The deBroglie-Bohm interpretation

Louis deBroglie  
(1892-1987)  

David Bohm  
(1917-1992)  

“I saw the impossible done...”  
John Bell
Responses to the measurement problem

1. Deny realism
   - Purely operational account of quantum theory

2. Deny the universality of unitary dynamics
   - Dynamical collapse theories

3. Deny that \( \psi \) is a complete representation of reality
   - Hidden variable models
     - Models of reality beyond hidden variables?

4. Deny indeterminism and discontinuity, except as subjective illusions
   - Everett’s relative state interpretation, or “many worlds”
The deBroglie-Bohm interpretation for a single particle

The ontic state: \( (\psi(r), \zeta) \)

- Wavefunction
- Particle position

\[ \psi(r, t) \]

\[ \zeta(t) \]
The deBroglie-Bohm interpretation for a single particle

The ontic state: \((\psi(r), \zeta)\)

Wavefunction \(\psi(r,t)\)

Particle position \(\zeta(t)\)

The evolution equations:

\[
i\hbar \frac{\partial \psi(r,t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi(r,t) + V(r)\psi(r,t)
\]

Schrödinger's eq'n
The deBroglie-Bohm interpretation for a single particle

The ontic state: \((\psi(\mathbf{r}, t), \zeta)\)

Wavefunction \[ \psi(\mathbf{r}, t) \]

Particle position \[ \zeta(t) \]

The evolution equations:

\[
\frac{i\hbar}{\partial t} \frac{\partial \psi(\mathbf{r}, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi(\mathbf{r}, t) + V(\mathbf{r})\psi(\mathbf{r}, t)
\]

Schrödinger’s eq’n

\[
\frac{d\zeta(t)}{dt} = \frac{1}{m} \left[ \nabla S(\mathbf{r}, t) \right]_{\mathbf{r}=\zeta(t)}
\]

The guidance eq’n

where

\[
\psi(\mathbf{r}, t) = R(\mathbf{r}, t)e^{iS(\mathbf{r}, t)/\hbar}
\]
The deBroglie-Bohm interpretation for a single particle

The ontic state: \((\psi(r), \zeta)\)

Wavefunction \(\psi(r,t)\)
Particle position \(\zeta(t)\)

The evolution equations:

\[
\begin{align*}
i\hbar \frac{\partial \psi(r,t)}{\partial t} &= -\frac{\hbar^2}{2m} \nabla^2 \psi(r,t) + V(r)\psi(r,t) \quad \text{Schrödinger's eq'n} \\
\frac{d\zeta(t)}{dt} &= \frac{1}{m} \left[ \nabla S(r,t) \right]_{r=\zeta(t)} \quad \text{The guidance eq'n}
\end{align*}
\]

where \(\psi(r,t) = R(r,t)e^{iS(r,t)/\hbar}\)

Note: There is no back-action on the wave
The amplitude of the wave is irrelevant → a pilot wave
Given \( \psi(r, t) = R(r, t)e^{iS(r, t)/\hbar} \)

The real part of the Schrodinger eq'n is:

\[
\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + Q + V = 0
\]

where \( Q(r, t) \equiv -\frac{\hbar^2}{2m} \frac{\nabla^2 R(r, t)}{R(r, t)} \)

The "quantum potential"

The imaginary part of the Schrodinger eq'n is:

\[
\frac{\partial}{\partial t} \left( R^2 \right) + \nabla \left( \frac{R^2 \nabla S}{m} \right) = 0
\]
Newtonian form of the particle dynamics:

\[
m \frac{d^2 \zeta(t)}{dt^2} = -\left[ \nabla V(r) + \nabla Q(r, t) \right]_{r = \zeta(t)}
\]

where \( Q(r, t) \equiv -\frac{\hbar^2}{2m} \frac{\nabla^2 R(r, t)}{R(r, t)} \) \( \quad \) The “quantum potential”

