Abstract: I will discuss progress on a non-perturbative approach to the study of string sigma-models relevant in AdS/CFT which exploits lattice field theory techniques.
AdS/CFT and string sigma-model non-perturbatively

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Framework

String/gauge correspondence, addresses together
- understanding gauge theories at all values of the coupling
- understanding string theories in non-trivial backgrounds

Type IIB strings in $AdS_5 \times S^5$

"Quark-antiquark" potential

$Z_{\text{string}}|_C = \int DXD\Psi e^{-S_{\text{string}}}$

$\mathcal{N} = 4$ super Yang-Mills in 4d

$\langle W[C] \rangle = \frac{1}{N} \text{Tr} \mathcal{P} e^{\int (iA_\mu \dot{x}^\mu + \Phi \Phi^* \dot{y}^i) ds}$
AdS/CFT and exact results

Impressive progress in obtaining results exact in the coupling (here, planar AdS$_5$/CFT$_4$)

\[ f(g) \]

\[ f(g) = a g^2 + b g^3 + \ldots \]

\[ g := \frac{\sqrt{\lambda}}{4\pi} = \frac{\sqrt{g_{YM}^2 N}}{4\pi} = \frac{R^2}{4\pi \alpha'} \]

- from integrability
- from supersymmetric localization
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Impressive progress in obtaining results exact in the coupling (here, planar AdS$_5$/CFT$_4$)

\[ f(g) = a g^2 + b g^4 + \ldots \]

Perturbative gauge theory

Integrability/Localization

Perturbative string sigma model

\[ f(g) = c g + d + \frac{e}{g} + \ldots \]

\[ g := \frac{\sqrt{\lambda}}{4\pi} = \frac{\sqrt{g_{YM}^2 N}}{4\pi} = \frac{R^2}{4\pi \alpha'} \]

- from integrability
- from supersymmetric localization
AdS/CFT and exact results

Impressive progress in obtaining results exact in the coupling (here, planar $\text{AdS}_5/\text{CFT}_4$)

\[
f(g) = \alpha g^2 + b g^4 + \ldots
\]

\[
g := \frac{\sqrt{\lambda}}{4\pi} = \frac{\sqrt{g_{YM}^2 N}}{4\pi} = \frac{R^2}{4\pi \alpha'}
\]

- from integrability (assumed)
- from supersymmetric localization (supersymmetric observables)

In the world-sheet string theory integrability only classically, localization not formulated.

The relevant string sigma-model (Green-Schwarz superstrings in $AdS$ backgrounds with RR-fluxes) is a complicated interacting 2d field theory which has subtleties also perturbatively.

Call for genuine 2d QFT to cover the finite-coupling region.
Motivation

Impressive progress in obtaining results exact in the coupling (here, planar AdS$_5$/CFT$_4$)

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f(g) = a g^2 + b g^4 + \ldots
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f(g) = e g + d + \frac{c}{g} + \ldots
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Call for genuine 2d QFT to cover the finite-coupling region.

with the aim of eventually extending this study to non-exact backgrounds!
Lattice techniques in AdS/CFT

Features:
- **2d**: computationally cheap
- no supersymmetry (only as flavour symmetry, Green-Schwarz)
- all gauge symmetries are fixed (no formulation à la Wilson), only scalar fields (some of which anti-commuting)

Non-trivial 2d qft with strong coupling analytically known, finite-coupling (numerical) prediction.

[previous study: Roiban McKeown 2013]
Green-Schwarz string in $AdS_5 \times S^5 +$ RR flux

$\tau^A$

AdS$_5 \times S^5$

Symmetries:
- global $PSU(2,2|4)$, local bosonic (diffeomorphism) and fermionic ($\kappa$-symmetry)
- classical integrability

manifest when written as sigma-model action on $G/H = \frac{PSU(2,2|4)}{SO(1,4) \times SO(5)}$. 
Green-Schwarz string in $\text{AdS}_5 \times S^5 + \text{RR flux}$ perturbatively

Highly non-linear, to quantize it use \textbf{semiclassical methods}

\[
X = X_{c1} + \tilde{X} \quad \rightarrow \quad \Gamma = g \left[ \Gamma_0 + \frac{\Gamma_1}{g} + \frac{\Gamma_2}{g^2} + \ldots \right]
\]

\begin{itemize}
  \item General analysis of fluctuations in terms of background geometry,
    \cite{Drukker Gross Tseytlin 00} \cite{Buchbinder Tseytlin 14} \cite{VF Giangreco Griguolo Seminara Vescovi 15}
  
