Abstract: I'll discuss recent work on finding time-dependent solutions of a black hole interacting with a scalar field. I'll discuss two distinct cases where the back-reaction of the scalar can be found. First, in the case that the scalar is slowly rolling (such as in inflation) the scalar field can be found in terms of super-advanced time coordinate, regular on both horizons. The scalar back-reacts on the geometry, with the black hole accreting and growing more or less as expected. The second case I'll describe briefly is scalar hair in modified gravity - giving a scalar-dressed rotating black hole that is finite on both horizons.
Slow Roll with a Black Hole

And other Scalar Tails …

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Black holes are fascinating test grounds for GR.

They represent regions of very strong gravity, and constrain any candidate to challenge GR.

Dark Energy in cosmology is a challenge for particle physics – and unless \( \Lambda \), a challenge to mix with a black hole.

Dark Energy provides a concrete way of varying \( \Lambda \), so how does this affect the black hole, and how far can we get analytically?
How does a Black Hole respond to a cosmological scalar?

- Story 1: some de Sitter stuff
- The rolling scalar – slow roll?
- The full geometry
- Story 2: some hair theorems
- A finite scalar hair
Black Hole Theorems

Black holes in 4D obey a set of theorems: We know they are spherical, that they obey laws of thermodynamics, and that they are characterized by relatively few “numbers” – or “Black Holes Have No Hair”.

i.e. electrovac solutions are uniquely specified by 3 parameters: M, Q, and J
No Scalar Hair

The essence of “no hair” is that the scalar field must have finite energy, and fall off at infinity. Integrating the equation of motion gives a simple relation, only satisfied for \( \varphi = \varphi' = 0 \)

\[
\int_{2GM}^{\infty} \left\{ r^2 V_{r,\phi}^2 + r(r - 2GM)\phi' \frac{d}{dr} (V_{r,\phi}) \right\} = [V_{r,\phi}(r - 2GM)\phi']_{2GM} = 0.
\]
NO-NO HAIR!

But this is highly idealised:

- Static
- Vacuum
- Convex potential
- $\Lambda$ not negative
- Einstein (or similar) gravity

And no hair came to mean the much stronger “no field profiles”.

Moral: Unlearn what you have learned!
VARYING SCALAR

One of the clearest ways of evading the no-hair axioms is to have time-dependence.

A rolling scalar corresponds to many ideas of dark energy.

The black hole should absorb scalar energy and grow, while the cosmological constant drops.
Black Hole in de Sitter

A black hole with cosmological constant is given by the Schwarzschild de Sitter solution:

$$ds^2 = f dt^2 - \frac{dr^2}{f} - r^2 d\Omega_{II}^2$$

Where

$$f = 1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2$$

(This is a static solution)
**Accreting Black Hole?**

The SdS black hole has two horizons:

\[ r_b \sim 2M \left( 1 + \mathcal{O} \left( M^2 \Lambda \right) \right) \]

\[ r_c \sim \sqrt{\frac{3}{\Lambda}} - M \]

As the scalar rolls, the black hole accretes, \( M \) increases, \( \Lambda \) decreases. It's obvious that \( r_b \) increases, but \( r_c \)?

Aim: Find analytic description of time dependent black hole
Controlling Lambda

In slow roll inflation, lambda varies gradually, while our universe is quasi-de Sitter.

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{1}{3M_p^2} \left[ \frac{\dot{\phi}^2}{2} + W(\phi) \right]
\]

\[3H\dot{\phi} = -\frac{\partial W}{\partial \phi}\]

Small slow-roll parameters ensure that inflation in maintained:

\[\varepsilon = \frac{M_p^2}{2} \frac{W'}{W^2}, \quad \eta = M_p^2 \frac{W''}{W}\]
Add Black Hole:

But a key difference is that SDS geometry is not explicitly time dependent – so what does “slow roll” mean? SDS has a cosmological horizon, but is a static slicing of de Sitter

Without the black hole, have a “static patch” de Sitter potential:

\[ f = 1 - H^2 r^2 \]

