Abstract: I will overview recent results on the defect CFT corresponding to Wilson loop operators in N=4 SYM theory. In particular, I will review the calculation of defect correlators at strong coupling using the AdS2 string worldsheet, and I will present exact results for correlation functions in a subsector of the defect CFT using localization. I will also discuss a defect RG flow from the BPS to the ordinary Wilson loop, which can be used to provide a test of the "defect F-theorem" for one-dimensional defects.
Wilson Loops and Defect CFT

Simone Giombi

Princeton University

Boundaries and Defects in Quantum Field Theory
Perimeter Institute, Aug. 8, 2019

Based mainly on SG, Roiban, Tseytlin arXiv: 1706.00756
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Outline

• Half-BPS Wilson loop in $N=4$ SYM as conformal defect

• Defect correlators at strong coupling from AdS$_2$ string worldsheet

• Exact results from localization

• Non-supersymmetric Wilson loop, defect RG flow and a test of “defect F-theorem”
Wilson loops in $N=4$ SYM

• In $N=4$ SYM, it is natural to study Wilson loop operators that include couplings to the six adjoint scalars $\Phi^I$ (“Maldacena-Wilson” loop)

$$ W = \text{tr} P e^{\int dt (i \dot{x}^\mu A_\mu + |\dot{x}| \theta^I \Phi^I)} $$

where $x^\mu(t)$ is a loop in spacetime and $\theta^I(t)$ a unit 6-vector.

• Special choices of $(x^\mu, \theta^I)$ lead to families of Wilson loop operators preserving various fractions of the superconformal symmetry

(Zarembo ‘02; Drukker, SG, Ricci, Trancanelli ‘07)
Half-BPS Wilson loop

- The most supersymmetric case is the 1/2-BPS Wilson loop:
  - $x^\mu(t)$: a circle, or infinite straight line in $R^4$ (related by conf. transf.)
  - $\theta^I$: a constant unit 6-vector

- E.g. take the line $x^0=t$, and $\theta^I=\delta^I_6$

\[
W = \text{tr} Pe^{\int dt (iA_t + \Phi^6)}
\]

- This preserves 8 Q’s and 8 S’s (superconformal charges): 1/2-BPS.
  (Similarly for a circle, but it preserves 16 lin. combinations of Q and S)
Half-BPS Wilson line as conformal defect

• Let us recall the symmetries preserved by the 1/2-BPS Wilson line. The bosonic symmetries are
  ▪ $\text{SO}(3)$: rotations around the line ($i=1,2,3$)
  ▪ $\text{SO}(5)$: $R$-symmetry rotations of the five scalars $\Phi^a$, $a=1,\ldots,5$ that do not couple to the Wilson loop operator
  ▪ $\text{SL}(2,R)$: dilatation, translations and special conformal transformation on the line.
  
  **1d conformal symmetry**
  
  (Kapustin ‘05; Drukker, Kawamoto ‘06)

• Together with the 16 supercharges, these combine into the 1d superconformal group $\text{OSp}(4^*|4) \supset \text{SL}(2,R)\times\text{SO}(3)\times\text{SO}(5)$

• Since it preserves a 1d conformal subgroup of the 4d conformal symmetry, the 1/2-BPS Wilson loop can be viewed as a conformal defect of the 4d theory
Correlators on the defect

- As usual in defect CFT, we can study correlators of operators on the defect, of bulk operators, or mixed bulk/defect ones. We will mainly focus on correlators of operators on the defect.
- Given some local operators $O_i(t)$ in the *adjoint* of the gauge group, consider

$$
\langle \langle O_1(t_1)O_2(t_2)\cdots O_n(t_n) \rangle \rangle \equiv \frac{\langle \text{tr} \ P \left[ O_1(t_1) e^{\int dt (iA_t + \Phi^6)} O_2(t_2) e^{\int dt (iA_t + \Phi^6)} \cdots O_n(t_n) e^{\int dt (iA_t + \Phi^6)} \right] \rangle}{\langle \langle W \rangle \rangle}
$$

$$
\equiv \frac{\text{tr} \ P \left[ O_1(t_1)O_2(t_2)\cdots O_n(t_n) e^{\int dt (iA_t + \Phi^6)} \right]}{\langle \langle W \rangle \rangle}
$$

- Such defect correlators arise naturally when we consider small deformations of the Wilson loop (Polyakov, Rychkov ‘00; Dukker, Kawamoto ‘06)
- They encode information on expectation value of Wilson loops of more general shapes and scalar coupling
Correlators on the defect

• These defect correlators are constrained by the SL(2,R) 1d conformal symmetry as in general CFT_d

• Defect operator can be organized in primaries and descendants. Primaries are labelled by their scaling dimension $\Delta$ and SO(3)xSO(5) representation

