Title: Non-invertible anomalies and Topological orders

Speakers: Wenjie Ji

Series: Condensed Matter

Date: November 19, 2019 - 3:30 PM

URL: http://pirsa.org/19110125

Abstract:

It has been realized that anomalies can be classified by topological phases in one higher dimension. Previous studies focus on 't Hooft anomalies of a theory with a global symmetry that correspond to invertible topological orders and/or symmetry protected topological orders in one higher dimension. In this talk, I will introduce an anomaly that appears on the boundaries of (non-invertible) topological order with anyonic excitations [1]. The anomalous boundary theory is no longer invariant under a re-parametrization of the same spacetime manifold. The anomaly is matched by simple universal topological data in the bulk, essentially the statistics of anyons. The study of non-invertible anomalies opens a systematic way to determine all gapped and gapless boundaries of topological orders, by solving simple eigenvector problems. As an example, we find all conformal field theories (CFT) of so-called ``minimal models'', except four cases, can be the critical boundary theories of $\mathbb{Z}_2$ topological order (toric code). The matching of non-invertible anomaly have wide applications. For example, we show that the gapless boundary of double-semion topological order must have central charge $c_L=c_R \geq 25/28$. And the gapless boundary of the non-Abelian topological order described by $S_3$ topological quantum field theory can be three-state Potts CFT, $\text{su}(2)_4$ CFT, etc. [1] WJ, Xiao-Gang Wen, arXiv: 1905.13279, Phys. Rev. Research 1,033054
Non-invertible Anomalies and Topological Order

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Perimeter Institute, Waterloo 2019

[WJ, Xiao-Gang Wen, Phys. Rev. Research 1, 033054]
Acknowledgment

Xiao-Gang Wen (MIT)      Shu-Heng Shao (IAS)
\( d \)-dimensional \( \leftrightarrow \) \( d + 1 \)-dimensional

Boundary states ? topological phases
Any additional properties = Anomaly

$H_{\text{bdy}} = i \int dx \psi^\dagger \partial_x \psi$

**Bulk**  Topological orders without anyons

**Boundary**  e.x. Integer quantized electric Hall conductance
Chiral Luttinger liquid

Bulk Topological orders with anyons

Boundary e.x. Fractional quantized electric Hall conductance
thermal Hall conductance
Boundary $\leftrightarrow$ Bulk

No thermal Hall conductance  non-chiral topological order

anything special ?  with anyons
Application

Boundary $\leftrightarrow$ Bulk

Gapped_1, Gapped_2, \ldots
CFT_1, CFT_2, \ldots
\hline
two gapped boundaries
all unitary CFTs
with central charge $c_L = c_R < 1$,
except four cases

Anyon models

Toric code
**Application**

Boundary of anyon model $\leftrightarrow$ Purely 1D

Stability? Different most relevant perturbation

<table>
<thead>
<tr>
<th>Ising CFT</th>
<th>Transverse Ising model</th>
</tr>
</thead>
<tbody>
<tr>
<td>on boundary of toric code</td>
<td>at critical point</td>
</tr>
<tr>
<td>Majorana mass term</td>
<td>spin operator</td>
</tr>
<tr>
<td>scal. dim. = 1</td>
<td>scal. dim. = $\frac{1}{8}$</td>
</tr>
</tbody>
</table>
Purely 1D system

Low energy description?

**Gapped** Count states in the ground states

**Gapless/critical** Conformal field theory (CFT)

- spin-$\frac{1}{2}$ Heisenberg model $\rightarrow SU(2)$ CFT $c = 1$
- Transverse Ising model $\rightarrow$ Ising CFT $\mathcal{M}(3, 4) \quad c = \frac{1}{2}$

predict specific heat $c_T = \frac{1}{2}$ in certain unit

Tricritical Ising model $\rightarrow$ Tricritical Ising CFT $\mathcal{M}(4, 5) \quad c = \frac{7}{10}$

CFT predicts specially discrete values of specific heat for 1d critical models.
1D gapless/ critical system

Universal study of 1d critical system

Minimal models $\mathcal{M}(p, p + 1)$

<table>
<thead>
<tr>
<th>$p$</th>
<th>$c$</th>
<th>lattice model</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$\frac{1}{2}$</td>
<td>Ising</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{7}{10}$</td>
<td>Tricritical Ising</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{4}{5}$</td>
<td>Tetracritical Ising, 3-state Potts</td>
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<tr>
<td>...</td>
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How are they determined?
1+1 d gapless/ critical system

**Power** Complete spectrum solved, given by partition function on a torus.

\[
Z(\tau, \bar{\tau}) = \text{Tr} \ e^{- (\text{Im} \tau H - i \text{Re} \tau P)} = \sum_{|\phi_i\rangle} \langle \phi_i | e^{- (\text{Im} \tau \epsilon_i - i \text{Re} \tau p_i)} |\phi_i\rangle
\]
1+1d CFT on a torus $\tau$

Re-parametrize the same torus, pick a different spacetime unit cell

\[ Z(\tau, \bar{\tau}) \]

\[ \mathcal{T} : Z(\tau, \bar{\tau}) \rightarrow Z(\tau + 1, \bar{\tau} + 1) \]

\[ S : Z(\tau, \bar{\tau}) \rightarrow Z \left( -\frac{1}{\tau}, -\frac{1}{\bar{\tau}} \right) \]

\[ Z(\tau + 1, \bar{\tau} + 1) = Z(\tau, \bar{\tau}) \quad Z(-1/\tau, -1/\bar{\tau}) = Z(\tau, \bar{\tau}) \]

$\Rightarrow$ Modular invariant
What modular invariance can do?