  \item Explicit analytic form of one-loop partition function $Z = \det O_F / \sqrt{\det O_B}$
    for a class of effectively one-dimensional problems.
    Several “vacua” (GKP string, quark-antiquark potential, generalized cusp)
    have been “solved” this way at one loop, and agree with predictions.
    \cite{Drukker Gross Tseytlin Frolov VF Beccaria Dunne Giangreco, Ohlson Sax, Griguolo Seminara Vescovi ....}
    In BPS cases (e.g. dual to circular Wilson loop) more care needed:
    - avoid measure ambiguities, considering ratio of partition functions
    - choose suitable regularization scheme
      \cite{Kruczenski Tirziu 08} \cite{Kristjansen Makeenko 12} \cite{Buchbinder Tseytlin 14}
      \cite{VF, Giangreco, Griguolo, Seminara, Vescovi 15} \cite{Pando-Zayas Trancanelli et al.16}
      \cite{VF, Vescovi, Tseytlin 17} \cite{Cagnazzo, Medina-Rincon, Zarembo 17} \cite{Medina-Rincon, Tseytlin, Zarembo 18}
\end{itemize}
Green-Schwarz string in $AdS_5 \times S^5 + RR$ flux perturbatively

Highly non-linear, to quantize it use \textit{semiclassical methods}

$$X = X_{cl} + \tilde{X} \quad \rightarrow \quad \Gamma = g \left[ \Gamma_0 + \frac{\Gamma_1}{g} + \frac{\Gamma_2}{g^2} + \ldots \right]$$

2 loops is current limit: “homogenous” configs, “AdS light-cone” gauge-fixing

\[ \text{[Metsaev, Tseytlin]} \text{[Metsaev, Thorn, Tseytlin]} \]

\[ \text{[Giombi Ricci Roiban Tseytlin 09]} \text{[Bianchi² Bres VF Vescovi 14]} \]

UV divergences: set to zero power-divergent massless tadpoles (as in \textit{dimreg}),
all remaining log-divergent integrals cancel out in the sum (no need of reg. scheme).

$AdS_5 \times S^5$: non-trivial test of UV finiteness and quantum integrability.

$AdS_4 \times CP^3$: quantum consistency of the string action proposed in \[\text{[Uvarov 09,10]}\],
non-trivial test of all-order conjecture for $h(\lambda)$ in ABJM magnon dispersion relation.

\[ \text{[Gromov, Syzov 14]} \]
Green-Schwarz string in $AdS_5 \times S^5 + RR$ flux perturbatively

Highly non-linear, to quantize it use **semiclassical methods**

\[ X = X_{cl} + \tilde{X} \quad \rightarrow \quad \Gamma = g \left[ \Gamma_0 + \frac{\Gamma_1}{g} + \frac{\Gamma_2}{g^2} + \ldots \right] \]

Efficient alternative to Feynman diagrams for **on-shell** objects (worldsheet S-matrix)

unitarity cuts (on-shell methods) in $d=2$

[Bianchi VF Hoare 2013][Englund Roiban 2013] [Bianchi Hoare 14]
The cusp anomaly of $\mathcal{N} = 4$ SYM from string theory

Ubiquitous observable in gauge field theory, it governs:
- the renormalization of cusped lightlike Wilson loops
- the IR structure of gluon scattering amplitudes
- the large-spin ($x \to 1$) behavior of anomalous dimensions for “twist-operators”
  (OPE description of deep inelastic scattering)
The cusp anomaly of $\mathcal{N} = 4$ SYM from string theory

Completely solved via integrability. [Beisert Eden Staudacher 2006]

Expectation value of a light-like cusped Wilson loop

$$\langle W[C_{\text{cusp}}]\rangle \sim e^{-f(g)\phi}\ln\frac{L_{\text{IR}}}{L_{\text{UV}}}$$

$$Z_{\text{cusp}} = \int [D\delta X][D\delta \theta] e^{-S_{\text{11B}}(X_{\text{cusp}} + \delta X, \delta \theta)}$$

String partition function with “cusp” boundary conditions.

