Abstract: In quantum cosmology wave functions are traditionally generated either from path integrals or from solving the Wheeler-DeWitt equation. In the first part of the talk I discuss what is required in these approaches in order to meet the usual requirements of Hilbert space quantum mechanics, namely, the specification of an inner product structure and classes of states and operators of interest. The Wheeler-DeWitt operator must be self-adjoint in this approach which has consequences for both the path integral and Wheeler-DeWitt account of the much-studied de Sitter minisuperspace model, since it is usually formulated in terms of a scale factor which must be non-negative, hence one is really doing quantum mechanics on the half-line. In the second part of the talk I discuss the types of amplitudes one is interested in from the perspective of the decoherent histories approach to quantum cosmology, which describe whether the trajectory of a cosmological model passes through various regions of minisuperspace. They are different in form to the simplest path integral constructions in quantum cosmology and most closely resemble scattering amplitudes.
QUANTUM MECHANICAL ASPECTS
OF QUANTUM COSMOLOGY

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Imperial College

• with J. Hartle, T. Hertog

• J JH, Phys. Rev. D80, 124032 (2009)
  arXiv: 1108.5991
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“I have often been made to eat my own words. And I have found them to be a most nutritious meal”.

W. CHURCHILL
QUANTUM COSMOLOGY

- Wheeler-DeWitt Eq: \( \mathcal{H}(\mathcal{W}) = 0 \)
  (a) \( \mathcal{W} \) is a solution to a differential eq
  (b) \( \mathcal{W} \) is an eigenstate of a self-adjoint operator \( \mathcal{K} \)

- Path integral
  \[ \mathcal{W} = \int Dg_{\mu\nu} e^{iS} \]
  (a) "\( \Pi \) is a standalone object"
  (b) "\( \Pi \) is tied to the Hilbert space structure of QM"
WHEELER-DEWITT QUANTIZATION

Mini-superspace coords \( q^2 \equiv (q, \phi, \ldots) \)

WD eq

\[ H \Psi = \left( -\nabla^2 + U(q) \right) \Psi = 0 \]

\( (-+++ \ldots) \)

- Inner Products

\[ (\Psi, \phi) = \int d^n q \, \Psi^* (q) \phi (q) \]

\[ (\Psi, \phi)_{k_1} = -i \int d^{n-1} q \, \Psi^* \nabla_n \phi \]

- Induced Inner Product

\[ H |\psi_{Ek}\rangle = E |\psi_{Ek}\rangle \quad k = \text{degen} \]

\[ \langle \psi_{Ek} | \psi_{Ek'} \rangle_{s} = S(E-E') S(k-k') \]

\( \overset{\uparrow}{\text{Drop to obtain}} \)

\[ \langle \phi | \phi \rangle_i \]
Minisuperspace coords \( q = (q, \phi, \ldots) \)

\[ \text{WD eq} \quad H \psi = \left( -\nabla^2 + U(q) \right) \psi = 0 \]

\[ (-+++) \]

- **Inner Products**
  \[ (\psi, \phi)_s = \int d^n q \, \psi^*(\phi) \, \phi(q) \]
  \[ (\psi, \phi)_k = -i \int d^n q \, \psi^* \frac{\delta}{\delta n} \phi \]

- **Induced Inner Product**
  \[ H \psi_{Ek} > = E \psi_{Ek} > \quad k = \text{cogen} \]
  \[ \langle \psi_{Ek} | \psi_{Ek'} \rangle_s = S(E-E') \quad S(k-k') \]
  \[ \Rightarrow \text{DROP to obtain} \]
  \[ <+1 \phi> \]

- In RQM \( (\psi, \phi)_I = (\psi^+, \phi^-)_{k} - (\psi^-, \phi^+)_{k} \)
H is SELF-ADJOINT

$dS$ mini-superspace model

$$ds^2 = -N^2 dt^2 + q(t) d\Omega_3^2 \quad q = q^2$$

- WD eq

$$\left[ \frac{d^2}{dq^2} + 12 \Pi^4 (\Lambda q - 1) \right] \Psi = 0$$

$q > 0 \quad QM \text{ on } \mathbb{R}^+$

- H self adjoint $\Rightarrow$ conditions on $\Psi(q)$

$$\Psi(q) + b \Psi'(q) = 0 \quad \text{at } q = 0$$

for any $b \in \mathbb{R} \; (b \neq 0)$
In the gauge $N = 0$

- $\Psi(q) = \int_{c} dq e^{iS[q,N]}$

  for some contour $c$

- $S[q,N] = 2\pi \int_{0}^{1} dt \left[ -\frac{3q^2}{4N} - N(\Lambda q - 3) \right]$