• Correlation functions constrained by conf. symmetry:

$$\langle O_{\Delta}(t_1)O_{\Delta}(t_2) \rangle = \frac{1}{t_{12}^{2\Delta}}$$

$$\langle O_1(t_1)O_2(t_2)O_3(t_3) \rangle = \frac{c_{123}}{t_{12}^{\Delta_1+\Delta_2-\Delta_3}t_{23}^{\Delta_2+\Delta_3-\Delta_1}t_{31}^{\Delta_3+\Delta_1-\Delta_2}}$$

$$\langle O_{\Delta}(t_1)O_{\Delta}(t_2)O_{\Delta}(t_3)O_{\Delta}(t_4) \rangle = \frac{1}{(t_{12}t_{34})^{2\Delta}} G(\chi)$$

$$\chi = \frac{t_{12}t_{34}}{t_{13}t_{24}}$$
Defect fermion/boson description

- There is no local 1d lagrangian describing these CFT$_1$ correlators, but there is a description in terms of N 1d fermions $\chi^i$ (or bosons) coupled to the bulk $N=4$ SYM fields (Gomis, Passerini ’06)

$$\text{tr}_{A_k} P e^{\int dt (iA + \phi)} = \int D\chi D\bar{\chi} D\alpha e^{-S_x} \quad \text{A}_k : \text{rank}-k \text{ antisymmetric repr.}$$

$$S_x = \int dt \left[ \bar{\chi} (\partial_t - ia) \chi + i \bar{\chi} (iA + \phi) \chi \right] + ik \int dt a$$

- Similarly, 1d bosons give rank-$k$ symmetric representation

- Defect local operators are the gauge invariant objects like $\bar{\chi}_i O^i_j \chi^j$ inserted on the line. General defect correlators are obtained by inserting such operators and computing path integral over 4d and 1d fields with the defect-CFT action

$$S_{\text{defect-CFT}} = S_{N=4} + S_x$$
Correlators on the defect

- As usual in defect CFT, we can study correlators of operators on the defect, of bulk operators, or mixed bulk/defect ones. We will mainly focus on correlators of operators on the defect.
- Given some local operators $O_i(t)$ in the adjoint of the gauge group, consider

$$\langle \langle O_1(t_1)O_2(t_2)\cdots O_n(t_n) \rangle \rangle \equiv \frac{\langle trP[O_1(t_1) e^{\int dt(iA_t+\Phi^6)} O_2(t_2) e^{\int dt(iA_t+\Phi^6)} \cdots O_n(t_n) e^{\int dt(iA_t+\Phi^6)}]\rangle}{\langle W \rangle}$$

$$\equiv \frac{\langle trP[O_1(t_1)O_2(t_2)\cdots O_n(t_n)e^{\int dt(iA_t+\Phi^6)}]\rangle}{\langle W \rangle}$$

- Such defect correlators arise naturally when we consider small deformations of the Wilson loop (Polyakov, Rychkov ‘00; Drukker, Kawamoto ‘06).
- They encode information on expectation value of Wilson loops of more general shapes and scalar coupling.
The “super-displacement” multiplet

• Among the possible defect primaries, a special role is played by a set of $8_B + 8_F$ “elementary insertions” forming a short multiplet of $Osp(4\ast |4)$.

• The 8 bosonic insertions are
  - The 5 scalars not coupled to the loop
  - The “displacement operator”
    
    $$\Phi^a, \quad a = 1, \ldots, 5 \quad \Delta = 1$$
    
    $$F_{ti} \equiv iF_{ti} + D_t \Phi^6, \quad i = 1, 2, 3 \quad \Delta = 2$$

• These operators have protected scaling dimensions, due to being in a short multiplet

The displacement operator, which is related to deformations of the defect in the transverse directions, has in fact, more generally, protected scaling dimension $\Delta = 2$ for any line defect, independently from supersymmetry
Two-point functions

- Because they have protected scaling dimensions, their exact 2-point functions take the form

\[
\langle \langle \Phi^a(t_1)\Phi^b(t_2) \rangle \rangle = \delta^{ab} \frac{C_\Phi(\lambda)}{t_{12}^2}, \quad \langle \langle F_i(t_1)F_j(t_2) \rangle \rangle = \delta_{ij} \frac{C_{F}(\lambda)}{t_{12}^4}
\]

- The normalization factors are related to the so-called “Brehmsstrahlung function” \(\text{(Correa, Maldacena, Sever '12)}\), and can be determined exactly using supersymmetric localization. In the planar limit:

\[
C_\Phi(\lambda) = 2B(\lambda), \quad C_F(\lambda) = 12B(\lambda)
\]

\[
B(\lambda) = \frac{\sqrt{\lambda} I_2(\sqrt{\lambda})}{4\pi^2 I_1(\sqrt{\lambda})}
\]
Defect chiral primaries

• A more general class of protected defect operators is given by the products (inserted inside WL trace as usual):

\[ \Phi^{(a_1 \ldots a_J)} \]

in the symmetric traceless of SO(5). They are in short multiplets and have protected scaling dimension \( \Delta = J \)

• Analogous to the familiar single-trace chiral primaries \( \text{tr}(Z^J) \)

• Exact results for correlation functions of these operators can be obtained from localization (SG, Komatsu ’18)
Defect correlators at strong coupling

• In general correlation functions of operator insertions on the Wilson loop are non-trivial functions of position and of the coupling constant

