Abstract: Alday-Gaiotto-Tachikawa connect instanton counts in gauge theory with conformal blocks for W-algebras. We realize this mathematically by relating q-deformed W-algebras with the affine q-Yangians that control gauge theory, thus offering an affine, q-deformed generalization of the well-known Brundan-Kleshchev construction in type A.
AGT from an algebra-geometric perspective
smooth projective $S = H$ ample

Def.: let $\mathcal{M}_S$ be the moduli
stable sheaves $F$ on $S$,
of rank $(F) = r$ and fixed $c_1$
but arbitrary $c_2 \in \mathbb{Z}$

Assume $K_S = C_2$ or $c_1(K_S) \geq 0$
$\mathcal{M}_S$ smooth

Assume $\gcd(r, a(F) + 1) = 1$

Exist a universal sheaf $U \rightarrow F$

$M \times S \rightarrow \mathcal{F}_S$

Thm 1 (N): the $K$-theory group $K(M)$
is a module for the algebra $\mathcal{W}_{\alpha, \beta}(g \mathfrak{h})$

Abelian group homomorphism
$\mathcal{W}_{\alpha, \beta}(g \mathfrak{h}) \rightarrow \text{Hom}(K(M), K(M_S))$
given by explicit generators and relations
satisfying compatibility conditions: \( f_{ab} = \phi_a \circ \phi_b \circ f_{ab} \cdot \phi_c \cdot \phi_d \cdot f_{cd} = 0 \)

\[
\begin{align*}
\phi_{ab} &= K(M) \xrightarrow{\phi_a} K(M \times S) \xrightarrow{\phi_a \times \text{Id}_S} K(M \times S \times S) \xrightarrow{\text{Id}_M \times \delta} K(M \times S) \\
\phi_{cd} &= K(M) \xrightarrow{\phi_c} K(M \times S) \xrightarrow{\text{Id}_M \times \text{mult} \text{ by } f(c)} K(M \times S)
\end{align*}
\]

where \( f : \mathbb{Z}[g_1, g_2]^m \to K(S) \) is a \( \mathbb{Z} \)-algebra morphism. The \( \mathbb{Z} \)-algebra \( g_1, g_2 \to [g_1, g_2]_S \) is the ring homomorphism sending \( g_1, g_2 \to [g_1, g_2]_S \).

\[
\begin{vmatrix}
\phi_{ab} & \phi_{cd}
\end{vmatrix} = \Delta_\phi \left( \begin{vmatrix}
\phi_a & \phi_d
\end{vmatrix}_{\left( \frac{\partial g_1}{\partial x} \right)_{(1,2)}}
\right)
\]

where the LHS is the difference of the following two compositions:

\[
\begin{align*}
&K(M) \xrightarrow{\phi_d} K(M \times S) \xrightarrow{\phi_d \times \text{Id}_S} K(M \times S \times S) \xrightarrow{\text{Id}_M \times \delta} K(M \times S) \\
&K(M) \xrightarrow{\phi_a} K(M \times S) \xrightarrow{\phi_a \times \text{Id}_S} K(M \times S \times S) \xrightarrow{\text{Id}_M \times \delta} K(M \times S)
\end{align*}
\]
Thm 2 (N) the bifundamental matter contribution to gauge, which is encoded by the Ext operator $A_m K(M_n) \to K(M_n)$ "commutes" with the $W_{d+2}(\mathfrak{g}_n)$ algebra action of $K(M_n)$
\[
\begin{align*}
E_m & \twoheadrightarrow \text{RHom}(\mathcal{F}, \text{Sym}^m \mathcal{F}) \\
M \times M' & = (\mathbb{F}, \mathbb{F}')
\end{align*}
\]

Fundamental

\[
A_m = \pi_\mathcal{F} \left( \left[ \pi_\mathcal{E} \left[ \sum_{\text{various } l's} \right] \right] \right)
\]

\[
E(x) = \frac{(1-x_1)(1-x_2)}{(1-x)(1-x_2)} \quad \text{for } x_1, x_2 \in \mathbb{F}
\]

\[
W_k(x) W_L(y) \prod_{l=\text{root}(q,l)} \frac{\psi(x_l)}{x_l} - W_L(y) W_k(x) \prod_{l=\text{root}(q,l)} \frac{\psi(x_l)}{x_l} = \\
= (1-x_1)(1-x_2) \sum_{\text{various } l's} W_k(x) W_{k+L}(y) \frac{S(x_l)}{x_l^2} \text{ const}_{k,L}...
\]