– not the familiar (flat) cosmological coordinates

\[ d\tau^2 - e^{2H\tau} dx^2 \]
The transformation to cosmological time is nontrivial:

$$\tau_{cos} = t_s + \frac{1}{2H} \log(1 - H^2 r_s^2)$$

$$\rho_{cos} = \frac{r_s e^{-H t_s}}{\sqrt{1 - H^2 r_s^2}}$$

And with black hole, even this relative simplicity is lost.
**Black Hole-Slow Roll Approximation**

With a black hole, intuition is that the geometry is approximately SDS, the scalar still slow-rolls, but that this produces a sub-leading effect on the background black hole geometry. The spacetime slides from one Lambda to a lower one, and the black hole accretes a little mass.

\[
\phi = \phi_0 + \delta \phi_{SR}
\]

\[
g_{\mu\nu} = g_{0\mu\nu} + \delta g_{SR\mu\nu}
\]

\[
\mathcal{O}(\delta \phi_{SR})^2
\]

SDS
Scalar Field Eqn

Idea is to turn e.o.m for $\phi$

$$\frac{\phi_{tt}}{f} - \frac{1}{r^2} \left( r^2 f \phi_r \right)_r = - \frac{\partial W}{\partial \phi}$$

into something like a slow roll equation by assuming $\phi = \phi(T)$, where

$$T = t + \xi(r)$$

$T$ is constructed so that $\phi$ is regular at both horizons, with only in(out) going modes at black hole (cosmological) horizon.
Substitute in: \( \phi = \phi(t + \xi(r)) \)

\[
\frac{1}{r^2} \left( r^2 f \xi' \right)' \phi - \frac{\phi}{f} \left( 1 - t^2 \xi''^2 \right) = \frac{\partial W}{\partial \phi}
\]

Dropping second term, and remember \( \phi = \phi(t) \), we must have

\[
\frac{1}{r^2} \left( r^2 f \xi' \right)' = -3\gamma
\]

\( \gamma \) constant, and hence

\[
\xi' = \frac{1}{f} \left( -\gamma r + \frac{\beta}{r^2} \right)
\]
Find $\gamma$ and $\beta$ by regularity: $\phi(\tau)$ must be ingoing on event horizon and outgoing on cosmological horizon. Final answer gives $T$:

$$T = t - \frac{1}{2\kappa_c} \log \left| \frac{r - r_c}{r_c} \right| + \frac{1}{2\kappa_b} \log \left| \frac{r - r_b}{r_b} \right|$$

$$+ \frac{r_b r_c}{r_c - r_b} \log \frac{r}{r_0} + \left( \frac{r_c}{4\kappa_b r_b} - \frac{r_b}{4\kappa_c r_c} \right) \log \left| \frac{r - r_n}{r_n} \right|$$

For those familiar with Kruskals, $T$ looks like $V$ at the black hole horizon ($r_b$) and $U$ at the cosmological horizon ($r_c$).
Substitute in:  \[ \phi = \phi(t + \xi(r)) \]

\[ \frac{1}{r^2} \left( r^2 f \xi' \right)' \phi - \frac{\ddot{\phi}}{f} \left( 1 - f^2 \xi'^2 \right) = \frac{\partial W}{\partial \phi} \]

Dropping second term, and remember \( \phi = \phi(T) \), we must have

\[ \frac{1}{r^2} \left( r^2 f \xi' \right)' = -3\gamma \]

\( \gamma \) constant, and hence

\[ \xi' = \frac{1}{f} \left( -\gamma r + \frac{\beta}{r^2} \right) \]
T looks like an Eddington-Finkelstein coord on each horizon, at $r_h$ a fn of $v$, and at $r_c$ a fn of $U$. 
The T Coordinate

The T coordinate is timelike at each horizon, and could be a cosmological time asymptotically.
BACK-REACTION

Given this Eddington-Finkelstein behaviour, look at SDS metric in (T,r) coords:

\[ ds^2 = f(r, T) \, dT^2 - 2h(r, T) \, dT \, dr - \frac{dr^2}{f} \left( 1 - h^2 \right) - r^2 \, d\Omega^2 \]