(Note independence of quantum potential on magnitude)
Newtonian form of the particle dynamics:

\[ m \frac{d^2 \zeta(t)}{dt^2} = -\left[ \nabla V(\mathbf{r}) + \nabla Q(\mathbf{r}, t) \right]_{\mathbf{r} = \zeta(t)} \]

where \[ Q(\mathbf{r}, t) \equiv -\frac{\hbar^2}{2m} \frac{\nabla^2 R(\mathbf{r}, t)}{R(\mathbf{r}, t)} \]  The "quantum potential"

(Note independence of quantum potential on amplitude)

How else does deBroglie-Bohm differ from Newtonian mechanics?
Newtonian form of the particle dynamics:

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The “quantum potential”

(Note independence of quantum potential on amplitude)

How else does deBroglie-Bohm differ from Newtonian mechanics?

The dynamics are \textit{fundamentally first order}

\[ \frac{d\zeta(t)}{dt} = \frac{1}{m} \left[ \nabla S(r, t) \right]_{r=\zeta(t)} \]
Acting the $\nabla$ operator on the real part of the Schrodinger eq'n gives:

$$\nabla \left[ \frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + Q + V \right] = 0$$

$$\left( \frac{\partial}{\partial t} + \frac{\nabla S \cdot \nabla}{m} \right) \nabla S = -\nabla (Q + V)$$

Taking the time derivative of the guidance equation gives:

$$\frac{d\zeta(t)}{dt} = \frac{1}{m} [\nabla S(r,t)]_{r=\zeta(t)}$$

$$\frac{d^2\zeta(t)}{dt^2} = \frac{1}{m} \left( \frac{\partial}{\partial t} + \frac{d\zeta}{dt} \cdot \nabla \right) \nabla S$$

Thus

$$m \frac{d^2\zeta(t)}{dt^2} = -[\nabla V(r) + \nabla Q(r,t)]_{r=\zeta(t)}$$
Epistemic state (assuming perfect knowledge of $\psi(r, t)$)

$\rho(\zeta)d\zeta = \text{the probability the particle is within } d\zeta \text{ of } \zeta.$

The "standard distribution"

$$\rho(\zeta, t) = |\psi(\zeta, t)|^2$$

Note: it is preserved by the dynamics:

if $\rho(\zeta, 0) = |\psi(\zeta, 0)|^2$ then $\rho(\zeta, t) = |\psi(\zeta, t)|^2$
Proof of the preservation of the standard distribution:

The velocity field is

\[ \mathbf{v}(\mathbf{r}, t) = \frac{1}{m} [\nabla S(\mathbf{r}, t)] \]

The probability current density is:

\[ \mathbf{j}(\mathbf{r}, t) = \rho(\mathbf{r}, t) \mathbf{v}(\mathbf{r}, t) \]

Conservation of probability implies

\[ \frac{\partial \rho(\mathbf{r}, t)}{\partial t} = -\nabla \cdot \mathbf{j}(\mathbf{r}, t) = -\nabla \cdot \left( \frac{\rho(\mathbf{r}, t) \nabla S(\mathbf{r}, t)}{m} \right) \]
Proof of the preservation of the standard distribution:

The velocity field is

$$\mathbf{v}(\mathbf{r}, t) = \frac{1}{m} \left[ \nabla S(\mathbf{r}, t) \right]$$

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Recall the imaginary part of the Schrodinger eq'n:

$$\frac{\partial}{\partial t} \left( R^2 \right) = -\nabla \cdot \left( \frac{R^2 \nabla S}{m} \right)$$
Proof of the preservation of the standard distribution:

The velocity field is

\[ \mathbf{v}(\mathbf{r}, t) = \frac{1}{m} \nabla S(\mathbf{r}, t) \]

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Conservation of probability implies

\[ \frac{\partial \rho(\mathbf{r}, t)}{\partial t} = -\nabla \cdot \mathbf{j}(\mathbf{r}, t) = -\nabla \cdot \left( \frac{\rho(\mathbf{r}, t) \nabla S(\mathbf{r}, t)}{m} \right) \]

