Title: Covariant observables and (quantum) extension theorems
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Abstract:
Covariant observables and (quantum) extension theorems

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\[ Z[g] = \int d[g] \exp \left( \frac{iS[g]}{\hbar} \right) \]
What are the Covariant Beables or Observables in Quantum Gravity?

- Intrinsic teleology in GR
  - Blackholes
  - Cauchy hypersurface: $(\Sigma, h)$
  - Global Causality Conditions

Covariant observables seem to require a "fully-evolved" spacetime
What are the Covariant Beables or Observables in Quantum Gravity?

- Intrinsic teleology in GR suggests we need “fully-evolved” spacetimes
- Access only to limited spacetime regions
- “Acorn’s intrinsic telos is to become a fully grown oak tree” — Aristotle
- What are the covariant “acorns” of quantum gravity?
Quantum measure formulation

Quantum Theory is a Generalisation of Classical Stochastic Dynamics

Sorkin (1994)
Classical Stochastic Dynamics given by a Probability Measure Space: \((\Omega, \mathcal{A}, \mu)\)

- **Sample Space** \(\Omega\): space of histories.

- **Event Algebra** \(\mathcal{A}\): set of all beables or propositions about the system.
  
  \((These\ are\ the \ "acorns" - to\ be\ identified)\)

  - \(\alpha \in \mathcal{A}, \alpha \subseteq \Omega\)
  - \(\mathcal{A}\) is closed under finite set union, intersection and complementation.
  - \(\emptyset, \Omega \in \mathcal{A}\)

- **Probability Measure** \(\mu : \mathcal{A} \to [0, 1]\): finitely additive

  Kolmogorov Sum Rule: \(\mu(\alpha_1 \sqcup \alpha_2) = \mu(\alpha_1) + \mu(\alpha_2)\)
Quantum Dynamics is given by a Quantum Measure Space \((\Omega, \mathcal{A}, \mu)\)

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- **Quantum Measure \(\mu : \mathcal{A} \rightarrow \mathbb{R}^+\):**

  Quantum Sum Rule:

  \[
  \mu(\alpha_1 \cup \alpha_2 \cup \alpha_3) = \mu(\alpha_1 \cup \alpha_2) + \mu(\alpha_1 \cup \alpha_3) + \mu(\alpha_2 \cup \alpha_3) - \mu(\alpha_1) - \mu(\alpha_2) - \mu(\alpha_3).
  \]
The Quantum Measure

Decoherence Functional $D : \mathcal{A} \times \mathcal{A} \to \mathbb{C}$

- $D(\alpha, \beta) = D^*(\beta, \alpha)$: Hermitian
- $D(\alpha, \cup_i^k \beta_i) = \sum_i D(\alpha, \beta_i)$: finitely biadditive
- $D(\Omega, \Omega) = 1$
- $M_{ij} \equiv D(\alpha_i, \alpha_j)$ for any finite collection $\{\alpha_i\}$ is positive semi-definite: $D$ is strongly positive.

$\mu(\alpha) \equiv D(\alpha, \alpha)$
An example: particle mechanics in $\mathbb{R}^N$

$$D(\gamma, \gamma') = \psi^*(\gamma(0)) \exp(-iS[\gamma]) \delta(\gamma(T) - \gamma'(T)) \exp(iS[\gamma']) \psi(\gamma'(0))$$

- $\bar{\alpha} = (\alpha_1, \alpha_2 \ldots \alpha_T), \alpha_i$: Lebesgue measurable sets of $\mathbb{R}^N$.

- $D(\alpha, \beta) = \int_\alpha d[\gamma] \int_\beta d[\gamma'] D(\gamma, \gamma') = \langle \psi_\alpha, \psi_\beta \rangle$

- $\psi_{\bar{\alpha}}(\bar{x}_T, T) = \chi_{\alpha_N} \int_{\alpha_{N-1}} d\bar{x}_{N-1} \ldots \int_{\alpha_1} d\bar{x}_0 K(\bar{x}_T, T|\bar{x}_N-1, t_{N-1}) \ldots K(\bar{x}_1, t_1|\bar{x}_0, 0) \psi(x, 0)$
Observables or Beables are measurable sets $\alpha \in \mathcal{A}$.

$\mathcal{O} \rightarrow \alpha$
Observables or Beables are measurable sets $\alpha \in A$.

$\hat{O} \rightarrow \alpha$

Hats off to Quantum Gravity!!
An Illustration: The Random Walk

- $\Omega$: set of all infinite time trajectories on $\mathbb{Z} \times \mathbb{Z}^+$.  

- $\mathcal{A}$: “measurable” events

  Event: The walker is at $x = a$ at $t = 3$.  

  $\alpha$: set of all paths which are at $x = +a$ at time $t = 3$.  

  ![Diagram showing a grid with a path at $(x=a, t=3)$]

- $\mu$: Probability on $\mathcal{A}$.  

  For unbiased walker, $\mu(\alpha) = 3/8$ is the probability for this event.
Constructing $\mathcal{A}$ from Finite Time Paths

- $\gamma^T$: finite time path
- $cyl(\gamma^T) \equiv \{ \gamma \in \Omega | \gamma|_{[0,T]} = \gamma^T \}$
- $\mathcal{A}$: generated by taking finite unions, intersections and complementations of cylinder sets.
- $\mu(cyl(\gamma^T)) \equiv Prob(\gamma^T) = 2^{-T}$. 
What about infinite time events? These are analogs of covariant events.

