruction

Given a fusion category $\mathcal{C}$ with a monoidal functor $\mathcal{C} \xrightarrow{F} \text{Vect}$

$\Rightarrow$ The following vector space is a unimodular Hopf algebra:

$a$ scalar 2-morphism in $3\text{Vect}$

$\Rightarrow$ a vector space
Tannaka reconstruction

Given a fusion category $\mathcal{C}$ with a monoidal functor $\mathcal{C} \xrightarrow{F} \mathbf{Vect}$

$\Rightarrow$ The following vector space is a unimodular Hopf algebra:

This is a version of Tannaka reconstruction:
If $\mathcal{C} = \text{Rep}(H) \xrightarrow{\text{forget}} \mathbf{Vect}$, this recovers the Hopf algebra $H$. 
Tannaka reconstruction

Given a fusion category $\mathcal{C}$ with a monoidal functor $\mathcal{C} \xrightarrow{F} \text{Vect}$
⇒ The following vector space is a unimodular Hopf algebra:

This is a version of Tannaka reconstruction:
If $\mathcal{C} = \text{Rep}(H)$ \xrightarrow{\text{forget}} \text{Vect}, this recovers the Hopf algebra $H$.
Conversely, any fusion category with fibre functor $\mathcal{C} \rightarrow \text{Vect}$ is of the form $\text{Rep}(H)$ with $H$ constructed as above.
Tannaka reconstruction

Given a fusion category $\mathcal{C}$ with a monoidal functor $\mathcal{C} \xrightarrow{F} \textbf{Vect}$
⇒ The following vector space is a unimodular Hopf algebra:

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Conversely, any fusion category with fibre functor $\mathcal{C} \to \textbf{Vect}$ is of the form $\text{Rep}(H)$ with $H$ constructed as above.

Proof. Follows from an old result of M. Müger.\footnote{Theorem 6.20 in [Müger, From subfactors to categories and topology I, 2003]}
Tannaka reconstruction

Given a fusion category $\mathcal{C}$ with a monoidal functor $\mathcal{C} \xrightarrow{F} \text{Vect}$
⇒ The following vector space is a unimodular Hopf algebra:

This is a version of Tannaka reconstruction:
If $\mathcal{C} = \text{Rep}(H)$
Conversely, a $\text{Rep}(H)$ with

**Question.**
Is there a completely graphical proof, independent of the target $\text{3Vect}$?

**Proof.** Follows from an old result of M. Müger.\(^1\)

\(^1\)Theorem 6.20 in [Müger, *From subfactors to categories and topology I*, 2003]
Part 6

Lattice models
Lattice models and $3\text{Vect}$

Kitaev or Levin-Wen lattice models with boundaries $\Rightarrow$ defect TQFTs
Lattice models and $3\text{Vect}$

Kitaev or Levin-Wen lattice models with boundaries $\Rightarrow$ defect TQFTs

bulk of lattice $\leftrightarrow$ a fusion category $\mathcal{C}$
boundary of lattice $\leftrightarrow$ a $\mathcal{C}$-module category $\mathcal{M}$
Lattice models and 3Vect

Kitaev or Levin-Wen lattice models with boundaries $\Rightarrow$ defect TQFTs

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Lattice models and $3\text{Vect}$

Kitaev or Levin-Wen lattice models with boundaries $\Rightarrow$ defect TQFTs

bulk of lattice $\leftrightarrow$ a fusion category $\mathcal{C}$
boundary of lattice $\leftrightarrow$ a $\mathcal{C}$-module category $\mathcal{M}$

Ground space $\begin{pmatrix} \text{grid} \end{pmatrix} = \begin{array}{c} \mathcal{N} \\ \mathcal{M} \end{array}$

$\begin{array}{c} \mathcal{N} \\ \mathcal{M} \\ \mathcal{M} \end{array}$
Lattice models and \textbf{3Vect}

Kitaev or Levin-Wen lattice models with boundaries $\Rightarrow$ defect TQFTs

bulk of lattice $\leftrightarrow$ a fusion category $\mathcal{C}$
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Ground space

\[
\begin{array}{c}
\text{a scalar 2-morphism in 3Vect} \\
= \text{a vector space}
\end{array}
\]
Lattice models and 3Vect