$X_{\text{cusp}}$ is the minimal surface

$$ds^2_{\text{AdS}_5} = \frac{dz^2 + dx^+ dx^- + dx^* dx}{z^2}$$

$$z = \sqrt{\frac{\tau}{\sigma}}, \quad x^+ = \tau, \quad x^- = -\frac{1}{2\sigma}, \quad x^+ x^- = -\frac{1}{2}z^2$$

ending on a null cusp, since $x^+ x^- = 0$ at the boundary $z = 0$.

[Giombi Ricci Roiban Tseytlin 2009]
The cusp anomaly of $\mathcal{N} = 4$ SYM from string theory

Completely solved via integrability. [Beisert Eden Staudacher 2006]

Expectation value of a light-like cusped Wilson loop

$$\langle W [C_{\text{cusp}}] \rangle \sim e^{-f(g) \Phi \ln \frac{L_{1\text{IB}}}{\epsilon_{\text{UV}}}}$$

$$Z_{\text{cusp}} = \int [D\delta X] [D\delta \theta] e^{-S_{1\text{IB}}(X_{\text{cusp}}+\delta X, \delta \theta)} = e^{-\Gamma_{\text{eff}}} \equiv e^{-f(g) V_2}$$

String partition function with “cusp” boundary conditions .

Perturbatively

$$f(g)|_{g \to 0} = 8g^2 \left[ 1 - \frac{\pi^2}{3} g^2 + \frac{11 \pi^4}{45} g^4 - \left( \frac{73}{315} + 8 \zeta_3 \right) g^6 + \ldots \right] \quad [\text{Bern et al. 2006}]$$

$$f(g)|_{g \to \infty} = 4g \left[ 1 - \frac{3 \ln 2}{4\pi} \frac{1}{g} - \frac{K}{16\pi^2} \frac{1}{g^2} + \ldots \right] \quad [\text{Gubser Klebanov Polyakov 02}] [\text{Frolov Tseytlin 02}] [\text{Giombi et al. 2009}]$$
Green-Schwarz string in the null cusp background

The (AdS lightcone) gauge-fixed action for fluctuations above the null cusp is

\[ S_{\text{cusp}} = g \int dt ds \mathcal{L}_{\text{cusp}} \]

\[ \mathcal{L}_{\text{cusp}} = |\partial_t x + \frac{1}{2} z|^2 + \frac{1}{z^2} |\partial_s x - \frac{1}{2} z|^2 + \left( \partial_t z^M + \frac{1}{2} z^M + \frac{i}{z^2} \eta_i (\rho^{MN})^i_j \eta^j \right)^2 + \frac{1}{z^4} \left( \partial_s z^M - \frac{1}{2} z^M \right)^2 \]

\[ + i \left( \theta^i \partial_t \theta^i + \eta^i \partial_t \eta_i + \theta_i \partial_t \theta^i + \eta_i \partial_t \eta^i \right) - \frac{1}{z^2} (\eta^i \eta_i)^2 \]

\[ + 2i \left[ \frac{1}{z^3} z^M \eta^i (\rho^M)_{ij} \left( \partial_s \theta^j - \frac{1}{2} \theta^j \eta^i \left( \partial_s x - \frac{1}{2} z \right) \right) + \frac{1}{z^4} z^M \eta_i (\rho^M)^{ij} \left( \partial_s \theta_j - \frac{1}{2} \theta_j \eta^i \left( \partial_s x - \frac{1}{2} z \right) \right) \right] \]

- 8 bosons: \(x, x^*, z^M (M = 1, \cdots, 6), z = \sqrt{z_M z^M} \);
- 8 fermions: \(\theta^i = (\theta_i)^\dagger, \eta^i = (\eta_i)^\dagger, i = 1, 2, 3, 4\), complex Grassmann;
- \(\rho^M\) are off-diagonal blocks of \(SO(6)\) Dirac matrices
- \((\rho^{MN})^i_j\) are the \(SO(6)\) generators

Remnant global symmetry is \(SO(6) \times SO(2)\).
Fermionic interactions at most quartic.
Lattice QFT basics

Discretize Euclidean worldsheet in a grid of lattice spacing $a$, size $L = N a$.

Fields $\phi \equiv \phi_n$ defined at $\xi = (an_1, an_2) \equiv a n$.

a) natural cutoff $-\frac{\pi}{a} < p_\mu \leq \frac{\pi}{a}$

b) path integral measure $[D\phi] = \prod_n d\phi_n$.