  NB proposal $\Rightarrow q(0) = 0, \ q(1) = q_i$

- $\int dq e^{iS[q,N]} = \frac{1}{N^{\frac{1}{2}}} e^{iS_0(q_i,N)}$

- $S_0 = 2\pi \left[ \frac{\Lambda^2}{36} N^3 + (3 - \Lambda q) N - \frac{3q_i^2}{4N} \right]$

- Evaluate $dN$ over various contours

- so far ignoring $q \geq 0$
In the gauge $N = 0$

- \[ \Psi(q_1) = \int dN \int dq \, e^{i S[q, N]} \]
  for some contour $C$

- \[ S[q, N] = 2\pi^2 \int_0^1 dt \left[ -\frac{3q^2}{4N} - N(\Lambda q - 3) \right] \]

NB proposal $\Rightarrow q(0) = 0, \; q(1) = q_1$

- \[ \int dq \, e^{i S[q, N]} = \frac{1}{N^{n/2}} \]

- \[ S_0 = 2\pi^2 \left[ \frac{\Lambda^2}{36} N^3 + (3 - \Lambda q_1)N - \frac{3q_1^2}{4N} \right] \]

- Evaluate $dN$ over various contours
  - so far ignoring $q > 0$
PATH INTEGRALS II

\[ H \psi = 0 \] is solved by \[ \psi(q) = \int dq e^{iS[q]} \]

for some contour \( C \)

- In QM, \[ \int dq e^{iS} = \langle q, 1 e^{-iHN} 1q_0 \rangle \]

Then with \( C = \mathbb{R} \)

\[ \psi(q) = \int dq \sum_{E, k} e^{-iNE} u_{Ek}(q) u_{Ek}^*(q_0) \]

\[ = 2\pi \sum_{E, k} \delta(E) u_{Ek}(q) u_{Ek}^*(q_0) \]

- \( \psi(q) = 2\pi \sum_k u_{Ek}(q) u_{Ek}^*(q_0) \)

- NOT UNIQUE without B.C.s on \( u(q) \)
A REAL LORENTZIAN contour can give one of many solutions to $H \psi = 0$
depending on B.C.s at $q = 0$

Ex. In dS MSS model:

$u(0) = -u'(0) \Rightarrow \psi = e^{-q} (T)$

$u(0) = u'(0) \Rightarrow \psi = e^{-\frac{q}{2}} (H11)$

- ALTERNATIVELY, ignore $q > 0$ in P.I.
  and take

$\psi(q_i) = \sum_i \chi_i \int dN \int B q e^{iS(q_i, N)}$

Choose $\chi_i$ according to B.C.s
MATTER FLUCTUATIONS

\[ H = H_{\text{ds}}(q) + H_{\text{m}}(q, \phi) \quad \phi \quad \text{small} \]

- PI analysis yields
  \[ \psi(q, \phi) = e^{\pm \frac{1}{\Lambda} + i S(q) + \phi^2} \]

- Is \( e^\phi^2 \) consistent with normalize? 

E.g. \( e^{\phi^2} \approx e^{\phi^2 - \phi^4} \) = normalizable.

- Mode \( f^2 \) analysis \( \Rightarrow \) Yes

- Can choose mode \( f^5 \) or \( +\phi^2 \) sums of contours to avoid \( e \)
WKB Interpretation

\[ \Psi = C e^{iS} \]

- classical paths \( p = \Delta S \)

\[ \text{prob}(\Delta) = \int_{\Sigma_{in}} d^nq \ u \cdot \nabla S \ |c|^2 \]

= INGOING boundary flux

What is the QM origin of this?
**OBSERVABLES**

- Reparametrization invariance
  \[ \Rightarrow \text{observables } A \text{ obey } [H, A] = 0 \]
  
  E.g. \[ A = \int_{-\infty}^{\infty} B(t) \]
  
  or \[ A = \frac{1}{\pi} \int_{t=-\infty}^{+\infty} P(t) \]

  \( \text{projector or POVM} \)

- Classically this means \( A \) is a function of the entire classical path
**Expected Answer**

\[ P = \int_\Delta dq \langle q \rangle \langle q | \]

\[ \dot{P} = (\dot{P})_{in} - (\dot{P})_{out} \quad \text{possible on given } \Sigma \]

\[ p_n < 0 \quad p_n > 0 \]