• At weak coupling, they can be computed in perturbation theory
  *(Cooke, Dekel, Drukker, ‘17; Kyriu, Komatsu ‘18)*

• At strong coupling, they can be computed from string theory using the AdS$_2$ worldsheet dual to the Wilson loop *(SG, Roiban, Tseytlin ‘17)*
Wilson loop from string theory

• In AdS/CFT dictionary, the Wilson loop operator is dual to a minimal string surface ending on the contour defining the operator at the boundary

\[
\langle W \rangle = Z_{\text{string}} = \int DX^M \mathcal{D}\psi e^{-S_{\text{string}}}
\]

\[
\langle W \rangle \xrightarrow{\lambda \to \infty} e^{-S_{\text{class.}}} = e^{-\frac{\sqrt{\lambda}}{2\pi} A_{\text{reg}}}
\]

• The bosonic part of the AdS$_5 \times $S$^5$ string action reads, taking Poincare coordinates and using Nambu-Goto form (we omit fermions):

\[
S_B = \frac{\sqrt{\lambda}}{2\pi} \int d^2\sigma \sqrt{\text{det} \left[ \frac{1}{z^2} \left( \partial_{\mu} x^r \partial_{\nu} x^r + \partial_{\mu} z \partial_{\nu} z \right) + \frac{\partial_{\mu} y^a \partial_{\nu} y^a}{(1 + \frac{1}{4} y^2)^2} \right]}
\]

where $\sigma^\mu=(t,s)$ are worldsheet coordinates, $r=(0,i)$, $i=1,2,3$ label the coordinates of the (Euclidean) boundary, and $a=1,\ldots,5$ are $S^5$ directions
AdS$_2$ minimal surface

- The minimal surface dual to the 1/2-BPS Wilson line is given by

$$z = s, \quad x^0 = t, \quad x^i = 0, \quad y^a = 0$$

- The induced metric is just that of AdS$_2$ in Poincare coordinates

$$ds^2_2 = \frac{1}{s^2} (dt^2 + ds^2)$$

- Similarly, one can describe the minimal surface for the circular Wilson loop, which is given by AdS$_2$ with the hyperbolic disk coordinates

$$ds^2_2 = \frac{d\sigma^2 + d\tau^2}{\sinh^2 \sigma}$$
Wilson loop from string theory

• In AdS/CFT dictionary, the Wilson loop operator is dual to a minimal string surface ending on the contour defining the operator at the boundary

\[ \langle W \rangle = Z_{\text{string}} = \int_{X^M|_{\partial \Sigma} = (x^r(t), \theta^I(t))} \mathcal{D}X^M \mathcal{D}\psi e^{-S_{\text{string}}} \]

\[ \langle W \rangle \xrightarrow{\lambda \to \infty} e^{-S_{\text{class.}}} = e^{-\frac{\sqrt{\lambda}}{2\pi} A_{\text{reg}}} \]

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where \( \sigma^\mu = (t, s) \) are worldsheet coordinates, \( r = (0, i) \), \( i = 1, 2, 3 \) label the coordinates of the (Euclidean) boundary, and \( a = 1, \ldots, 5 \) are \( \text{S}^5 \) directions
AdS$_2$ minimal surface

- So the minimal surface dual to 1/2-BPS Wilson loop is an AdS$_2$ worldsheet embedded in AdS$_5$, and sitting at a point on S$^5$
- It preserves the same superconformal symmetry OSp(4* |4) as the dual Wilson loop operator
- SL(2,R) is just realized as the isometry of AdS$_2$
- The SO(3)xSO(5) correspond to rotations of the transverse coordinates $x^i(t,s)$ (i=1,2,3) and $y^a(t,s)$ (a=1,...,5)
- By expanding the string sigma model around this minimal surface, we can study the dynamics of small fluctuations of the worldsheet
Worldsheet fluctuations as fields in AdS$_2$

- It is convenient to adopt a *static gauge* where $x^0$ and $z$ (which are identified with the AdS$_2$ worldsheet coordinates) do not fluctuate.
- Then we get a Lagrangian for the 8 transverse fluctuations $x^i(t,s)$ and $y^a(t,s)$, which can be viewed as fields propagating in a rigid AdS$_2$

\[ S_B = \frac{\sqrt{\lambda}}{2\pi} \int d^2\sigma \sqrt{g} L_B \]

\[ L_B = 1 + L_2 + L_4 + \ldots \]

\[ L_2 = \frac{1}{2} g^{\mu\nu} \partial_\mu x^i \partial_\nu x^i + x^i x^i + \frac{1}{2} g^{\mu\nu} \partial_\mu y^a \partial_\nu y^a \]

\[ L_{4y} = y^2 (\partial y)^2 + (\partial y)^4 + \ldots \]

etc.
Worldsheet fluctuations as fields in AdS$_2$

• From the quadratic Lagrangian

\[ L_2 = \frac{1}{2} g^{\mu \nu} \partial_\mu x^i \partial_\nu x^i + x^i x^i + \frac{1}{2} g^{\mu \nu} \partial_\mu y^a \partial_\nu y^a \]

we find

- 5 massless scalars $y^a$
- 3 scalars $x^i$ with $m^2 = 2$

• Since these may be viewed as scalar fields in AdS$_2$, they should be dual to operators inserted at the d=1 boundary, with dimension given by

\[ \Delta (\Delta - 1) = m^2 \]