\text{Def (Feigin-Frenkel-Reshetikhin, Awata-Kubo-Odake-Shiraishi)}

\[
W_{\mathcal{F}_0}(g_0) \text{ in the } \mathbb{Z}[[g_1, g_2]] \text{ algebra generated by } \{W_{d,k} \}_{d,k \geq 0}
\]

\text{modular quadratic relations (write } W_{d,0} = \sum_{d \geq 2} W_{d,k} x^k)

\text{expanded in } N \in \mathbb{N}

\text{for } \text{various } l's

\text{const}_{k,L}...
Thm 2: (N) the bifundamental matter contribution to gauge, which is encoded by the Ext operator \( A_m : K(M^*) \rightarrow K(M) \) "commutes" with the \( W_{\theta_1, \theta_2}(q^{\tau_1}) \) algebra action of \( K(M^*) \).

\[
A_m \circ W_{\theta}(x) = \frac{m^k}{\theta} \left[ \frac{1}{\theta_1} \right] W_{\theta}(x) \left( 1 - \frac{x \theta_1}{\theta} \right) \circ A_m
\]

as an identity of operators \( K(M^*) \rightarrow K(M \times S) \)

where \( \theta = \frac{m^{\tau_1}}{\theta_1} \frac{\det \frac{\partial}{\partial x^i}}{\det \frac{\partial}{\partial y^i}} \).
Def (Feigin-Frenkel-Reshetikhin, Awata-Kubo-Okado-Shiraishi) $W_{a_1, a_2} (g_{P})$ is the $\mathbb{Z}[[a_1, a_2]]$-algebra generated by $\{W_{d, k} \mid d \in \mathbb{Z}, 1 \leq k \leq 10\}$ modulo quadratic relations (write $\mathcal{W}_C(x) = \sum \frac{W_{d,k}}{x^d}$)

\[
W_k(x) W_L(y) \prod_{l = \max(q, l-k)}^{k-1} \mathcal{Y}(\frac{y^q}{x^q}) - W_L(y) W_k(x) \prod_{l = \max(q, q, l-k)}^{L-1} \mathcal{Y}(\frac{x}{y^q}) = \]

\[
= (1-xz_2)(1-z_2) \sum \text{Various l's}
\]

\[
\text{Expand in } |y| < |x|
\]

\[
\text{const}_{G_{1,2}}^{[a_1, a_2]} \mathbb{Z}[[a_1, a_2]]
\]
The function $f_{c_2}$ (for $\mathcal{G}_{c_2}^g$) has

given by

$$E_{d,k} E_{d',k'} = (1-g_{1})(1-g_{2}) \sum \frac{E_{d_{1},k_{1}} \ldots E_{d_{n},k_{n}} \text{ const}}{\text{determined recursively}}$$

$$\text{like } U_{d_{1},d_{2}}^{g_{2}} [g_{2}^{\text{sym}}]$$

and

$$\mathcal{Z}_{u_{1},u_{2}}^{g_{2}}$$
\[ \begin{align*}
\text{allow infinite sum} \\
\text{has } (g^k_1) \text{ given by} \\
\left[ E_{dk}, E_{d'k'} \right] = (1-g_2)(1-g_3) \sum \frac{d_i}{K_i} \leq \frac{d_j}{K_j} \leq \frac{d_m}{K_m} = \frac{d}{k} \\
\text{determined recursively} \\
\end{align*} \]
allow infinite sum

\[
E_{d,k}, E_{d',k'} = \sum_{(d',k') \leq (d,k), (d',k') \leq (d,k)} E_{d',k'} E_{d,k} \text{ const}
\]

Thm 3: (N) define the elements

\[
W_{dk} = \sum E_{d,k} E_{d,k} (-1)^{\gamma}
\]

\[
Z[\hat{\alpha}, \hat{\beta}][\hat{\gamma}]^{\text{sym}}
\]
\[ E_{dk}, \text{ const} \]
\[ \sum_{E_{dk}, \text{ const}} \]
\[ Z \mathbb{Z}_l, l \in \mathbb{Z}_l \]

**Corollary.** \( \mathbb{Z}_l \mathbb{Z}_l \mathbb{Z}_l \mathbb{Z}_l \) has a \( \mathbb{Z}[l^*, l^*] \)-basis given by

\[ \begin{align*}
W_{dk}, W_{dk, n}, d_k, e_k, & = \frac{d_n}{k_n}, \\
\text{thick lines look like } U_{3, 3}, (g) \end{align*} \]

\[ D_k (\text{Buchh"{o}m-Schiffmann}) \]
\[ U_{3, 3}, (g^+) \text{ in the } \mathbb{Z}[l^*^*, l^*^*] \text{ sym} \]

\[ W_k = \sum_{E_{dk}, \text{ const}} E_{dk} \]

