Abstract: Can a relativistic quantum field theory be consistently described as a theory of localizable particles? There are many well-known obstructions to such a description. Here, we trace exactly how such obstructions arise in the regime between nonrelativistic quantum mechanics and relativistic quantum field theory. Perhaps unexpectedly, we find that in the nonrelativistic limit of QFT, there are persisting issues with the localizability of particle states. Related via the Reeh-Schlieder theorem, we also show that the fate of ground state entanglement and the Unruh effect is nontrivial in the nonrelativistic limit.
Relativity, Particle localizability and Entanglement

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What is a particle?

→ Entity which is **localizable**

Other considerations:
- Can be counted \( \mathcal{F}[\mathcal{H}] = \mathbb{C} \oplus \mathcal{H} \oplus (\mathcal{H} \otimes^2)_{S,A} \oplus \cdots \)
  - observable number operator, Fock space
- Relativistic dispersion relation \( E = \sqrt{p^2 + m^2} \)
- ...
Where is a particle?

- Operators are local
  \[ d(x, y) \text{ spacelike } \implies [\phi(x), \phi(y)] = 0 \]

- But particles are represented by states!
  \[ \mathcal{F}[\mathcal{H}] = \mathbb{C} \oplus \mathcal{H} \oplus (\mathcal{H}^\otimes 2)_{S,A} \oplus \cdots \]

→ Can we find states that describe localizable particles?
How can we characterize particle localizability in QFT?

Position operator? Local number operator?
Position operator in QFT?

- Analogue of $|\psi(x)|^2$, $X = \int dx\ x |x\rangle \langle x|$?

- (No-go) Malament theorem

  Hilbert space $\mathcal{H}$, Projs. $\Delta \rightarrow P_\Delta$, Transl. rep. $x \rightarrow U(x)$

  1. **Translation covariance**: $P_{\Delta+x} = U(x)P_\Delta U(-x)$

  2. **Energy condition**: time transl $U(x) = e^{-itH(x)}$ s.t. $H(x) > 0$

  3. **Localizability**: $\Delta_1, \Delta_2$ disjoint $\implies P_{\Delta_1}P_{\Delta_2} = P_{\Delta_2}P_{\Delta_1} = 0$

  4. **Locality**: $\Delta_1, \Delta_2$ spacelike $\implies P_{\Delta_1}P_{\Delta_2} = P_{\Delta_2}P_{\Delta_1}$

  $\implies P_\Delta = 0$

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Local number operator?

- Count particles in finite volume? \[ N_L = \int_V dx a_x^\dagger a_x \]

- (No-go) Reeh-Schlieder (1961)
  
  In AQFT, \( \mathcal{O} \to \mathcal{A}(\mathcal{O}) \)
  
  e.g., generated by \( \phi(f) = \int dx f(x)\phi(x) \)

  **cor.** If \( A \in \mathcal{A}(\mathcal{O}) \) and \( A|0\rangle_G = 0 \), then \( A = 0 \).

  \[ \therefore a_x|0\rangle_G = 0 \implies a_x = 0, \ a_x^\dagger = 0 \]
Concrete attempts at localization

- How do obstructions appear in a concrete scenario?
- Schemes for attempting localization for free KG field
  1) Fourier transform of standard Fock states

\[ H = \int \frac{dk}{(2\pi)^n} \hbar \omega_k a_k^\dagger a_k \]

\[ a_x = \int \frac{dk}{(2\pi)^n} e^{i k \cdot x} a_k \]

\[ |\psi\rangle = \int dx \psi(x) a_x^\dagger |0\rangle_G \]
Alternative schemes?

2) Local harmonic oscillators

\[ c^{-2} \partial_t^2 \phi(t, x) - \nabla^2 \phi(t, x) + \left( \frac{mc}{\hbar} \right)^2 \phi(t, x) = 0 \]

Generate Fock space using:

\[ b_x := \sqrt{\frac{m}{2\hbar^2}} \phi(x) + \frac{i}{\sqrt{2m}} \pi(x) \]

\[ H = \int dx \left[ mc^2 b_x^\dagger b_x - \frac{\hbar^2}{4m} (b_x + b_x^\dagger) \nabla^2 (b_x + b_x^\dagger) \right] \]
What does “local” particle number in these schemes represent?