• So we recover the 8 bosonic operators in the super-displacement multiplet

\[
y^a \quad \leftrightarrow \quad \Phi^a \quad \Delta = 1
\]

\[
x^i \quad \leftrightarrow \quad F_{ti} \quad \Delta = 2
\]
Four-point functions

- The four-point functions of the dual operators at strong coupling can then be obtained from familiar AdS/CFT techniques by computing Witten diagrams in AdS$_2$

- E.g., the leading tree level connected term just involves contact 4-point interactions, with Witten diagram
Some comments

- These calculations are technically very similar to Witten diagram calculations in SUGRA in $\text{AdS}_5 \times S^5$, but the interpretation is a bit different.

- In the SUGRA case, one computes correlation functions of single trace local operators like, $\text{tr}Z^j$, dual to closed string states. The expansion parameter is $G_N \sim 1/N^2$

- In our case, we compute correlators of insertions inside the Wilson loop trace (it is an expectation value of a single trace, non-local operator), dual to open string fluctuations. The expansion parameter is the worldsheet sigma-model coupling, i.e. string tension or $1/\sqrt{\lambda}$
Summary of 4-point function result from string theory

- Let us consider just the 4-point function of the $S^5$ fluctuations $y^a$, dual to the $\Delta=1$ operator insertions $\Phi^a$ on the line defect.

- The 4pt function is specified by a function of cross ratio (multiplied by the fixed prefactor $1/(t_{12}t_{34})^2$). We can decompose it in the singlet $(S)$, symmetric traceless $(T)$ and antisymmetric $(A)$ channels of $SO(5)$. Calculation of $AdS_2$ Witten diagrams gives

$$G_S^{(1)}(\chi) = -\frac{2(\chi^4 - 4\chi^3 + 9\chi^2 - 10\chi + 5)}{5(\chi - 1)^2} + \frac{\chi^2 (2\chi^4 - 11\chi^3 + 21\chi^2 - 20\chi + 10)}{5(\chi - 1)^3} \log |\chi|$$

$$- \frac{2\chi^4 - 5\chi^3 - 5\chi + 10}{5\chi} \log |1 - \chi|,$$

$$\chi = \frac{t_{12} t_{34}}{t_{13} t_{24}}$$

$$G_T^{(1)}(\chi) = -\frac{\chi^2 (2\chi^2 - 3\chi + 3)}{2(\chi - 1)^2} + \frac{\chi^4 (\chi^2 - 3\chi + 3)}{(\chi - 1)^3} \log |\chi| - \chi^3 \log |1 - \chi| ,$$

$$G_A^{(1)}(\chi) = \frac{\chi (-2\chi^3 + 5\chi^2 - 3\chi + 2)}{2(\chi - 1)^2} + \frac{\chi^3 (\chi^3 - 4\chi^2 + 6\chi - 4)}{(\chi - 1)^3} \log |\chi| - (\chi^3 - \chi^2 - 1) \log |1 - \chi|$$

(SG, Roiban, Tseytlin '17)
Extracting OPE data

• From the small $\chi$ expansion we can read off the anomalous dimensions and OPE coefficients of “two-particle” operators appearing in the OPE

$$G(\chi) = \sum_{h} c_{\Delta,\Delta'; h} \chi^{h} 2F_{1}(h, h, 2h, \chi)$$

• The lowest-lying unprotected operator is the singlet “2-particle” bound state $y^{a} y^{a}$, whose dimension turns out to be

$$\Delta_{y^{a} y^{a}} = 2 - \frac{5}{\sqrt{\lambda}} + \ldots .$$
The dimension of the $\Phi^6$ insertion

- At weak coupling, the lowest dimension singlet, unprotected operator in the defect primary spectrum is $\Phi^6$: this is the insertion of the scalar that appears in the Wilson loop exponent

- Its dimension is known to 1-loop order \((\text{Alday, Maldacena `07; Polchinski,Sully `11})\)

\[
\Delta_{\Phi^6} = 1 + \frac{\lambda}{4\pi^2} + \ldots
\]

- It is natural to expect that this operator smoothly goes to the lowest unprotected singlet at strong coupling, i.e. the “2-particle” operator $y^a y^a$. So we expect at strong coupling

\[
\Delta_{\Phi^6} = 2 - \frac{5}{\sqrt{\lambda}} + \ldots
\]
Exact results from localization

• It turns out to be possible to derive a number of exact results for the correlators of a special type of protected insertions on the Wilson loop.