The energy momentum of the scalar has 2 independent cpts:

\[ T_{TT} = \left( W(\phi) + \frac{1 + h^2}{2f} \phi^2 \right) |g_{TT}|, \]

\[ T_{ab} = \left( -W(\phi) + \frac{1 - h^2}{2f} \phi^2 \right) g_{ab} \]
Which we relate to the Einstein tensor:

\[
G_{TT} = \left[ \frac{1}{r^2} (1 - f - rf') - \frac{hf}{rf} \right] |g_{TT}| \\
G_{rr} = \left[ -\frac{1}{r^2} (1 - f - rf') + \frac{hf}{rf} + \frac{2h}{r(1-h^2)} \right] g_{rr} \\
G_{rT} = \left[ -\frac{1}{r^2} (1 - f - rf') - \frac{(1-h^2)f}{rhf} \right] g_{rT} \\
G_{\theta\theta} = \left[ \frac{f''}{2} + \frac{f'}{r} - \frac{hf'}{2f} + \frac{hf'}{2f} + \frac{h}{r} + \dot{h} + \frac{1}{2} \left( \frac{(h^2-1)}{f} \right) \right] g_{\theta\theta} = \frac{G_{\phi\phi}}{\sin^2 \theta}
\]
**Slow Roll with a Black Hole**

Need to have control of the slow-roll approximation to identify the key dependences in these equations:

- **Scalar:**
  \[
  \frac{1 - h^2}{f} \ddot{\phi} - \frac{(r^2 h)'}{r^2} \dot{\phi} = -W'(\phi)
  \]

- **Einstein:**
  \[
  \left[ \frac{1}{r^2} (1 - f - rf') + \frac{(1 - h^2)\dot{f}}{rh f} \right] = \frac{1}{M_p^2} \left( W(\phi) - \frac{1 - h^2}{2f} \dot{\phi}^2 \right)
  \]
**Slow Roll with a Black Hole**

Taking the same general slow roll requirements, these now depend on position:

\[
\frac{1 - h^2}{f} \dot{\phi}^2 \ll W, \quad \frac{1 - h^2}{f} \ddot{\phi} \ll \frac{1}{r^2} \left| (r^2 h)' \right| \dot{\phi}
\]

As usual in slow-roll, take background values of metric functions, and can bound these r-dependent background functions to the usual slow roll type parameters

\[
\varepsilon = \frac{M_p^2}{2} \frac{W''}{W^2} \ll 1, \quad \eta = M_p^2 \frac{W'''}{W} \ll 1
\]
Saying that geometry is dominated by potential energy gives a spatially dependent epsilon parameter:

\[
\frac{1 - h^2}{f} \phi^2 \ll W
\]

\[
M_p^2 \frac{W' f}{W^2} \frac{(1 - h^2)}{f} \frac{H^2}{3 \gamma^2} \ll 1
\]

\[
\varepsilon = M_p^2 \frac{W' f}{W^2} \ll 1
\]
**Slow roll conditions:**

Geometry should be dominated by the potential energy:

\[
\frac{1 - h^2}{f} \phi^2 \ll W \Rightarrow \frac{1 - (rf)'}{r^2} = 3H^2 = \frac{W}{M_p^2}
\]

And the scalar motion by friction:

\[
\frac{1 - h^2}{f} \ddot{\phi} \ll 3\gamma \dot{\phi} \Rightarrow -3\gamma \dot{\phi} = W'(\phi)
\]

Take each in turn.
Saying that geometry is dominated by potential energy gives a spatially dependent epsilon parameter:

\[
\frac{1 - h^2}{f} \phi^2 \ll W
\]

\[
M_p^2 \frac{W'^2}{W^2} \frac{(1 - h^2)}{f} \frac{H^2}{3\gamma^2} \ll 1
\]

\[
\varepsilon = M_p^2 \frac{W'^2}{W^2} \ll 1
\]
SCALAR

\[
\frac{1 - h^2}{f} \phi \ll 3 \gamma \phi
\]

Use scalar equation to find

\[
M_p^2 \frac{W''}{W} \left(1 - h^2\right) \frac{H^2}{3 \gamma^2} \ll 1
\]

..the same spatial dependence as before, giving a bound

\[
\eta = M_p^2 \frac{W''}{W} \ll 1
\]

The eta parameter.
This allows us to solve the Einstein equations to leading order in the slow-roll parameters.

\[ \mathcal{TT} + \mathcal{Tr}: \quad \dot{f} = -r h \frac{\dot{\phi}^2}{M_p^2} \]

implies

\[ f(r, T) = f_0(r) + \delta f(r, T) - r h_0 \int \frac{\dot{\phi}^2}{M_p^2} \]