\[ N_1 = \int d\mathbf{x} \ a_x^\dagger a_x \quad N_2 = \int d\mathbf{x} \ b_x^\dagger b_x \]
Attempts at local number operators

- How do localizability obstructions appear for number operators in these two schemes?
  1) Fourier transformed: \( a_x = \int \frac{dk}{(2\pi)^n} e^{ik \cdot x} a_k \)
     - Non-locally related to field operators

\[
a_y = \int dx \left[ F_+ (y - x) b_x + F_- (y - x) b_x^\dagger \right]
\]

\[
F_\pm \sim e^{-k_c |y - x|}, \quad \text{recall: } b_x := \sqrt{\frac{m}{2\hbar^2}} \phi(x) + \frac{i}{\sqrt{2m}} \pi(x)
\]

\[\implies |\psi\rangle = \int dx \, \psi(x) a_x^\dagger |0\rangle_G \quad \text{is non-local}\]
Attempts at local number operators

- How do localizability obstructions appear for number operators in these two schemes?

2) Local oscillators: $b_x := \sqrt{\frac{m}{2\hbar^2}} \phi(x) + \frac{i}{\sqrt{2m}} \pi(x)$

$$a_y = \int dx \left[ F_+(y - x)b_x + F_-(y - x)b_x^\dagger \right]$$

- $a_x, b_x$ generate different Fock spaces: $b_x |0\rangle_G \neq 0$
- Representations are unitarily inequivalent

$$\text{tr}(\beta \beta^\dagger) = \int dx dy |F_-(x - y)|^2 = \infty$$
1) \[ a_\mathbf{x} = \int \frac{dk}{(2\pi)^n} e^{i\mathbf{k} \cdot \mathbf{x}} a_\mathbf{k} \]

- Non-local in space
- \[ a_\mathbf{y} = \int d\mathbf{x} \left[ F_+(\mathbf{y} - \mathbf{x}) b_\mathbf{x} \right. \\
\left. + F_-(\mathbf{y} - \mathbf{x}) b_\mathbf{x}^\dagger \right] \]

- \[ [H, N] = 0 \]
- \[ E = \sqrt{p^2 + m^2} \]

2) \[ b_\mathbf{x} := \sqrt{\frac{m}{2\hbar^2}} \phi(\mathbf{x}) + \frac{i}{\sqrt{2m}} \pi(\mathbf{x}) \]

- Label dofs in space
- Not preserved in time
- Dispersion relation
Impact of special relativity?

**Relativistic QFT**
- ☒ Position operator
- ☒ Local number op

**Non-rel. approx**
- $v \ll c$

**Fix particle number,** $N$

**NR-QFT**

$\approx$ **NRQM**

**Non-relativistic QM**
- ☑ Position operator
- ☑ Local number op

**Concretely:** How does the tension between the two schemes for attempted localization subside?
How to take non-relativistic approx?

- “c → ∞” : |v| ≪ c gives |k| ≪ k_c := mc/ℏ
  \[
  \frac{|v|}{c} = \frac{|p|}{mc} = \frac{ℏ|k|}{mc} ≪ 1
  \]

- hard cutoff |k| ≤ Λ ≪ k_c, expand up to (Λ/k_c)^2
  since
  \[
  E = \sqrt{p^2c^2 + m^2c^4} \approx mc^2 + \frac{p^2}{2m}
  \]
How are schemes related after NR approx?

- Does the Bogoliubov transformation become the identity?
- No! Remaining discrepancy between schemes

\[ a_y = b_y + \int dx \ F^{(2)}_-(y - x) b_x^\dagger \]

Also remain unitarily inequivalent

- Fully recovering particle localizability in NRQM involves more than just removing (special) relativistic features.
How to recover non-relativistic QM?

- Use $a_y$, $a_y^\dagger$ to recover NRQM

\[
|\Psi(t)\rangle := \frac{1}{\sqrt{N!}} \int dy_1 \cdots dy_N \Psi(y_1, \ldots, y_N; t) a_{y_1}^\dagger \cdots a_{y_N}^\dagger |0\rangle_G
\]

\[
i\hbar \partial_t \Psi(y_1, \ldots, y_N; t) = \left( E_0 + mc^2 N - \frac{\hbar^2}{2m} \sum_{i=1}^N \nabla_i^2 \right) \Psi(y_1, \ldots, y_N; t)
\]

\[
X = \frac{1}{N} \int dy y a_{y}^\dagger a_{y}, \quad [X_i, P_j] = i\hbar \delta_{ij}
\]
How to bridge the gap?