• To use localization, we consider the 1/2-BPS circular loop rather than straight line. Correlators on the circle are related to those on the line by a conformal transformation, e.g.

\[
\langle \langle O_\Delta(t_1)O_\Delta(t_2) \rangle \rangle_{\text{line}} = \frac{C_O}{t_{12}^{2\Delta}} \quad \rightarrow \quad \langle \langle O_\Delta(\tau_1)O_\Delta(\tau_2) \rangle \rangle_{\text{circle}} = \frac{C_O}{(2 \sin \frac{\tau_{12}}{2})^{2\Delta}}
\]

and similarly for 4-point functions, with cross ratio now given by

\[
\chi = \frac{\sin \frac{\tau_{12}}{2} \sin \frac{\tau_{34}}{2}}{\sin \frac{\tau_{13}}{2} \sin \frac{\tau_{24}}{2}}
\]
Exact results from localization

• Recall that the expectation value of the circular loop is given exactly by the Gaussian matrix model (Erickson, Semenoff, Zarembo ‘00; Drukker, Gross ‘00; Pestun ‘09)

\[ \langle W_{\text{circle}} \rangle = \int D M \frac{1}{N} \text{tr} e^{M} e^{-\frac{2N}{\lambda} \text{tr} M^2} N \xrightarrow{\rightarrow} \infty \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) \]

• To derive exact results for correlators of insertions on the Wilson loop, we will need to consider a more general family of 1/8-BPS Wilson loops constructed in Drukker, SG, Ricci, Trancanelli ‘07

• These Wilson loops are defined on generic contours on an \( S^2 \) subspace of \( R^4 \) (or \( S^4 \)), and couple to three of the six scalar fields, say \( \Phi^1, \Phi^2, \Phi^3 \)
The 1/8-BPS Wilson loops

- Explicitly, take an $S^2$ given by $x_1^2 + x_2^2 + x_3^2 = 1$ in Cartesian coordinates, and define the Wilson loop operator

$$\mathcal{W} \equiv \frac{1}{N} \text{tr } P \left[ e^{\oint_C (i A_j + \epsilon_{kjl} x^k \Phi^l) dx^j} \right]$$

- This preserves 1/8 of the superconformal symmetries for generic contour

- The 1/2-BPS circle is a special case: it corresponds to the contour being a great circle of $S^2$

- It was conjectured in Drukker et al ’07, and essentially proved in Pestun ’09 by localization, that the expectation value, as well as correlators of any number of Wilson loops on the $S^2$, is captured by 2d YM theory (more precisely its truncation to the “zero-instanton” sector)
The 1/8-BPS Wilson loops

• This in particular implies that the expectation value only depends on the area singled out by the loop on $S^2$

• The expectation value is given by the same function as for the 1/2-BPS circular loop, but with a rescaled coupling constant. E.g. in the planar limit:

$$\langle W(A) \rangle = \frac{2}{\sqrt{\lambda'}} I_1(\sqrt{\lambda'}) , \quad \lambda' \equiv \frac{A(4\pi - A)}{4\pi^2} \lambda$$

with $A=2\pi$ being the 1/2-BPS case
1/8-BPS Wilson loops and local operators

- More generally, localization applies to general correlation functions of Wilson loops and local operators (SG, Pestun ‘09–’12)

\[ \langle W_{R_1}(C_1) W_{R_2}(C_2) \cdots O_{J_1}(x_1) O_{J_2}(x_2) \cdots \rangle_{4d} \]
\[ = \langle W_{R_1}^{2d}(C_1) W_{R_2}^{2d}(C_2) \cdots \text{tr} \, F_{2d}^{J_1}(x_1) \, \text{tr} \, F_{2d}^{J_2}(x_2) \cdots \rangle_{2d \, \text{YM}} \]

- The relevant local operators may be inserted inside the Wilson loop trace, or in the “bulk”, and they involve the position-dependent combination of scalars

\[ (x_1 \Phi_1 + x_2 \Phi_2 + x_3 \Phi_3 + i \Phi_4)^J \equiv \tilde{\Phi}^J, \quad x_1^2 + x_2^2 + x_3^2 = 1 \]
1/8-BPS Wilson loops and local operators

- These are just chiral primaries of the form $(Y \cdot \Phi)^J$, with $Y$ a null vector which is taken to be position dependent. They were first studied in Drukker, Plefka ’09.

- A crucial property is that their correlation functions are *position independent*.

- In the localization approach, they are mapped to insertions of powers of the Hodge dual of the 2d YM field strength:

  $\tilde{\Phi} \Leftrightarrow i \ast F_{2d}$
Correlators on the Wilson loop

- Focusing on our problem of defect correlators on the circular loop, this means that localization allows us to study correlators of the operators

\[ \tilde{\Phi}^J = (Y_i(\tau_i) \cdot \Phi(\tau_i))^J \quad Y_i = (\cos \tau_i, \sin \tau_i, 0, i, 0, 0) \]