Where \( \delta f \) is order \( \epsilon \eta \), but slowly varying. We can then integrate the \( \phi \) kinetic energy:

\[ \int_{T_0}^{T} \dot{\phi}^2 dT' = -\frac{1}{3\gamma} \left\{ W(\phi(T)) - W(\phi(T_0)) \right\} = -\frac{\delta W}{3\gamma} \]
And end up with a familiar expression:

\[
f(r, T) = 1 - \frac{\Lambda(T)}{3M_p^2} r^2 - \frac{2GM(T)}{r} + \tilde{\delta f}
\]

With

\[
\Lambda(T) = W[\phi(T)] \quad M(T) = M_0 - 4\pi \beta \frac{\delta W}{3\gamma}
\]

A somewhat finicky argument shows that \(\delta f\) is transient, and of sub-leading order (\(\epsilon \eta\)) to the changes in \(\Lambda\) and \(M\).
We can solve for \( h(r, T) \) as well, and we find that to leading order, for a slow roll scalar the black hole geometry takes its “Schwarzschild” form in the scalar \( T \)-coordinate (regular on both horizons) but with \( \Lambda \) and \( M \) now time varying. We get a remarkably simple expression for the time-dependence of the horizon areas:

\[
\dot{A}_h = \frac{A_h}{|\kappa_h|} \frac{\dot{\phi}^2}{M_p^2}
\]
**Φ Equation**

Back to the scalar equation, a standard slow-roll type

\[ 3\gamma \dot{\phi}(T) = -\frac{\partial W}{\partial \phi} \]

but with friction parameter modified from H:

\[ \gamma = \frac{r_c^2 + r_h^2}{r_c^3 - r_h^3} = \frac{A_{TOT}}{3V} \]

Physical effect of black hole is to add friction to roll, or to slow down the scalar.
Or is it?? The T coordinate is useful for finding backreaction, but the black hole should have only a small impact asymptotically in dS – need to explore cosmological coordinates.

\[ dT = dt + \xi' dr \]

\[ dR = dt + \frac{dr}{f^2 \xi'} \]

The R coordinate is orthogonal to T, and useful for extending beyond the cosmological horizon.
Then –

\[
\frac{d\phi}{dT} = \frac{H}{\gamma} \frac{d\phi}{d\tau}
\]

Hence at large \( r \), the standard friction dominated slow roll equation is recovered.

\[
-3H \frac{d\phi}{d\tau} = W'(\phi)
\]
New metric:

\[ ds^2 = \frac{f}{1 - h^2} \left( dT^2 - h^2 dR^2 \right) - r^2 d\Omega^2 \]

Outside \( r_c \) both \( f \) and \( (1-h^2) \) change sign, so \( T \) remains timelike and \( R \) spacelike.
$r$ is a function of $T-R$, and at large $r$

$$T - R \approx \frac{\gamma}{H^2} \log r$$

$$ds^2 \approx \frac{H^2}{\gamma^2} dT^2 - H^2 r^2 dR^2 - r^2 d\Omega^2$$

Suggesting-

$$\tau_{\cos} = \frac{HT}{\gamma}$$

$$r^2 \sim e^{2H^2(T-R)/\gamma} = e^{2H\tau} e^{-2H^2 R/\gamma} e^{\rho^2}$$
r is a function of T-R, and at large r

\[ T - R \approx \frac{\gamma}{H^2} \log r \]

\[ ds^2 \approx \frac{H^2}{\gamma^2} dT^2 - H^2 r^2 dR^2 - r^2 d\Omega^2 \]

Suggesting -

\[ \tau_{\cos} = \frac{HT}{\gamma} \]

\[ r^2 \sim e^{2H^2(T-R)/\gamma} = e^{2H\tau} e^{-2H^2 R/\gamma} \]

\[ \rho^2 \]
Then –

\[ \frac{d\phi}{dT} = \frac{H}{\gamma} \frac{d\phi}{d\tau} \]

Hence at large \( r \), the standard friction dominated slow roll equation is recovered.