Then $\int \prod_n d\phi_n e^{-S_{\text{discr}}}$ via Monte Carlo: generate an ensemble $\{\Phi_1, \ldots, \Phi_K\}$ of field configurations, each weighted by $P[\Phi_i] = \frac{e^{-S_E[\Phi_i]}}{Z}$.

Ensemble average $\langle A \rangle = \int [D\Phi] P[\Phi] A[\Phi] = \frac{1}{K} \sum_{i=1}^{K} A[\Phi_i] + O\left(\frac{1}{\sqrt{K}}\right)$
Lattice QFT basics

Discretize Euclidean worldsheet in a grid of lattice spacing $a$, size $L = Na$.

Fields $\phi \equiv \phi_n$ defined at $\xi = (an_1, an_2) \equiv n$.

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Ensemble average $\langle A \rangle = \int [D\Phi] P[\Phi] A[\Phi] = \frac{1}{K} \sum_{i=1}^{K} A[\Phi_i] + O\left(\frac{1}{\sqrt{K}}\right)$

Graßmann-odd fields are formally integrated out: $P[\Phi_i] = \frac{e^{-S_E[\Phi_i]} \det O_F}{Z}$

- action must be quadratic in fermions
- determinant must be positive definite

\[ \det O_F \rightarrow \sqrt{\det(O_F^{\dagger} O_F)} \equiv \int D\zeta \bar{D} \zeta e^{-\int d^2 \xi \bar{\zeta} (O_F^{\dagger} O_F)^{-\frac{1}{2}} \zeta} \]
Lattice QFT basics

Discretize Euclidean worldsheet in a grid of lattice spacing $a$, size $L = N a$.

Fields $\phi \equiv \phi_n$ defined at $\xi = (an_1, an_2) \equiv n$.

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Ensemble average $\langle A \rangle = \int [D\Phi] \ P[\Phi] \ A[\Phi] = \frac{1}{K} \sum_{i=1}^{K} A[\Phi_i] + \mathcal{O}\left(\frac{1}{\sqrt{K}}\right)$

Grassmann-odd fields are formally integrated out: $P[\Phi_i] = \frac{e^{-S_E[\Phi_i]} \det \mathcal{O}_F}{Z}$

- action must be quadratic in fermions

- determinant must be positive definite

\[ \text{Pf } O_F \rightarrow (\det O_F^\dagger O_F)^{\frac{1}{4}} \equiv \int D\zeta \ D\bar{\zeta} \ e^{-\int d^2 \zeta \ (O_F^\dagger O_F)^{-\frac{1}{4}} \zeta} \]
Four-fermion interactions

Linearization via Hubbard-Stratonovich transformation

\[
\exp \left\{ -g \int dt \, ds \, \mathcal{L}_4 \right\} \sim \int d\phi \, d\phi^M \, \exp \left\{ -g \int dt \, ds \, \mathcal{L}_{\text{aux}} \right\}
\]

\[
\exp \left\{ -g \int dt ds \left[ -\frac{1}{z^2} (\eta^i \eta_i)^2 + \left( \frac{i}{z} \bar{z}_N \eta_i \rho^{MN_i} \eta^j \right)^2 \right] \right\}
\]

\[
\sim \int D\phi D\phi^M \, \exp \left\{ -g \int dt ds \left[ \frac{1}{2} \phi^2 + \frac{\sqrt{2}}{z} \phi \eta^2 + \frac{1}{2} (\phi_M)^2 - i \frac{\sqrt{2}}{z^2} \phi^M \left( \frac{i}{z} \bar{z}_N \eta_i \rho^{MN_i} \eta^j \right) \right] \right\} .
\]