Kitaev or Levin-Wen lattice models with boundaries $\Rightarrow$ defect TQFTs

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Ground space

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\mathcal{M} & \mathcal{M} \\
\mathcal{N} & \\
\end{array}
\end{pmatrix}
\]

surface codes

$\mathcal{C} = \text{Vect}_{\mathbb{Z}_2}$
Lattice models and $\mathbf{3Vect}$

Kitaev or Levin-Wen lattice models with boundaries $\Rightarrow$ defect TQFTs

bulk of lattice $\leftrightarrow$ a fusion category $\mathcal{C}$
boundary of lattice $\leftrightarrow$ a $\mathcal{C}$-module category $\mathcal{M}$

Ground space $\left( \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \right) = \mathbf{Vect}_{\mathbb{Z}_2}$

$\mathbf{Vect}$ $\mathbf{Vect}_{\mathbb{Z}_2}$$\mathbf{Vect}_{\mathbb{Z}_2}$

$\mathbf{Vect}$

surface codes

<table>
<thead>
<tr>
<th>two possible boundaries: $\text{smooth}$ and $\text{rough}$</th>
<th>two module categories: $\mathbf{Vect}_{\mathbb{Z}_2}$ and $\mathbf{Vect}$</th>
</tr>
</thead>
</table>

$\mathcal{C} = \mathbf{Vect}_{\mathbb{Z}_2}$

David Reutter

Hopf algebras and 3-categories

August 3, 2017 31 / 34
Lattice surgery

topologically protected operations on surface codes via
splitting or merging of lattices along smooth or rough boundaries\(^2\)

\(^2\)Horsman et al., *Surface code quantum computing by lattice surgery*, 2012
Lattice surgery

topologically protected operations on surface codes via *splitting* or *merging* of lattices along smooth or rough boundaries\(^2\)

\(^2\)[Horsman et al., *Surface code quantum computing by lattice surgery*, 2012]
Lattice surgery

topologically protected operations on surface codes via
splitting or merging of lattices along smooth or rough boundaries²

²[Horsman et al., Surface code quantum computing by lattice surgery, 2012]
Lattice surgery

topologically protected operations on surface codes via splitting or merging of lattices along smooth or rough boundaries

\[2\text{[Horsman et al., Surface code quantum computing by lattice surgery, 2012]}\]
Lattice surgery

topologically protected operations on surface codes via
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\(^2\)Horsman et al., *Surface code quantum computing by lattice surgery, 2012*
The End

- Many open questions:
  - Can we drop dualizabilities in $\mathbb{H}$ to get more general Hopf algebras?
  - Can we make $\mathbb{H}$ into a symmetric monoidal 3-category with duals to talk about actual fully extended defect TQFTs?
  - For a Frobenius algebra in a monoidal category $\mathcal{C}$, there is a 2-category $\mathcal{C} \leftrightarrow \hat{\mathcal{C}}$ such that the Frobenius algebra comes from a dualizable 1-morphism in $\hat{\mathcal{C}}$. Is something similar true for Hopf algebras?
  - ...

- Maybe most interestingly:
  For a defect TQFT, what is the physical meaning of the conditions:

\[ \begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{example1.png}}
\end{array} \]
The End

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```
=   =
```

Thanks for listening!
Weak Hopf algebras

If we *drop* the second condition

![Diagram of Weak Hopf algebras]
Weak Hopf algebras

If we \textit{drop} the second condition

we only obtain a \textit{weak} Hopf algebra on

but have more functors $\mathbb{H} \rightarrow 3\text{Vect}$. 
Weak Hopf algebras

If we \textit{drop} the second condition

we only obtain a \textit{weak} Hopf algebra on

but have more functors $\mathbb{H} \to 3\text{Vect}$.

In fact, \textit{every} fusion category induces such a functor. The corresponding Hopf algebra coincides with the Kitaev-Kong construction.
The End

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```
\[ \begin{array}{ccc}
\text{\includegraphics[width=0.3\textwidth]{image1.png}} & = & \text{\includegraphics[width=0.3\textwidth]{image2.png}} \\
\end{array} \]
```

Thanks for listening!