Recall the imaginary part of the Schrodinger eq’n:

\[ \frac{\partial}{\partial t} \left( R^2 \right) = -\nabla \cdot \left( \frac{R^2 \nabla S}{m} \right) \]

Therefore, if \( \rho(\mathbf{r}, t) = R^2(\mathbf{r}, t) \) then

\[ \frac{\partial}{\partial t} \left( \rho(\mathbf{r}, t) - R^2(\mathbf{r}, t) \right) = 0 \]
\[ \psi = \sum_j c_j \psi_j \]

“waves” of the decomposition

Spatial support of \( \psi_j = \{ \mathbf{r} : \psi_j(\mathbf{r}) \neq 0 \} \)

\( \zeta \subseteq \) Spatial support of \( \psi_j \) \( j \)th wave is occupied

\( \zeta \nsubseteq \) Spatial support of \( \psi_j \) \( j \)th wave is empty

If only the \( k \)th wave is occupied

Then the guidance equation depends only on the \( k \)th wave
Proof of ineffectiveness of empty waves

\[ \psi = \psi_a + \psi_b \]

\[ \text{Re} e^{i\phi/\hbar} = R_a e^{iS_a/\hbar} + R_b e^{iS_b/\hbar} \]

\[ R^2 = R_a^2 + R_b^2 + 2R_a R_b \cos\left(\frac{(S_a - S_b)}{\hbar}\right) \]

\[ \nabla S = R^{-2} \left\{ R_a \nabla S_a + R_b \nabla S_b + R_a R_b \cos\left(\frac{(S_a - S_b)}{\hbar}\right) \nabla (S_a + S_b) \right\} \]

\[ -\hbar \left[ R_a \nabla R_b - R_b \nabla R_a \right] \sin\left(\frac{(S_a - S_b)}{\hbar}\right) \]
Proof of ineffectiveness of empty waves

\[ \psi = \psi_a + \psi_b \]

\[ R e^{i\frac{S}{\hbar}} = R_a e^{i\frac{S_a}{\hbar}} + R_b e^{i\frac{S_b}{\hbar}} \]

\[ R^2 = R_a^2 + R_b^2 + 2 R_a R_b \cos \left( \frac{(S_a - S_b)}{\hbar} \right) \]

\[ \nabla S = R^{-2} \left\{ R_a^2 \nabla S_a + R_b^2 \nabla S_b + R_a R_b \cos \left[ \frac{(S_a - S_b)}{\hbar} \right] \nabla (S_a + S_b) \right\} \]

\[ - \frac{\hbar}{2} \left[ R_a \nabla R_b - R_b \nabla R_a \right] \sin \left[ \frac{(S_a - S_b)}{\hbar} \right] \]

If \( R_a R_b \approx 0, \quad R_a \nabla R_b \approx 0, \quad R_b \nabla R_a \approx 0 \)

then \( R^2 = R_a^2 + R_b^2 \) and \( \nabla S = \frac{R_a^2 \nabla S_a + R_b^2 \nabla S_b}{R_a^2 + R_b^2} \)
Proof of ineffectiveness of empty waves

\[ \psi = \psi_a + \psi_b \]

\[ \text{Re} e^{iS/a} = R_a e^{iS_a/h} + R_b e^{iS_b/h} \]

\[ R^2 = R_a^2 + R_b^2 + 2R_a R_b \cos \left( \frac{(S_a - S_b)}{\hbar} \right) \]

\[ \nabla S = R^{-2} \left\{ R_a \nabla S_a + R_b \nabla S_b + R_a R_b \cos \left( \frac{(S_a - S_b)}{\hbar} \right) \nabla (S_a + S_b) \right\} \]

\[ -\hbar \left[ R_a \nabla R_b - R_b \nabla R_a \right] \sin \left( \frac{(S_a - S_b)}{\hbar} \right) \]