- +7 bosonic auxiliary fields $\phi, \phi^M \ (M = 1, \cdots, 6)$
Phase problem

Even with \( \text{Pf}(O_F) = e^{i\theta} \left( O_F O_F^\dagger \right)^{\frac{1}{4}} \), vev's can be still obtained via reweighting:

\[
\langle \mathcal{A} \rangle = \frac{\int D\Phi \, \mathcal{A} \text{Pf}(O_F) \, e^{-S[\Phi]}}{\int D\Phi \, \text{Pf}(O_F) \, e^{-S[\Phi]}}
\]

\[
= \frac{\int D\Phi \, D\zeta \, D\bar{\zeta} \, \mathcal{A} e^{i\theta} \, e^{-S[\Phi]} - \int d^2 \xi \, \bar{\zeta} (O_F O_F^\dagger)^{-\frac{1}{4}} \zeta}{\int D\Phi \, D\zeta \, D\bar{\zeta} \, e^{i\theta} \, e^{-S[\Phi]} - \int d^2 \xi \, \bar{\zeta} (O_F O_F^\dagger)^{-\frac{1}{4}} \zeta} = \langle \mathcal{A} e^{i\theta} \rangle_{\theta=0} / \langle e^{i\theta} \rangle_{\theta=0}
\]

It gives meaningful results as long as the phase does not averages to zero.
Phase problem

Even with $\text{Pf}(O_F) = e^{i\theta} (O_F O_F^\dagger)^{\frac{1}{4}}$, vev's can be still obtained via reweighting:

$$
\langle A \rangle = \frac{\int D\Phi \ A \text{Pf}(O_F) \ e^{-S[\Phi]}}{\int D\Phi \ \text{Pf}(O_F) \ e^{-S[\Phi]}}
$$

$$
= \frac{\int D\Phi \ D\zeta \ D\bar{\zeta} \ A e^{i\theta} \ e^{-S[\Phi]} - \int d^2 \xi \ \bar{\zeta}(O_F O_F^\dagger) \ \frac{1}{4} \zeta}{\int D\Phi \ D\zeta \ D\bar{\zeta} \ e^{i\theta} \ e^{-S[\Phi]} - \int d^2 \xi \ \bar{\zeta}(O_F O_F^\dagger) \ \frac{1}{4} \zeta} = \frac{\langle A \ e^{i\theta} \rangle_{\theta=0}}{\langle e^{i\theta} \rangle_{\theta=0}}
$$

It gives meaningful results as long as the phase does not averages to zero.

Dedicated algorithms: active field of study, no general proof of convergence.
Alternative linearization

The phase is implicit in the linearization, like \( e^{-\frac{b^2}{4a}} = \int dx \, e^{-a \, x^2 + i \, b \, x} \).

Consider a simple SO(4) invariant four-fermion interaction [Catterall 2015]

\[
\mathcal{L}_{4F} = \frac{1}{2} \epsilon_{abcd} \, \psi^a(x) \, \psi^b(x) \, \psi^c(x) \, \psi^d(x) \equiv \Sigma^{ab} \, \Sigma^{ab}
\]

where \( \Sigma^{ab} = \psi^a \psi^b \), \( \Sigma^{ab} = \frac{1}{2} \epsilon_{abcd} \psi^c \psi^d \). Introducing \( \Sigma^{ab} = \frac{1}{2} \left( \Sigma^{ab} \pm \Sigma^{cd} \right) \), rewrite

\[
\mathcal{L}_{4F} = \pm \, 2 \left( \Sigma^{ab}_{\pm} \right)^2
\]

just exploiting the Grassmann character of the underlying fermions.
Alternative linearization

In our case, \((\rho^M)^{im}(\rho^M)^{kn} = 2e^{imkn}\), we analogously rewrite

\[
\mathcal{L}_{F4} = -\frac{1}{z^2} (\eta^2)^2 \mp \frac{2}{z^2} (\eta^2)^2 \mp \frac{1}{z^2} \Sigma^i_{\pm i} \Sigma^i_{\pm j}
\]

\[
\Sigma_i^j = \eta_i \eta^j, \quad \Sigma_j^i = (\rho^N)^{ik} n_N (\rho^L)^{j\ell} n_L \eta_k \eta^\ell, \quad \Sigma^i_{\pm i} = \Sigma^i_i \pm \Sigma^i_i
\]

Choosing the good sign \((-\)), new set of \(1 + 16\) real auxiliary fields

\[
\mathcal{L}_{aux} = \frac{12}{z} \eta^2 \phi + 6\phi^2 + 2\Sigma^i_{\pm j} \phi^i_j + \phi^i_j \phi^i_j \quad \mathcal{L}^\dagger_{aux} = \mathcal{L}_{aux}
\]