\[ -3H \frac{d\phi}{d\tau} = W'(\phi) \]
THERMODYNAMICS AND LAMBDA

Now vary the Schwarzschild potential with lambda:

\[
\delta f(\mathfrak{r}_+ + \delta \mathfrak{r}_+) = -\frac{\delta \Lambda}{3} \mathfrak{r}_+^2 - \frac{2\delta M}{\mathfrak{r}_+} + f'(\mathfrak{r}_+) \delta \mathfrak{r}_+ = 0
\]

Rearrange to

\[
\delta M = T \delta S - \frac{4\pi \mathfrak{r}_+^3}{3} \delta \left( \frac{\Lambda}{8\pi} \right)
\]
Dynamical Temperature

Hayward et al suggested a dynamical temperature

\[ \kappa_{dyn} = \frac{1}{2} \star d \star dr \]

Which we can calculate for our solution

\[ \kappa_{dyn}(T) = (f' + \dot{h}) = \kappa_b(T) + \mathcal{O}(\varepsilon \eta) \]

i.e. the instantaneous temperature of the time-dependent SdS potential.
Tail 2: Modified Gravity

Modified gravity arises from the challenge of explaining the late time acceleration of our Universe, which is gently accelerating with an effective cosmological constant of $10^{-30} \text{ g cm}^{-3}$. The challenge is to modify gravity in IR without spoiling UV (or mid UV) success.
DHOST Modified Gravity

Work with Degenerate Higher Order Scalar-Tensor gravity. Further – impose $c_T=1$ (gravity propagates at $c$), and no Ostrogradski instabilities:

$$\mathcal{L} = K(X) + G(X) R + A_3 \phi^\mu \phi_{\mu\nu} \phi_{\nu} \Box \phi$$

$$+ A_4 \phi^\mu \phi_{\mu\rho} \phi_{\rho\nu} \phi_{\nu} + A_5 (\phi^\mu \phi_{\mu\nu} \phi_{\nu})^2 , \quad c_T=1$$

with:

$$A_4 = -A_3 + \frac{1}{8G} (4G X + A_3 X) (12G X + A_3 X),$$

$$A_5 = \frac{A_3}{2G} (4G X + A_3 X), \quad \text{OSTRO}$$
Self-Tuning

Look for an Einstein metric solution:

Where the scalar has a constant kinetic energy, or $X=X_0$:

$$\left[ A_3(X_0) (\mathcal{E}_3 - \Lambda X_0) - 2 \left( K_X + 4 \Lambda G_X \right) \right] \phi_{\mu} \phi_{\nu} + (K + 2 \Lambda G) g_{\mu \nu} = 0$$

$$A_3(X_0) (\mathcal{E}_4 + 2 R_{\mu \nu \rho \sigma} \phi^{\rho \sigma} \phi^\mu \phi^\nu - 3 \Lambda X_0 \Box \phi) - 2 \left( K_X + 4 \Lambda G_X \right) \Box \phi = 0$$

A bit daunting, but remember evaluated at $X_0$, so get a tuned cosmological constant:

$$\Lambda = -K/(2G) \bigg|_{X_0}$$

$$\mathcal{E}_3 \equiv (\Box \phi)^2 - (\phi_{\mu \nu})^2 ,$$

$$\mathcal{E}_4 \equiv (\Box \phi)^3 - 3 \Box \phi (\phi_{\mu \nu})^2 + 2 (\phi_{\mu \nu})^3 .$$
DHOST MODIFIED GRAVITY

Work with Degenerate Higher Order Scalar-Tensor gravity. Further – impose $c_T=1$ (gravity propagates at c), and no Ostrogradski instabilities:

$$\mathcal{L} = K(X) + G(X)R + A_3 \phi^\mu \phi_{\mu \nu} \phi^\nu \Box \phi$$

$$+ A_4 \phi^\mu \phi_{\mu \rho} \phi^\rho \phi_{\nu} + A_5 (\phi^\mu \phi_{\mu \nu} \phi^\nu)^2 ,$$

$c_T=1$

with:

$$A_4 = -A_3 + \frac{1}{8G} (4G_X + A_3 X)(12G_X + A_3 X),$$

$$A_5 = \frac{A_3}{2G} (4G_X + A_3 X),$$
Self-Tuning

Look for an Einstein metric solution:

Where the scalar has a constant kinetic energy, or $X=X_0$:

\[
A_3(X_0) \left( \mathcal{E}_3 - \Lambda X_0 \right) - 2 \left( K_X + 4\Lambda G_X \right) \bigg|_{X_0} \phi_{\mu\nu} + (K + 2\Lambda G) \bigg|_{X_0} g_{\mu\nu} = 0 \\
A_3(X_0) \left( \mathcal{E}_4 + 2R_{\mu\nu\rho\sigma} \phi^\rho \phi^\sigma \phi_{\mu} \phi_{\nu} - 3\Lambda X_0 \Box \phi \right) - 2 \left( K_X + 4\Lambda G_X \right) \bigg|_{X_0} \Box \phi = 0
\]

A bit daunting, but remember evaluated at $X_0$, so get a tuned cosmological constant:

\[
\Lambda = -\frac{K}{(2G)} \bigg|_{X_0} = \mathcal{E}_3 \equiv (\Box \phi)^2 - (\phi_{\mu\nu})^2 , \\
\mathcal{E}_4 \equiv (\Box \phi)^3 - 3\Box \phi (\phi_{\mu\nu})^2 + 2 (\phi_{\mu\nu})^3 .
\]
**Self-Tuned Solutions**

\[
\left[ A_3(X_0) \left( \mathcal{E}_3 - \Lambda X_0 \right) - 2 \left( K_X + 4 \Lambda G_X \right) \right]_{x_0} \phi_\mu \phi_\nu + (K + 2 \Lambda G)_{x_0} g_{\mu\nu} = 0 \\
A_3(X_0) \left( \mathcal{E}_4 + 2 R_{\mu\nu\rho\sigma} \phi^{\mu\sigma} \phi^{\alpha \beta} \phi^{\rho} - 3 \Lambda X_0 \Box \phi \right) - 2 \left( K_X + 4 \Lambda G_X \right)_{x_0} \Box \phi = 0
\]

To get a self-tuned solution, look for \( X_0 \) a zero of \( A_3 \), and also with

\[
\left( K_X + 4 \Lambda G_X \right)_{x_0} = 0
\]

This restricts us within the space of possible theories, to those where \( G \) satisfies the above relation at \( X_0 \).
Black Hole Hair?

These self-tuned cosmological solutions seem very special – can a black hole support this scalar?

There are known black hole solutions with shift-symmetric scalars. The scalar rolls on the horizon, and throughout the geometry, but does not backreact, since the Einstein equations are explicitly assumed.

The scalar also diverges at infinity, or the cosmological horizon, but because of shift-symmetry this is not a singularity. (However, challenges continuation beyond the horizon.)
Finding Stealth Hair

The common factor in known stealth solutions is that $X_0$ is a constant, and only spherically symmetric solutions have been found. We keep the former, but not the latter.

Note that a constant length vector is reminiscent of the tangent vector for an affinely parametrised geodesic, which for Kerr can be solved via a Hamilton-Jacobi potential

$$\frac{\partial S}{\partial x^\mu} = p_\mu = g_{\mu\nu} \frac{dx^\nu}{d\lambda}$$
Kerr Geodesics

Want a rotating black hole with stealth hair:

\[ ds^2 = -\frac{\Delta_r}{\Xi^2 \rho^2} \left[ dt - a \sin^2 \theta d\varphi \right]^2 + \rho^2 \left( \frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta} \right) + \frac{\Delta_\theta \sin^2 \theta}{\Xi^2 \rho^2} \left[ a \, dt - (r^2 + a^2) \, d\varphi \right]^2 \]

\[ \Delta_r = \left( 1 - \frac{r^2}{\ell^2} \right) (r^2 + a^2) - 2Mr, \quad \Xi = 1 + \frac{a^2}{\ell^2}, \quad \Delta_\theta = 1 + \frac{a^2}{\ell^2} \cos^2 \theta, \quad \rho^2 = r^2 + a^2 \cos^2 \theta \]

Idea of Hamilton-Jacobi is to identify constants of motion, usually associated with Killing vectors. Here \( t \) and \( \phi \) give two constants. A third is the “Carter constant”

\[ Q = \Delta_\theta p_\theta^2 + m^2 a^2 \cos^2 \theta - \Xi^2 \left[ (a \, E - L_z)^2 - \frac{\sin^2 \theta}{\Delta_\theta} \left( a \, E - \frac{L_z}{\sin^2 \theta} \right)^2 \right] \]
Finding Stealth Scalar