- Local dofs in QFT (even after NR approx) vs. localization in recovered NRQM
- Foldy-Wouthuysen trsf implicit in Bjorken & Drell (1964)

\[ a_y = U^\dagger b_y U = b_y + \int dx \ F^{(2)}_-(y - x) b_x^\dagger \]

(note: unitary inequivalence)

- Extra transformation step involves a non-local reshuffling of the degrees of freedom.
Recap

- Obstructions to particle localizability (Malament, Reeh-Schlieder) do not subside in non-relativistic approximation of QFT

- Recovery requires non-local reshuffling of degrees of freedom
Entanglement and particle localizability

- Entanglement gives vacuum fundamentally non-local character. Is this obstructing particle localizability?
- What happens to entanglement in non-relativistic limit?

- Entanglement in QFT of independent interest for: Unruh/Hawking effects, holography, condensed matter, quantum information, …

- Here, examining role in foundational aspects of QFT
Revisiting Reeh-Schliieder

• How is entanglement related to localizability?

• Recall corollary

\[ \mathcal{O} \rightarrow \mathcal{A}(\mathcal{O}) \]

If \( A \in \mathcal{A}(\mathcal{O}) \) and \( A|0\rangle_G = 0 \), then \( A = 0 \).

• Reeh-Schliieder theorem (1961)

For any \( \mathcal{O} \), \( |0\rangle_G \) is cyclic for \( \mathcal{A}(\mathcal{O}) \).

i.e., can approximate any state by acting \( \mathcal{A}(\mathcal{O}) \) on \( |0\rangle_G \)

\[ \rightarrow \text{Related to vacuum entanglement!} \]
Redhead’s intuition\(^1\)

- How are state cyclicity and entanglement related?
- Can reproduce with two qubits!
  \[ \mathcal{H}_1 \otimes \mathcal{H}_2 = \mathbb{C}^2 \otimes \mathbb{C}^2 \]
- “Baby” Reeh-Schlieder theorem
  \[ |\Psi\rangle = |01\rangle - |10\rangle \text{ is cyclic for } \mathcal{B}(\mathcal{H}_1) \]

  idea: any state in \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) can be written
  \[ (\alpha 1 \otimes 1 + \beta X \otimes 1 + \gamma Z \otimes 1 + \delta XZ \otimes 1)|\Psi\rangle \]

Persisting entanglement?

- Entanglement between local dofs $b_x, b_x^\dagger$
  - Cutoff $|k| \leq \Lambda \ll k_c$ (sampling theory$^1$)

1. Can represent fields on a lattice
2. Local dofs are at lattice points

Persisting entanglement?

- Formally remains entangled

\[
|0\rangle_G^A = \mathcal{N} \exp \left[ -\frac{1}{2} \sum_{m, m' \in \mathbb{Z}^n} (\alpha^{-1} \beta)_{mm'} b_m^\dagger b_{m'}^\dagger \right] |0\rangle_L^A
\]

\[
= |00 \cdots \rangle - \frac{1}{2} \sum_{m \neq m'} (\alpha^{-1} \beta)_{mm'} |0 \cdots 01_m 0 \cdots 01_{m'} 0 \cdots \rangle
- \frac{1}{\sqrt{2}} \sum_m (\alpha^{-1} \beta)_{mm} |0 \cdots 02_m 0 \cdots \rangle
\]

- But unitary inequivalence!
Persisting entanglement?

- Can we quantify entanglement more carefully?
- Temperature of single oscillator

\[ k_B T \sim \frac{mc^2}{\log[(\Lambda/k_c)^{-4}]} \]

- Logarithmic Negativity between two local oscillators
  - e.g., \( n = 1 \) dimensions, distance \( M \) lattice spacings

\[ E_N \sim \frac{1}{M^2 \pi^2} (\Lambda/k_c)^2 \]
Summary

- Obstructions in the extent to which QFT can describe localizable particles

- These obstructions persist after non-relativistic approximation

- Related to vacuum entanglement, which also persists in this regime