- These operators form a "topological subsector" of the defect CFT, since their \( n \)-point correlation functions

\[ \langle \tilde{\Phi}^{L_1}(\tau_1)\tilde{\Phi}^{L_2}(\tau_2)\cdots\tilde{\Phi}^{L_n}(\tau_n) \rangle_{\text{circle}} \]

are completely position independent
Defect CFT data from topological correlators

- Note that the 2-point and 3-point functions of the general defect chiral primaries are completely fixed by symmetries up to overall functions of the coupling

\[
\langle (Y_1 \cdot \Phi)^{L_1}(\tau_1) \ (Y_2 \cdot \Phi)^{L_2}(\tau_2) \rangle_{\text{circle}} = n_{L_1}(\lambda, N) \times \frac{\delta_{L_1,L_2}(Y_1 \cdot Y_2)^{L_1}}{(Y_1 \cdot Y_2)^{L_{12}[3]}(Y_2 \cdot Y_3)^{L_{23}[1]}(Y_3 \cdot Y_1)^{L_{31}[2]}} \times \frac{(Y_1 \cdot Y_2)^{L_{12}[3]}(Y_2 \cdot Y_3)^{L_{23}[1]}(Y_3 \cdot Y_1)^{L_{31}[2]}}{(2 \sin \frac{\tau_{12}}{2})^{2L_{12}[3]}(2 \sin \frac{\tau_{23}}{2})^{2L_{23}[1]}(2 \sin \frac{\tau_{31}}{2})^{2L_{31}[2]}}
\]

So we can use localization for the “topological correlators” to find the exact 2-point normalization and structure constants of the general chiral primaries on the defect.

- Of course, for higher-point functions, one cannot fully reconstruct the general correlators from the topological ones.
Correlators from localization

- Using the localization correspondence
  \[ \tilde{\Phi} \iff i \ast F_{2d} \]
  
  and area-preserving invariance in 2d YM, one can obtain correlators of $L$-point scalar insertions by taking multiple area-derivatives of the Wilson loop VEV

  \[ \langle \underbrace{\tilde{\Phi} \cdots \tilde{\Phi}}_{L}\rangle \bigg|_{\text{circle}} = \frac{\partial^L \langle W \rangle}{(\partial A)^L} \bigg|_{A=2\pi} \]

- E.g. for the 2-point function, we get for the normalized correlator

  \[ \langle \langle \tilde{\Phi}(\tau_1)\tilde{\Phi}(\tau_2) \rangle \rangle = \frac{\partial^2}{\partial A^2} \log \langle W(A) \rangle \bigg|_{A=2\pi} = -\frac{\sqrt{\lambda}I_2(\sqrt{\lambda})}{4\pi^2 I_1(\sqrt{\lambda})} \]

  which gives the Bremsstrahlung function $B(\lambda)$
The general composite operators

- For general operators $\tilde{\Phi}^J = (Y_i(\tau_i) \cdot \Phi(\tau_i))^J$, one needs to define the properly normal-ordered composite operators.

- This can be accomplished systematically by a Gram-Schmidt orthogonalization procedure \((SG, Komatsu \ '18)\)

- In the planar limit, one gets the explicit form of the charge-\(L\) operators as a determinant

$$
\begin{vmatrix}
\langle W \rangle & \langle W \rangle^{(1)} & \cdots & \langle W \rangle^{(L)} \\
\langle W \rangle^{(1)} & \langle W \rangle^{(2)} & \cdots & \langle W \rangle^{(L+1)} \\
\vdots & \vdots & \ddots & \vdots \\
\langle W \rangle^{(L-1)} & \langle W \rangle^{(L)} & \cdots & \langle W \rangle^{(2L-1)} \\
1 & \Phi & \cdots & \Phi^{L-1}
\end{vmatrix}
$$

$$
D_L = \begin{vmatrix}
\langle W \rangle & \langle W \rangle^{(1)} & \cdots & \langle W \rangle^{(L-1)} \\
\langle W \rangle^{(1)} & \langle W \rangle^{(2)} & \cdots & \langle W \rangle^{(L)} \\
\vdots & \vdots & \ddots & \vdots \\
\langle W \rangle^{(L-1)} & \langle W \rangle^{(L)} & \cdots & \langle W \rangle^{(2L-2)}
\end{vmatrix}
$$

with $\langle W \rangle^{(k)} \equiv (\partial_A)^k \langle W \rangle$

- This allows to obtain exact results for all correlation functions in the topological subsector.
Some explicit 3-point function results

\[
\langle \langle \Phi^2 \cdots \Phi^3 \rangle \rangle = \langle \langle \Phi^2 \cdots \Phi^2 \rangle \rangle
\]

\[
\langle \langle \Phi^2 \cdots \Phi^2 \cdots \Phi^2 \rangle \rangle = -\frac{\lambda^{3/2} I_0 \left( \sqrt{\lambda} \right)^3}{32 \pi^6 I_1 \left( \sqrt{\lambda} \right)^3} + \frac{51 \lambda}{32 \pi^6} - \frac{3 \lambda I_0 \left( \sqrt{\lambda} \right)^2}{8 \pi^6 I_1 \left( \sqrt{\lambda} \right)^2} - \frac{3 \sqrt{\lambda} (\lambda + 40) I_0 \left( \sqrt{\lambda} \right)}{32 \pi^6 I_1 \left( \sqrt{\lambda} \right)} + \frac{37}{4 \pi^6}
\]