Look for a scalar with the Kerr Hamilton Jacobi potential:

\[ S = -Et + L_z \varphi \pm \int \frac{\sqrt{R}}{\Delta_r} \, dr \pm \int \frac{\sqrt{\Theta}}{\Delta_\theta} \, d\theta \]

Given in terms of the Carter and other constants:

\[ R = \Xi^2 \left[ E \left( r^2 + a^2 \right) - a L_z \right]^2 - \Delta_r \left[ Q + \Xi^2 \left( a E - L_z \right)^2 + m^2 r^2 \right] \]

\[ \Theta = -\Xi^2 \sin^2 \theta \left( a E - \frac{L_z}{\sin^2 \theta} \right)^2 + \Delta_\theta \left[ Q + \Xi^2 \left( a E - L_z \right)^2 - m^2 a^2 \cos^2 \theta \right] \]
The scalar has to have a regular gradient, checking the axes gives $L_z=0$ and

$$Q = m^2 a^2 - \Xi^2 a^2 E^2 = m^2 a^2 (1 - \eta^2)$$

This gives a 2 parameter set of solutions, an overall scale $m$, plus a ‘relative energy’ $\eta$ that can range from $\eta_c$ to 1, in which range the potentials are positive-

$$\Theta = a^2 m^2 \sin^2 \theta (\Delta_\theta - \eta^2),$$

$$R = m^2 (r^2 + a^2) (\eta^2 (r^2 + a^2) - \Delta_r).$$

Note that the root of the potentials is what determines the scalar field, leading in general to 4 possible choices for the solution, but for $\eta_c$ or 1, a new possibility arises.
Focus on the $\Theta$-potential, contributes to $\phi$

$$\eta \log \left[ \frac{\sqrt{1 - \eta^2 + \frac{a^2}{\ell^2} \cos^2 \theta} + \frac{a}{\ell} \cos \theta}{\sqrt{(1 - \eta^2)\Delta \theta}} \right] - \log \left[ \frac{\sqrt{1 - \eta^2 + \frac{a^2}{\ell^2} \cos^2 \theta} + \frac{a}{\ell} \cos \theta}{\sqrt{1 - \eta^2}} \right]$$

This is asymmetric in general, but as $\eta$ tends to 1,

$$\Theta \to \frac{a^4}{\ell^2} \sin^2 \theta \cos^2 \theta$$

The fully differentiable root is $\cos$, giving

$$\phi \big|_\phi = \frac{m \ell}{2} \log \Delta \theta$$
Contrasting the symmetric and asymmetric angular behaviour
Similarly, as η tends to η_c the R-function has a zero, and the root changes sign:

\[ R = 2Mr(r^2 + a^2) + \frac{r^2}{\ell^2}(r^2 + a^2)^2 - (1 - \eta_c^2)(r^2 + a^2)^2 \]

\[ \propto \frac{1}{2}(r - r_0)^2 R_0'' \]

This now has an interesting consequence. Near each horizon, φ asymptotes either the retarded or advanced null coord:

\[ \phi \bigg|_r \sim \pm \eta r^* \]

If the root cannot change sign, then φ diverges on one or the other horizon.
But for the critical $\eta_c$ the field remains bounded throughout the spacetime.
To compare to the slow roll solution, let $a=0$, so that the black hole is SdS:

$$\eta_c^2 = 3 \frac{r_0^2}{\ell^2}, \quad r_0 = (m\ell^2)^{1/3}$$

$$\phi = -m \left[ \eta_c t + \int \frac{\ell(r - r_0)\sqrt{r(r + 2r_0)}}{(\ell^2 r - r^3 - 2M\ell^2)} \, dr \right]$$

$\phi_{MG} - \phi_{SR}$
Summary

- Have generalised slow-roll description to non-homogeneous black hole background. The black hole geometry is to a very good approximation quasi-Schwarzschild de Sitter.

- Explored dynamical temperature and First Law – holds dynamically during the flow.

- Found first example of stealth hair for rotating black hole and finite at both horizons.

- Both examples are time-dependent, both finite at each horizon, but different physics inbetween.