If \( R_a R_b \approx 0 \), \( R_a \nabla R_b \approx 0 \), \( R_b \nabla R_a \approx 0 \)

then \( R^2 = R_a^2 + R_b^2 \) and \( \nabla S = \frac{R_a^2 \nabla S_a + R_b^2 \nabla S_b}{R_a^2 + R_b^2} \)

\[ \frac{d\xi(t)}{dt} = \frac{1}{m} \left[ \nabla S(r, t) \right]_{r = \xi(t)} \]
Proof of ineffectiveness of empty waves

\[ \psi = \psi_a + \psi_b \]

\[ \text{Re} e^{i\psi/\hbar} = R_a e^{iS_a/\hbar} + R_b e^{iS_b/\hbar} \]

\[ R^2 = R_a^2 + R_b^2 + 2R_a R_b \cos \left( \frac{(S_a - S_b)}{\hbar} \right) \]

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\[ \left[ -\hbar [R_a \nabla R_b - R_b \nabla R_a] \sin \left( \frac{(S_a - S_b)}{\hbar} \right) \right] \]

If \( R_a R_b \approx 0, \quad R_a \nabla R_b \approx 0, \quad R_b \nabla R_a \approx 0 \)

then \( R^2 = R_a^2 + R_b^2 \) and \( \nabla S = \frac{R_a^2 \nabla S_a + R_b^2 \nabla S_b}{R_a^2 + R_b^2} \)

\[ \frac{d\xi(t)}{dt} = \frac{1}{m} \left[ \nabla S(\mathbf{r},t) \right]_{\mathbf{r} = \xi(t)} = \frac{\nabla S_a}{m} \quad \text{If } \xi \in \text{ Support of } \psi_a \]

\[ = \frac{\nabla S_b}{m} \quad \text{If } \xi \in \text{ Support of } \psi_b \]
Double slit experiment
Transmission through a barrier (probability ½)
Beam splitter experiment
The deBroglie-Bohm interpretation for many particles

The ontic state: \( \psi(r_1, r_2, \zeta_1, \zeta_2) \)

Wavefunction on configuration space  
Particle positions

\[ \psi(r_1, r_2) \]
The deBroglie-Bohm interpretation for many particles

The ontic state: \( (\psi(r_1, r_2), \xi_1, \xi_2) \)

Wavefunction on configuration space

Particle positions

The evolution equations:

**Schrödinger’s equation**

\[
\frac{i\hbar}{\hbar} \frac{\partial \psi(r_1, r_2, t)}{\partial t} = - \frac{\hbar^2}{2m_1} \nabla_1^2 \psi(r_1, r_2, t) - \frac{\hbar^2}{2m_2} \nabla_2^2 \psi(r_1, r_2, t) + V(r_1, r_2) \psi(r_1, r_2, t)
\]
The deBroglie-Bohm interpretation for many particles

The ontic state: \( \langle \psi(\mathbf{r}_1, \mathbf{r}_2), \zeta_1, \zeta_2 \rangle \)

Wavefunction on configuration space

Particle positions

The evolution equations:

\[ i\hbar \frac{\partial \psi(\mathbf{r}_1, \mathbf{r}_2, t)}{\partial t} = -\frac{\hbar^2}{2m_1} \nabla^2_1 \psi(\mathbf{r}_1, \mathbf{r}_2, t) - \frac{\hbar^2}{2m_2} \nabla^2_2 \psi(\mathbf{r}_1, \mathbf{r}_2, t) + V(\mathbf{r}_1, \mathbf{r}_2) \psi(\mathbf{r}_1, \mathbf{r}_2, t) \]

\[ \frac{d\zeta_1(t)}{dt} = \frac{1}{m_1} \left[ \nabla_1 S(\mathbf{r}_1, \mathbf{r}_2, t) \right]_{\mathbf{r}_1 = \zeta_1(t), \mathbf{r}_2 = \zeta_2(t)} \]

\[ \frac{d\zeta_2(t)}{dt} = \frac{1}{m_2} \left[ \nabla_2 S(\mathbf{r}_1, \mathbf{r}_2, t) \right]_{\mathbf{r}_1 = \zeta_1(t), \mathbf{r}_2 = \zeta_2(t)} \]