\[
\langle \langle \Phi^3 \cdots \Phi^2 \rangle \rangle = \langle \langle \Phi^3 \cdots \Phi^3 \rangle \rangle
\]

\[
\langle \langle \Phi^3 \cdots \Phi^3 \cdots \Phi^2 \rangle \rangle = -\frac{3 \lambda (5 \lambda + 72) I_0 \left( \sqrt{\lambda} \right)^4}{256 \pi^8 I_1 \left( \sqrt{\lambda} \right)^2 I_2 \left( \sqrt{\lambda} \right)^2} - \frac{3 \sqrt{\lambda} (127 \lambda + 1920) I_0 \left( \sqrt{\lambda} \right)^3}{128 \pi^8 I_1 \left( \sqrt{\lambda} \right) I_2 \left( \sqrt{\lambda} \right)^2} + \frac{3 \lambda (2 \lambda + 579) + 6192) I_0 \left( \sqrt{\lambda} \right)^2}{64 \pi^8 I_2 \left( \sqrt{\lambda} \right)^2} + \frac{3 \lambda (5 \lambda - 757) - 6336) I_1 \left( \sqrt{\lambda} \right) I_0 \left( \sqrt{\lambda} \right)}{32 \pi^8 \sqrt{\lambda} \left( I_0 \left( \sqrt{\lambda} \right) - \frac{2 I_1 \left( \sqrt{\lambda} \right)}{\sqrt{\lambda}} \right)^2} + \frac{3 \lambda (\lambda (9 \lambda - 112) + 4960) + 34176) I_1 \left( \sqrt{\lambda} \right)^2}{256 \pi^8 \lambda I_2 \left( \sqrt{\lambda} \right)^2}.
\]
Weak and Strong coupling checks

- One may test the exact results for the correlators
  \[
  \langle :\tilde{\Phi}^{L_1} : :\tilde{\Phi}^{L_2} : \ldots :\tilde{\Phi}^{L_m} : \rangle = \int d\mu \prod_{k=1}^{m} Q_{L_k}(x)
  \]
  in the weak and strong coupling limit (where $Q_{L}(x)$ turn out to be related to Chebyshev and Hermite polynomials respectively).

- We have checked that the first two orders in perturbation theory both at weak and strong coupling indeed precisely agree with the exact predictions.

- On the string theory side, we essentially need to compute correlation functions of products of $S^5$ fluctuations $Y(\tau) \cdot y)^L = \tilde{y}^L$, bringing the insertion points of these operators to the boundary of AdS$_2$. 
Non-supersymmetric Wilson loop

• It is natural to also study the ordinary Wilson loop operator

\[ \text{tr} P e^{\oint iA} \]

• For smooth contours, there are no logarithmic divergences in its expectation value

• When the contour is a line or circle, the operator preserves again an SL(2,R) conformal symmetry

• Correlation functions of operator insertions should then define a non-supersymmetric defect CFT with SL(2,R) x SO(3) x SO(6) symmetry
A defect RG flow

• It is useful to consider a more general operator interpolating between the ordinary Wilson loop and the Maldacena-Wilson loop

\[ W(\zeta) = \text{tr} P e^{\int iA + \zeta \Phi^6} \]

• For generic \( \zeta \), there are logarithmic divergences and \( \zeta \) develops a non-trivial beta function. At weak coupling it can be computed to be (Alday, Maldacena ‘07; Polchinski, Sully ‘11)

\[ \beta_\zeta = -\frac{\lambda}{8\pi^2} \zeta (1 - \zeta^2) + O(\lambda^2) \]

• \( \zeta = 0 \) and \( \zeta = 1 \) are conformal fixed points corresponding to the ordinary and 1/2-BPS Wilson loop
A defect RG flow

- The dimension of $\Phi^6$ at the fixed points at weak coupling can be found to be

\[ \Delta(1) = 1 + \frac{\lambda}{4\pi^2} + \ldots, \quad \Delta(0) = 1 - \frac{\lambda}{8\pi^2} + \ldots \]

- Since $\Phi^6$ is a slightly relevant defect operator at $\zeta = 0$, running of $\zeta$ can be interpreted as a defect RG flow between the non-susy Wilson loop in the UV and the 1/2-BPS Maldacena-Wilson loop in the IR

- It is natural to expect that $F = \log<W>$ plays the role of a d=1 “defect free energy”, that should satisfy

\[ F_{\text{UV}} > F_{\text{IR}} \]

(see Kobayashi, Nishioka, Sato, Watanabe ’18 and Nishioka’s talk)
• A direct perturbative calculation (Beccaria, SG, Tseytlin ’17) gives

\[ \langle W^{(\zeta)} \rangle = 1 + \frac{1}{8} \lambda + \left[ \frac{1}{192} + \frac{1}{128 \pi^2} (1 - \zeta^2)^2 \right] \lambda^2 + \mathcal{O}(\lambda^3) \]

which indeed satisfies

\[ \log \langle W^{(0)} \rangle > \log \langle W^{(1)} \rangle \]