Antisymmetry and \(\Gamma_5\)-hermiticity \((\Gamma_5^\dagger \Gamma_5 = 1, \Gamma_5^\dagger = -\Gamma_5)\)

\[
O_F^\dagger = \Gamma_5 O_F \Gamma_5, \quad O_F^T = -O_F
\]

ensure positive-definite determinant \((\text{Pf} O_F)^2 = \det O_F \geq 0\), and a real Pfaffian.
**Alternative linearization**

In our case, $(\rho^M)^{im}(\rho^M)^{kn} = 2\epsilon^{imkn}$, we analogously rewrite

$$ L_{F4} = -\frac{1}{z^2} (\eta^2)^2 \pm \frac{2}{z^2} (\eta^2)^2 \pm \frac{1}{z^2} \Sigma_+^i \Sigma_+^j $$

$$ \Sigma_i^j = \eta_i \eta^j, \quad \bar{\Sigma}_j^i = (\rho^N)^{ik} n_N (\rho^L)_{jl} n_L \eta_k \eta^l, \quad \Sigma_\pm^i = \Sigma_i^j \pm \bar{\Sigma}_i^j $$

Choosing the **good** sign ($-$), new set of $1 + 16$ real auxiliary fields

$$ L_{aux} = \frac{12}{z} \eta^2 \phi + 6 \phi^2 + \frac{2}{z} \Sigma_\pm^i \phi_\pm^i \phi_\pm^j \phi_\pm^j \quad L_{aux}^\dagger = L_{aux} $$

Antisymmetry and $\Gamma_5$-hermiticity ($\Gamma_5^\dagger \Gamma_5 = \mathbb{1}$, $\Gamma_5^\dagger = -\Gamma_5$)

$$ O_F^\dagger = \Gamma_5 O_F \Gamma_5, \quad O_F^T = -O_F $$

ensure positive-definite determinant $(\text{Pf} O_F)^2 = \det O_F \geq 0$, and a **real** Pfaffian.

In simpler models with four-fermion interactions, similar manipulations ensure a positive definite Pfaffian. [Catterall 2016, Catterall and Schaich 2016]  
Here, gain in computational costs but $\text{Pf} O_F = \pm \sqrt{\det O_F}$. 
Spectrum of $O_F$

From $\Gamma_5$-hermiticity and antisymmetry,

$$\mathcal{P}(\lambda) = \det(O_F - \lambda\mathbb{1}) = \det(\Gamma_5 (O_F - \lambda\mathbb{1}) \Gamma_5)$$

$$= \det(O_F^\dagger + \lambda\mathbb{1}) = \det(O_F + \lambda^* \mathbb{1})^* = \mathcal{P}(-\lambda^*)^*$$

Spectrum characterized by quartets $\{\lambda, -\lambda^*, -\lambda, \lambda^*\}$.

$$\det O_F = \prod_i |\lambda_i|^2 |\lambda_i^*|^2 \quad \rightarrow \quad \text{Pf}(O_F) = \pm \prod_i |\lambda_i|^2$$

Choosing a starting configuration with positive Pfaffian, no sign change possible.
Where are we sign-problem free?

Eigenvalue distribution of fermionic operators well separated from zero, no sign problem for $g \geq 10$, where nonperturbative physics is captured.
Guiding lines for discretization

- Lattice perturbation theory $\xrightarrow{\alpha \to \epsilon}$ continuum perturbation theory
- Preserve the symmetries of the model
- No complex phases
Guiding lines for discretization

- Lattice perturbation theory $\xrightarrow{a \to 0}$ continuum perturbation theory

A naive discretization $p_\mu \to \hat{p}_\mu \equiv \frac{1}{a} \sin(a p_\mu)$ leads to fermion doublers,

$$K_F = \begin{pmatrix}
0 & -\hat{p}_0 1 & (\hat{\rho}_1 - i \frac{m}{2})\rho^M u_M & 0 \\
-\hat{p}_0 1 & 0 & 0 & (\hat{\rho}_1 - i \frac{m}{2})\rho^M u_M \\
-(\hat{\rho}_1 + i \frac{m}{2})\rho^M u_M & 0 & 0 & -\hat{p}_0 1 \\
0 & -(\hat{\rho}_1 + i \frac{m}{2})\rho^M u_M & -\hat{p}_0 1 & 0
\end{pmatrix}$$

spoiling UV finiteness (effective 2d supersymmetry).
A Wilson-like fermion discretization