The guidance equation

where \( \psi(\mathbf{r}_1, \mathbf{r}_2, t) = R(\mathbf{r}_1, \mathbf{r}_2, t) e^{iS(\mathbf{r}_1, \mathbf{r}_2, t)/\hbar} \)
\[ \psi(r_1, r_2, t) = \phi^{(1)}(r_1, t) \chi^{(2)}(r_2, t) \tag{Product state} \]
\[ = R_1(r_1, t)e^{iS_1(r_1, t)/\hbar} R_2(r_2, t)e^{iS_2(r_2, t)/\hbar} \]

\[ S(r_1, r_2, t) = S_1(r_1, t) + S_2(r_2, t) \]

\[ \frac{d\xi_1(t)}{dt} = \frac{1}{m_1} \left[ \nabla_{r_1} S(r_1, r_2, t) \right]_{r_1 = \xi_1(t), r_2 = \xi_2(t)} = \frac{1}{m_1} \left[ \nabla_{r_1} S_1(r_1, t) \right]_{r_1 = \xi_1(t)} \]

\[ \frac{d\xi_2(t)}{dt} = \frac{1}{m_2} \left[ \nabla_{r_2} S(r_1, r_2, t) \right]_{r_1 = \xi_1(t), r_2 = \xi_2(t)} = \frac{1}{m_2} \left[ \nabla_{r_2} S_2(r_2, t) \right]_{r_2 = \xi_2(t)} \]

The two particles evolve independently
\[ \psi(r_1, r_2, t) = \sum_j c_j \phi_j^{(1)}(r_1, t) \chi_j^{(2)}(r_2, t) \quad \text{Entangled state} \]

\[(\zeta_1, \zeta_2) \in \text{support of } \phi_j^{(1)}(r_1, t) \chi_j^{(2)}(r_2, t) \quad j\text{th wave is occupied}\]
\[
\psi(r_1, r_2, t) = \sum_j c_j \phi_j^{(1)}(r_1, t) \chi_j^{(2)}(r_2, t)
\]

Entangled state

((\zeta_1, \zeta_2) \in \text{ support of } \phi_j^{(1)}(r_1, t) \chi_j^{(2)}(r_2, t)) \quad j\text{th wave is } \text{occupied}

((\zeta_1, \zeta_2) \notin \text{ support of } \phi_j^{(1)}(r_1, t) \chi_j^{(2)}(r_2, t)) \quad j\text{th wave is } \text{empty}

If only the kth wave is occupied

Then the particles evolve independently
$$\psi(r_1, r_2, t) = \sum_j c_j \phi_j^{(1)}(r_1, t) \chi_j^{(2)}(r_2, t)$$  \hspace{1cm} \text{Entangled state}$$

$$((\zeta_1, \zeta_2) \in \text{ support of } \phi_j^{(1)}(r_1, t) \chi_j^{(2)}(r_2, t) \quad \text{ jth wave is occupied}$$

$$((\zeta_1, \zeta_2) \notin \text{ support of } \phi_j^{(1)}(r_1, t) \chi_j^{(2)}(r_2, t) \quad \text{ jth wave is empty}$$

If only the kth wave is occupied

Then the particles evolve independently

But in general, they do not

This implies a failure of local causality and of Lorentz invariance at the ontological level
\[ \psi(r_1, r_2; t) = c_a \phi_a(r_1, t) \chi_a(r_2, t) + c_b \phi_b(r_1, t) \chi_b(r_2, t) \]
\[ \psi(r_1, r_2; t) = c_a \phi_a(r_1, t) \chi_a(r_2, t) + c_b \phi_b(r_1, t) \chi_b(r_2, t) \]
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\[
\psi(r_1, r_2; t) = c_a \phi_a(r_1, t) \chi_a(r_2, t) + c_b \phi_b(r_1, t) \chi_b(r_2, t)
\]