- Return event
- First passage
- Infinite return event
- Infinite time events requires countable set operations
- Extension question: Are infinite time events measurable?
- Example: Return event $R$
  - $R = \bigcup_{T \in \mathbb{Z}^+} R(T) \in \mathcal{A}$: return for the first time at $t = T$.
  - $\mu(R(T))$ is defined but is $\mu(R) = \sum_T \mu(R(T))$ well defined?
- $R \in \mathcal{S}$, sigma algebra $\mathcal{S} \supseteq \mathcal{A}$: closed under countable set operations
- Kolmogorov-Caratheodory-Hahn extension theorem:
  $(\Omega, \mathcal{A}, \mu)$ extends to $(\Omega, \mathcal{S}, \bar{\mu})$. 
Infinite time events requires countable set operations

Extension question: Are infinite time events measurable?

Eg: Return event $R$

$R \in S$, sigma algebra $S \supseteq \mathcal{A}$: closed under countable set operations

Kolmogorov-Caratheodory-Hahn extension theorem:

$(\Omega, \mathcal{A}, \mu)$ extends to $(\Omega, S, \bar{\mu})$.

Remark: Wiener measure

\[ p(\Delta t, \Delta x) = \left(4\pi\Delta t\right)^{-\frac{n}{2}} \exp\left(-\frac{(\Delta x)^2}{4\Delta t}\right) \]

defined only when the measure “extends” from $\mathcal{A}$ (generated by “window sets”) to $S$. 
Extension of the quantum measure in gravity tied to covariance of beables.

Is there an analog of the KCH extension for the quantum measure?

Geroch, unpublished notes (available on Sorkin’s webpage)

Complex Percolation Dynamics in Causal Set Quantum Gravity

Dowker, Johnston and Surya (2010), Sorkin and Surya (in prepn)
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Let's just focus on quantum mechanics for now..
The Quantum Measure as a Vector Measure

- A Histories Hilbert space $\mathcal{H}$:
  - $V$: free vector space of functions $f : \mathcal{A} \to \mathbb{C}$.
  - $\langle u, v \rangle_{V} \equiv \sum_{\alpha \in \mathcal{A}} \sum_{\beta \in \mathcal{A}} u^{\ast}(\alpha)v(\beta)D(\alpha, \beta)$
  - $\{u_{i}\} \sim \{v_{i}\}$ if $\lim_{i \to \infty} ||u_{i} - v_{i}||_{V} = 0$.
  - Hilbert Space $\mathcal{H} = V/\sim$, $\{\{u_{i}\}\} \in \mathcal{H}$.
The Quantum Measure as a Vector Measure

- A Histories Hilbert space $\mathcal{H}$:  
  
  Dowker, Johnston and Sorkin, (2010)

- $\mu_v : \mathcal{A} \to \mathcal{H}$ is a vector measure  
  Dowker, Johnston and Surya (2010)

\[
\mu_v(\alpha) \equiv [\chi_\alpha] \in \mathcal{H}.
\]

\[
\chi_\alpha(\beta) = \begin{cases} 
1 & \text{if } \beta = \alpha, \\
0 & \text{if } \beta \neq \alpha.
\end{cases}
\]

\[
\langle \mu_v(\alpha), \mu_v(\beta) \rangle = D(\alpha, \beta),
\]
The Quantum Measure as a Vector Measure

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  Dowker, Johnston and Sorkin, (2010)

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  Finite additivity:

  $$\mu_v(\cup_{i=1}^n \alpha_i) = \sum_{i=1}^n \mu_v(\alpha_i)$$

- Caratheodary-Hahn-Kluvanek theorem:
  An extension $(\Omega, \mathcal{A}, \mu_v) \to (\Omega, S_\mathcal{A}, \mu_v^*)$ exists and is unique
  provided $\mu_v$ satisfies certain convergence conditions

- Unique extension of $\mu_v$ gives a unique extension of decoherence functional $D$. 
Discrete finite unitary process

Extension of $\mu_\nu$ requires it to be of bounded total variation:

$$|\mu_\nu|(\alpha) = \sup_{\Pi(\alpha)} \sum_{\alpha_i \in \Pi(A)} |\mu(\alpha_i)| < \infty$$
Discrete finite unitary process

Extension of $\mu_\nu$ requires it to be of bounded total variation:

$$|\mu_\nu|(\alpha) = \sup \sum_{\alpha \in \Pi(A)} |\mu(\alpha_j)| < \infty$$

- $\mathcal{H} = \mathbb{C}^N$
- $\psi_m = U(m, m-1)U(m-1, m-2)\ldots U(2, 1)\psi$
- $D(\text{cyl}(\gamma^T), \text{cyl}(\gamma'^T)) = A^*(\gamma)A(\gamma)\delta(x(T) - x'(T))$
- $\tilde{\psi}_\gamma = P_mU(m, m-1)P_{m-1}U(m-1, m-2)\ldots U(2, 1)P_1\psi_{\text{initial}}$
- $\mu_\nu(\alpha) = \psi_\alpha$

The quantum vector measure for a "generic" finite unitary system with discrete time steps is not of bounded variation.

Dowker, Johnston and Surya (2010)
Is $S$ too large?!

Use physics to determine which infinite time events we want..

Return Question: in $\forall A \subset S$. Can we restrict to an $S'$ with $A \subset S' \subset S$?

Even convergence (conditional convergence) of the quantum measure v/s unconditional convergence.

Measure on return event can be defined in discrete unitary processes. \textit{Sorkin (2011)}

Open Question: What about the quantum particle in $\mathbb{R}^N$? Is there an analogue of the Wiener process?

Aspiration: quantum measure of gravity should evenly converge for a set of infinite time events of physical interest

Complex percolation dynamics for causal sets:
The quantum measure on the originary event is evenly convergent.
\textit{Sorkin and Surya (in preparation)}