- Lattice perturbation theory $\overset{a \to 0}{\longrightarrow}$ continuum perturbation theory
- Preserve $SO(6)$, breaks $U(1) \sim SO(2)$
- No complex phases: $(O_F^W)^\dagger = \Gamma_5 O_F^W \Gamma_5 \ , \ (O_F^W)^T = -O_F^W$

Add to the action a “Wilson term”, $K_F + W \equiv K_F^W$ to kill fermion doublers

$$K_F^W = \begin{pmatrix} W_+ & -\hat{p}_0^\ddagger & (\hat{p}_1 - i \frac{m}{2})\rho^M u_M & 0 \\ -\hat{p}_0 \ddagger & -W_+ & 0 & (\hat{p}_1 - i \frac{m}{2})\rho^\dagger_M u^M \\ -(\hat{p}_1 + i \frac{m}{2})\rho^M u_M & 0 & W_- & -\hat{p}_0 \ddagger \\ 0 & -(\hat{p}_1 + i \frac{m}{2})\rho^\dagger_M u^M & -\hat{p}_0 \ddagger & -W_- \end{pmatrix}$$

where $W_\pm = \frac{r}{2} (\hat{p}_0^2 \pm i \hat{p}_1^2) \rho^M u_M, |r| = 1$, and $\hat{p}_\mu \equiv \frac{2}{a} \sin \frac{p_\mu a}{2}$, leads to

$$\Gamma_{\text{LAT}}^{(1)} = \frac{V_2}{2a^2} \int_{-\pi}^{+\pi} \frac{d^2 p}{(2\pi)^2} \ln \left[ \frac{4^8 (\sin^2 \frac{p_0}{2} + \sin^2 \frac{p_1}{2})^5 (\sin^2 \frac{p_0}{2} + \sin^2 \frac{p_1}{2} + \frac{M^2}{8})^2 (\sin^2 \frac{p_0}{2} + \sin^2 \frac{p_1}{2} + \frac{M^2}{8})^2 (\sin^2 \frac{p_0}{2} + 4 \sin^2 \frac{p_1}{2} + \frac{M^2}{8})^2}{(\sin^2 p_0 + \sin^2 p_1 + \frac{M^2}{4} + 4 \sin^4 \frac{p_0}{2} + 4 \sin^4 \frac{p_1}{2})^8} \right]$$

$$\overset{a \to 0}{\longrightarrow} -\frac{3 \ln 2}{8\pi} V_2 m^2 \ , \ \text{cusp anomaly at strong coupling} \quad (|r| = 1, M = m a.)$$
Parameter space, continuum limit ($a \to 0$)

- Two bare parameters, $g = \frac{\sqrt{\lambda}}{4\pi}$ and $P^+ \sim m$, assume the only additional scale is $a$

$$F_{\text{LAT}} = F_{\text{LAT}}(g, M, N) \quad M = m \ a, \quad N = \frac{L}{a}$$

- The continuum limit must be taken along a line of constant physics: curve in $\{g, M, N\}$ where physical quantities are kept fixed as $a \to 0$.

E.g. 

$$m_x^2 = \frac{m^2}{2} \left(1 - \frac{1}{8g} + \mathcal{O}(g^{-2})\right)$$

$$L^2 m_x^2 = \text{const} \quad \rightarrow \quad (Lm)^2 \equiv (NM)^2 = \text{const}.$$ 

- For a generic observable

$$F_{\text{LAT}} = F_{\text{LAT}}(g, M, N) = F(g) + \mathcal{O}\left(\frac{1}{N}\right) + \mathcal{O}\left(e^{-MN}\right)$$

Recipe: fix $g$, fix $MN$ large enough, evaluate $F_{\text{LAT}}$ for $N = 6, 8, 10, 12, 16, \ldots$; Obtain $F(g)$ extrapolating to $N \to \infty$. 
Measurement I: $\langle x, x^* \rangle$ correlator

From the correlator of the $x$ fields

$$C_x(t; 0) = \sum_{s_1, s_2} \langle x(t, s_1) x^*(0, s_2) \rangle$$

$$t \gg 1 \quad e^{-t m_x \text{LAT}}$$

Consistent with large $g$ prediction, no clear signal of bending down.
No infinite renormalization occurring.
Measurement II: (derivative of the) cusp anomaly

We measure \( \langle S_{\text{cusp}} \rangle \equiv g \frac{V_2 m^2}{8} f'(g) \). At large \( g \),

\[
\langle S_{\text{LAT}} \rangle \equiv g \frac{N^2 M^2}{4} \left( 4 + \frac{c}{2} (2N^2) \right)
\]

quadratic divergences appear, with \( c = n_{\text{bos}} = 8 + 17 = 25 \).