occupied wave

\[
\frac{d\zeta_2(t)}{dt} = \frac{1}{m_2} \left[ \nabla_2 S(r_1, r_2, t) \right]_{r_1 = \zeta_1(t), r_2 = \zeta_2(t)}
\]
\[ \psi(r_1, r_2; t) = c_a \phi_a(r_1, t) \chi_a(r_2, t) + c_b \phi_b(r_1, t) \chi_b(r_2, t) \]

occupied wave

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\psi(r_1, r_2; t) = c_a \phi_a(r_1, t) \chi_a(r_2, t) + c_b \phi_b(r_1, t) \chi_b(r_2, t)
\]

both waves occupied
\[ \psi(r_1, r_2; t) = c_a \phi_a(r_1, t) \chi_a(r_2, t) + c_b \phi_b(r_1, t) \chi_b(r_2, t) \]

both waves occupied

Failure of local causality
Reproducing the operational predictions

Consider a measurement of $\mathcal{A}$ with eigenvectors $\phi_k(r)$

$$\phi_k(r)\chi(r') \rightarrow \phi_k(r)\chi_k(r')$$
Reproducing the operational predictions

Consider a measurement of $A$ with eigenvectors $\phi_k(r)$

$$\phi_k(r)\chi(r') \rightarrow \phi_k(r)\chi_k(r')$$

$$[\sum_k c_k\phi_k(r)]\chi(r') \rightarrow \sum_k c_k\phi_k(r)\chi_k(r')$$

Assumption: different outcomes of a measurement correspond to disjoint regions of the configuration space of the apparatus

$$\chi_j(r')\chi_k(r') \simeq 0 \text{ if } j \neq k$$
Reproducing the operational predictions

Consider a measurement of \( A \) with eigenvectors \( \phi_k(r) \)

\[
\phi_k(r) \chi(r') \rightarrow \phi_k(r) \chi_k(r')
\]

\[
[\sum_k c_k \phi_k(r)] \chi(r') \rightarrow \sum_k c_k \phi_k(r) \chi_k(r')
\]

Assumption: different outcomes of a measurement correspond to disjoint regions of the configuration space of the apparatus

\[
\chi_j(r') \chi_k(r') \approx 0 \quad \text{if} \quad j \neq k
\]

If the \( j \)th wave comes to be occupied, then one can postulate an effective collapse of the guiding wave

\[
\sum_k c_k \phi_k(r) \rightarrow \phi_j(r)
\]
Reproducing the operational predictions

Consider a measurement of $A$ with eigenvectors $\phi_k(r)$

$$\phi_k(r) \chi(r') \rightarrow \phi_k(r) \chi_k(r')$$

$$[\sum_k c_k \phi_k(r)] \chi(r') \rightarrow \sum_k c_k \phi_k(r) \chi_k(r')$$

Assumption: different outcomes of a measurement correspond to disjoint regions of the configuration space of the apparatus

$$\chi_j(r') \chi_k(r') \simeq 0 \text{ if } j \neq k$$

If the $j$th wave comes to be occupied, then one can postulate an effective collapse of the guiding wave

$$\sum_k c_k \phi_k(r) \rightarrow \phi_j(r)$$

Decoherence makes the process effectively irreversible
Criticisms

- Fails to satisfy the action-reaction principle

- The quantum state plays an epistemic role in determining the initial distribution but it also plays an ontic role in the guidance equation

- Underdetermination of preferred variable and of the form of the dynamics

- Lorentz-invariance at the operational level, failure of Lorentz invariance at the ontological level

- Involves more contextuality and nonlocality than necessary to avoid contradiction

- Everett in denial?
The “standard distribution” as quantum equilibrium

Figure 7. Smoothed $\rho$ ((a), (c) and (e)), compared with $|\psi|^2$ ((b), (d) and (f)), at times $t = 0$ ((a), (b)), $2\pi$ ((c), (d)) and $4\pi$ ((e), (f)). While $|\psi|^2$ recovers its initial value, the smoothed $\rho$ shows a remarkable evolution towards equilibrium.