The inequality can be also shown to hold at strong coupling, where one finds

\[ \langle W^{(0)} \rangle \sim \sqrt{\lambda} e^{\sqrt{\lambda}} \quad \langle W^{(1)} \rangle \sim \lambda^{-3/4} e^{\sqrt{\lambda}} \quad \lambda \gg 1 \]
Ordinary Wilson loop at strong coupling

- The dual of the ordinary Wilson loop $\text{tr}P\exp iA$ at strong coupling should be a string worldsheet with Neumann, rather than Dirichlet, boundary conditions on $S^5$ (Alday, Maldacena ‘07; Polchinski, Sully ’11)

- For circular/straight Wilson loop, the worldsheet is still AdS$_2$, but it does not sit at a point on $S^5$, rather we should integrate over 5 zero modes representing the position on $S^5$

- More explicitly, using $S^5$ embedding coordinates $Y^A$ ($Y^A Y^A = 1$), we may write

$$Y^A = \sqrt{1 - \zeta^2} n^A + \zeta^A$$

$$n^A \zeta^A = 0$$

with $\zeta^A$ ($\sigma$) fluctuations obeying Neumann conditions, and integration over the constant unit vector $n^A$ restores SO(6) symmetry
Ordinary Wilson loop at strong coupling

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Correlation functions at strong coupling

• Similarly to the case of the 1/2-BPS loop, one can compute 4-point functions of scalar insertions from the AdS$_2$ worldsheet theory. Considerably more complicated due to Neumann boundary conditions and logarithmic propagators

• Explicit result for scalar 4-point function

(Beccaria, SG, Tseytlin '19)

\[
\langle Y^A(t_1)Y^A(t_2)Y^B(t_3)Y^B(t_4) \rangle = \frac{1}{|t_{12}t_{34}|^{2\Delta}} G_S
\]

\[
G_S = 1 + \frac{10}{(\sqrt{\lambda})^2} \log^2(1 - \chi) + \frac{1}{(\sqrt{\lambda})^3} G_S^{(3)} + \mathcal{O}\left(\frac{1}{(\sqrt{\lambda})^4}\right),
\]

\[
G_S^{(3)} = 80 \left[ \text{Li}_3(\chi) + \text{Li}_3\left(\frac{\chi}{\chi - 1}\right) - \text{Li}_2(\chi) \log(1 - \chi) \right] + 40 \log \frac{\chi}{1 - \chi} \log^2(1 - \chi)
\]

\[ - 10 \frac{\chi^2}{1 - \chi} \log \chi + 5 \left( 5 - \frac{10}{\chi} - 2 \chi \right) \log(1 - \chi) - 50 + 4 d_2 \log^2(1 - \chi) \]
Conclusion

- Correlation functions on the straight/circular Wilson loop have the structure of a d=1 conformal system living on the defect
- Correlator of operator insertions on the loop are dual to AdS$_2$ amplitudes for the fluctuations of the open string worldsheet
  - Is there a manifestation of integrability in the AdS$_2$ amplitudes? AdS$_2$ analog of S-matrix factorization? (Mellin amplitudes for 1d CFT?)
- For the defect CFT on the 1/2-BPS Wilson loop, exact results may be obtained in a “topological subsector” of special operator insertions
- Perhaps combining information from localization, integrability and bootstrap techniques (Liendo, Meneghelli ’16; Liendo, Meneghelli, Mitev ’18. Also Mazac, Paulos ’18-’19…) one may be able to solve this defect CFT
- The non-supersymmetric Wilson loop in N=4 SYM define another interesting, non-supersymmetric defect CFT.
  - Integrability? (Correa, Leoni, Luque ’18)
Conclusion

- Some other directions
  - Higher-point functions (localization checks: signs of integrability?...)
  - Loops in the AdS$_2$ worldsheet theory ($1/\sqrt{\lambda}$ corrections to defect CFT$_1$ data)

- Wilson loops in more general representations
  - “Giant Wilson loops”: rank k-N symmetric/antisymmetric representations dual to D3/D5 branes with AdS$_2\times$S$^2$ and AdS$_2\times$S$^4$ worldvolumes
  - “Bubbling Geometries”
Bulk-defect correlators

- So far I focused mainly on correlators of defect operators, which are captured by the AdS$_2$ open string worldsheet theory.
- But one more generally can also consider also "bulk-defect" correlators: correlation functions of the Wilson and single-trace operators inserted away from the loop, e.g. $\langle W \text{tr} Z^I \rangle$, $\langle W[O(t)] \text{tr} Z^I \rangle$...
- This correspond to an "open-closed" string amplitude of the schematic form (to leading order):

\[
\left< \mathcal{W} \prod_{k=1}^{n} \tilde{\Phi}^{L_k} \text{tr} \tilde{\Phi}^{J} \right> \sim \int d\mu \ B_J(x) \prod_{k=1}^{n} Q_{L_k}(x)
\]

\[
B_J(x) = \frac{4\pi gx^{J+1}}{1 + x^2}
\]

Localization: (SG, Komatsu ’18)