- **Ingoing Intersection Number**

\[ I_{in}^\Sigma = \int_{-\infty}^{\infty} e^{i k \ell} (\dot{P})_{in} e^{-i k \ell} \]

classically

\[ \begin{cases} 1 & \text{path enters } \Delta \\ 0 & \text{path not enter } \Delta \end{cases} \]

\[ [I_{in}^\Sigma, H] = 0 \]

\[ \text{prob}(\Delta) = \langle \Psi | I_{in}^\Sigma | \Psi \rangle \]
PATH INTEGRAL APPROACH

\[ A_\Delta (\varphi'', \varphi') = \int d\varphi N \int_\Delta \delta \varphi^\Delta e^{i S(\varphi, N)} \]

\[ A_\Delta (\varphi'', \varphi') = \text{Sum over paths outside } \Delta \]

\[ K A (\varphi'', \varphi') \neq 0 \text{ on } \partial \Delta = \Sigma \]

Problems with Zeno effect
CLASS OPERATOR FOR NOT ENTERING $\Delta$

\[ P = \int_{\Delta} d^q \langle \Psi | \Phi \rangle \]

- $C_\Delta = \frac{1}{t} P(t)$, $\tilde{P} = 1 - P$
  - unphysical due to Zeno
  - $\tilde{P} \rightarrow P_{OVM}$

- $C_\Delta = \left( \lim_{t_2 \to \infty} e^{iHt_2} - (iH + V)(t_2 - t_1) - iHt_1 \right) e^{iHt_1}$
  - $t_1 \to -\infty$
  - $V = V_0 P$ absorbing potential

- $S$-matrix for scattering off $V$

- $\text{PROB}(\Delta) = \langle \Phi | C_\Delta^+ C_\Delta | \Phi \rangle$ (expected WKB result)
KEY POINTS

- $\Pi$ of the form
  $\psi = \int_0^\infty \int_\eta^\xi e^{\imath S}$

  are not uniquely defined without QM boundary conditions.

- Probabilities for histories in quantum cosmology lead to other types of $\Pi$
$H$ is SELF-ADJOINT

dS minisuperspace model
$ds^2 = -\frac{\dot{N}^2 dt^2}{q(t)} + q(t) d\Omega_3^2 \quad q = q^2$

- $\text{WD eq}^2 \left[ \frac{d^2}{dq^2} + 12\pi^4 (\Lambda q - 1) \right] \psi = 0$

$q \geq 0 \quad \text{QM on } \mathbb{R}^+$

- $H$ self adjoint $\Rightarrow$ conditions on $\psi(q)$

$\psi(q) + b \psi'(q) = 0$ at $q = 0$

for any $b \in \mathbb{R}$ ($b \neq 0$)
A real Lorentzian contour can give one of many solutions to $HY = 0$ depending on B.C.s at $q = 0$.

Ex. In dS MSS model:

- $u(0) = -u'(0) \Rightarrow \psi = e^{-q} (T)$
- $u(0) = u'(0) \Rightarrow \psi = e^{+q} (HH)$

Alternatively, ignore $q > 0$ in P.I. and take

$$\psi(q) \sim \sum_{i} \alpha_{i} \int dN \int \Theta q e^{iS(q, N)}$$

Choose $\alpha_{i}$ according to B.C.s.
A REAL LORENTZIAN contour can give one of many solutions to $H \psi = 0$ depending on B.C.'s at $q = 0$

Ex. In dS MSS model:

$u(0) = -u'(0) \Rightarrow \psi = e^{-\frac{q}{2}}$ (T)

$u(0) = u'(0) \Rightarrow \psi = e^{\frac{q}{2}}$ (III)

ALTERNATIVELY, ignore $q > 0$ in P.I and take

$\psi(q_1) = \sum_i c_i \int dN \int dq_2 e^{i \tilde{q} N}$

Choose $c_i$ according to B.C.'s
• A REAL LORENTZIAN contour can give one of many $S^1$'s to $H^p = 0$
depending on B.C.s at $q = 0$

Ex. In dS MSS model:

$u(t) = u'(t) \Rightarrow \psi = e^{-i\xi t}$

$u(0) = u'(0) \Rightarrow \psi = e^{i\gamma}$

• ALTERNATIVELY, ignore $q > 0$ in P.I.

and take

$\psi(q), \sum_i \int d^2 \theta \int \mathcal{D} \xi \mathcal{D}_{\xi} e^{i\xi \psi}$

Choose $\xi_i$ according to B.C.s
H is SELF-ADJOINT

ds minim superspace model

ds^2 = -N^2 c dt^2 + q(t) dΩ_3^2  \quad q = q^2

\begin{itemize}
  \item WD eq^A \left[ \frac{d^2}{dq^2} + 12π^4 (Λq - 1) \right] \psi = 0
  \item q > 0 \quad QM on R^+
  \item H self-adjoint \Rightarrow \text{conditions on } \psi(q)
\end{itemize}

\psi(q) + b \psi'(q) = 0 \quad \text{at } q = 0

\text{for any } b \in \mathbb{R} \quad (b \neq 0)