**Theorem 3** (N) define the elements

\[ \sum_{d_k \geq 0, k \geq 0} \frac{d_k}{k_n} \]

These \( \omega_k \) satisfy relations \( \ast \) instead of \( W_k \) and give \( \text{an iso} \)

\[ \omega_{3, 3}, (g^+) \]

\[ (W/d) \]

\[ \text{affine, } g \text{-deformed version of } \text{ Beaudon-Kheslev} \]

\[ \text{Yangian } \rightarrow \text{ finite W-algebra} \]
with a little bit of work, you can show that

\[ f = f' \text{ in that } f = f' \text{ and } f \text{ on } \mathbb{C} \]

**Thm 4** (Schellekens Variant, Feigin-Tsymlavce for \( A^2 \))

\( N \text{ for general } s \)

\[ \sum \text{ etc.} \]

\[ \text{gcd}(12, \ldots) \Rightarrow \exists \text{ an } \]

**Thm 1** (N): this is a module

\[ \text{Jacobian group hom.} \]

\[ W_3 (g_0) \to \ldots \]
with a little bit of work, you can show that

\[ W \delta \text{ acts on } K(M) \text{ on the following } \]

\[ \sum_{k_0, k_1, k_2 > 0} \sum_{d_0, d_1 > 0} (R_{\Omega})_{k_0} \left( \Lambda \left( \frac{M}{d_0} \right) \cdot \frac{1}{d_1} \right) P^+ \delta_{d_0} \delta_{d_1} = 0 \]

\[ \text{D}(\delta) \text{ for } M(M) \]

\[ \left\{ \left( F_s, F_{s_1}, F_{s_2}, F_{s_2'} \right) : \right\} \]

\[ P \left[ F_s \right] P_1 \left[ F_{s_1} \right] \quad \left[ F_{s_2} \right] \quad \left[ F_{s_2'} \right] \]

\[ F \in M \quad x \in S \quad M = F \]

\[ \text{Thm 1} (N): \text{ the group } \]

\[ \text{Thm 4} \quad \left( \text{Scheffers-Vermaut, Feigen-Tsymbalow for } A^2 \right) \]

\[ (N \text{ for general } s) \]

\[ \text{sums over } \text{gcd} (1, r) \quad \Rightarrow \exists \alpha \in \mathbb{C} \]

\[ U_{2,2} (\alpha) \sim K(M) \]

To deduce Thm 1, you must show that

\[ U_{2,2} (\alpha) \text{ acts correctly on } K(M) \]

(follow from \( r \) bounded below)

\[ \text{Work act as } 0 \text{ in } K(M) \]

\[ x_1 = \Gamma \left( \frac{F}{F'} \right) \quad \text{Univ. sheaf} \]

\[ x_2 = \Gamma \left( \frac{F'}{F} \right) \quad \text{parameterize } \frac{F}{F'} \]

\[ W_{3,2} (\partial \delta) \sim \Xi \]

\[ \text{Abelian group hom} \]
Fix smooth $D \subset S$

$\pi_1, \ldots, \pi_n \in \mathbb{N}$

$\mathcal{M}_{\pi_1, \ldots, \pi_n} = \{ \text{parabolic sheaves } \mathcal{F}_0 \}$

Expectation: $K(\mathcal{M}_{\pi_1, \ldots, \pi_n})$ should be a module for $N_{\text{sym}}$ ($\text{Frenkel-Tsymbaliuk for } \mathfrak{g}^2$)

$U_{\xi_1, \xi_2}(\mathfrak{sl}_n)_{\pi_1, \ldots, \pi_n}$ \textit{shift}

To Be Defined
\[ \sum P_{k} = 0 \]
\[ P_{k} = A_{L} - B_{k}, \quad A_{L} = B_{k} \]
\[ A_{L} = 0 \]

\[ \gcd(r_{L}, z_{i}(x^{j}) + i) = 1 \]

\[ \Rightarrow \exists \text{a universal sheaf } U \rightarrow \mathbb{F} \]
\[ M \times S_{L} \rightarrow \mathbb{F} \]

\[ \text{Thm. 1}(N) \text{ the } K\text{-theory group } K(M) \]
\[ \text{is a module for the algebra } W_{a,s}(g,h) \]

\[ [\phi_{a}, \phi_{b}] = \Delta_{a} (\phi_{a}) \]

\[ \text{where the LHS is the difference of the following } \]

\[ \text{homomorphism} \]
\[ \text{algebra} \]

\[ \text{abelian group homomorphism} \]

\[ W_{a,s}(g,h) \rightarrow \text{Hom}(K(M), K(M)) \]