Indeed, \( \langle S \rangle = -\frac{\partial \ln Z}{\partial \ln g} \) and \( Z \sim \Pi_{n_{\text{bos}}} \left( \det g \mathcal{O} \right)^{-\frac{1}{2}} \), so for each bosonic species there is a factor \( \sim g - \frac{(2N^2)}{2} \). In lattice codes, coupling omitted from fermionic part.
Measurement II: (derivative of the) cusp anomaly

We measure \( \langle S_{\text{cusp}} \rangle \equiv g \frac{V_2 m^2}{8} f'(g) \). At finite \( g \),

\[
\langle S_{\text{LAT}} \rangle \equiv g \frac{N^2 M^2}{4} f'_{\text{LAT}}(g) + \frac{c(g)}{2} (2N^2)
\]

In continuum, existing power divergences are set to zero (dim. reg.)

Here, expected mixing of the Lagrangian with lower dimension operator

\[
\mathcal{O}(\phi(s)) = \sum_{\alpha : [\mathcal{O}_\alpha] \leq D} Z_\alpha \mathcal{O}_\alpha(\phi(x)), \quad Z_\alpha \sim \Lambda^{(D-[\mathcal{O}_\alpha])} \sim a^{-(D-[\mathcal{O}_\alpha])}
\]
Recent simulations

- The fermionic correlator shows a linear ($\sim N, 1/\alpha$) divergence in the fermionic masses (in continuum $m_\pi^2 = \frac{m}{2}$ are protected).
  Strict analogy with lattice QCD in the case of Wilson fermions: for quark masses a power (linear) divergence in the lattice spacing appears, related to the fact that the lattice action for fermions breaks chiral symmetry.
  Our case: easy to relate to $\langle x \rangle \neq 0$, namely to the $SO(2)$ symmetry breaking by our Wilson-like fermions. We expect to cure it via additive renormalization, fine-tuning a critical mass imposing the corresponding Ward identity.

- We are extending the window of simulations to $g \leq 5$, where sign problem starts to be severe and instabilities appear, due to the zero eigenvalues of $O_F$.
  Simulations are done with a modified, positive-definite fermionic determinant 
  \begin{equation}
  (\det (\hat{O}_F \hat{O}_F^+) + \mu^2)^\frac{1}{4}
  \end{equation}
  (replacing the Pfaffian by its absolute value and adding a "twisted mass", infrared regulator standard in lattice QCD)

  \begin{align*}
  \langle O \rangle &= \frac{\langle OW \rangle_m}{\langle W \rangle_m}, \\
  W &= W^s W^\mu, \\
  W^s &= \text{sign Pf } \hat{O}_F \\
  W^\mu &= \frac{(\det \hat{O}_F \hat{O}_F)^\frac{1}{4}}{(\det(\hat{O}_F \hat{O}_F + \mu^2))^\frac{1}{4}}
  \end{align*}

  We notice that the sign-reweighting seems practically not to have effect on the measured observables. Indicates deeper motivation for being sign-problem free?
On the CFT side

Strong sign problem at strong coupling \((\lambda \gg 1)\), one tuning.
The control is in the perturbative region (matching with NNLO).

David Schaich, "Progress and prospects of lattice supersymmetry", talk at Lattice 2018
Conclusions

Addressed the problem of quantizing the superstring sigma-model in a non-trivial background at any value of the coupling.

Solving a non-trivial 4d QFT is hard \( \rightarrow \) reduce the problem via AdS/CFT:

solve (finding a good regulator for) a non-trivial 2d QFT.

I presented a study of lattice field theory methods for gauge-fixed string \( \sigma \)-models relevant in AdS/CFT: address ab initio, non-perturbative calculations within them.

- The model – GS string on GKP vacuum – is amenable to study using standard techniques (Wilson-like fermion discretizations, RHMC algorithm).
- We observe good agreement with expectation at large \( g \), and indications of non-perturbative physics;

Ongoing work on several open questions, which include the proper continuum limit.
Thanks for your attention.