Minimal models $\mathcal{M}(p, p + 1)$

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<td>:</td>
</tr>
</tbody>
</table>
Boundary of anyon models

Proper description: Vector of partition functions

Bulk  Boundary
Different anyon  Different excitations
$\mathbb{Z}_2$ topological order on lattice – Toric code

\[ H = - \sum_P g_P - \sum_S g_S \]

\[ \sigma^x_i \]

\[ \sigma^z_i \]
\( \mathbb{Z}_2 \) topological order / Toric code

**Bulk simple topological data**

- anyon \( i = 1 \ e \ m \ f \)  
- self & mutual statistics  
  \[ T^{\text{top}}_{ij} = \delta_{ij} \bigcirc_{/i} \bigcirc_{j} \]  
  \[ S^{\text{top}}_{ij} = \bigodot_{/i} \bigodot_{j} \]

\[
T_{\mathbb{Z}_2} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}
\]

\[
S_{\mathbb{Z}_2} = \frac{1}{2} \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1
\end{bmatrix}
\]
Effective boundary Hamiltonian of toric code

Toric code boundary Hamiltonian = Transverse Ising model

Bulk: vacuum

Boundary: Start with \( m \)-condensed boundary \( e \sim f \)
Effective Hamiltonian of energy gap \( U + \) hop around

\[
H = -\frac{U}{2} \sum_j \sigma_j^z - J \sum_j \sigma_j^x \sigma_{j+1}^x - \epsilon_0 L
\]

\[
\sigma_i^z = \begin{cases} 
1 & \text{empty} \\
-1 & \text{occupied by } e
\end{cases}
\]
Effective boundary Hamiltonian of toric code

Bulk: vacuum

 Boundary: Start with $m$-condensed boundary

Effective Hamiltonian of $e$ energy gap $U$ + hop around

$$H = -\frac{U}{2} \sum_j \sigma_j^z - J \sum_j \sigma_j^x \sigma_{j+1}^x - \epsilon_0 L$$

$$\sigma_i^z = \begin{cases} 
1 & \text{empty} \\
-1 & \text{occupied by } e 
\end{cases}$$

Global constraint

Total number of $e$ is even

Boundary condition

$$\prod_j \sigma_j^z = 1$$

$$\sigma_{N+1}^x = \sigma_1^x$$
Effective boundary Hamiltonian of toric code

Bulk: e-sector

Boundary:
- Global constraint
- Total number of e is odd
- Boundary condition

\[
\prod_j \sigma_j^z = -1
\]

\[
\sigma_{N+1}^x = \sigma_1^x
\]
Effective boundary Hamiltonian of toric code

Bulk: m-sector

Boundary:

Global constraint

Number of e is even

Boundary condition

\[ \prod_j \sigma_j^z = 1 \]

\[ \sigma_{N+1}^x = -\sigma_1^x \]
Boundary of $\mathbb{Z}_2$ topological order

\[ H_{\text{bdy}} = -\frac{U}{2} \sum_j \sigma_j^z - J \sum_j \sigma_i^x \sigma_{i+1}^x \]

<table>
<thead>
<tr>
<th>Bulk</th>
<th>Boundary constraint</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Periodic b.c. $\mathbb{Z}_2$ even</td>
</tr>
<tr>
<td>$e$</td>
<td>Periodic b.c. $\mathbb{Z}_2$ odd</td>
</tr>
<tr>
<td>$m$</td>
<td>Anti-Periodic b.c. $\mathbb{Z}_2$ even</td>
</tr>
<tr>
<td>$f$</td>
<td>Anti-Periodic b.c. $\mathbb{Z}_2$ odd</td>
</tr>
</tbody>
</table>

available states:

- $e = (1, -1)$
- $m = (-1, 1)$

Bulk Anyon = ( $\mathbb{Z}_2$ flux, $\mathbb{Z}_2$ charge )

Boundary states = ( Bdy condition, charge )
Boundary: vector of partition function

Low energy partition function

\[ Z_{\text{anyon}} = \text{Tr}_{\mathcal{H}_a} e^{-\beta H_a} \]

Low temperature limit \( \beta \to \infty \) with fixed \( \frac{\beta}{L} \)

\[ |\phi\rangle \text{ is gapped} \implies e^{-\beta E_{|\phi\rangle}} \to 0 \]
\[ |\phi\rangle \text{ is gapless} \implies e^{-\frac{\beta}{L} E_{|\phi\rangle}} \]
**Boundary partition function of $\mathbb{Z}_2$ topological order**

\[
H_{\text{bdy}} = -\frac{U}{2} \sum_j \sigma_j^z - J \sum_j \sigma_j^x \sigma_{j+1}^x - \epsilon_0 L
\]

\[
Z_{\text{anyon } a} = \text{Tr}_{\mathcal{H}_a} e^{-\beta H_a}
\]

**Gapped m-condensed boundary** $|J| < \frac{U}{2}$

<table>
<thead>
<tr>
<th>Bulk $a$</th>
<th>Boundary $Z_a$</th>
<th>states</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>single ground state $\epsilon_0 = 0$</td>
</tr>
<tr>
<td>$e$</td>
<td>0</td>
<td>gapped excitation</td>
</tr>
<tr>
<td>$m$</td>
<td>1</td>
<td>condensed $\epsilon_m = \epsilon_0$</td>
</tr>
<tr>
<td>$f$</td>
<td>0</td>
<td>gapped excitation</td>
</tr>
</tbody>
</table>

\[
Z^{\text{m-condensed}} = \begin{bmatrix}
1 \\
0 \\
1 \\
0
\end{bmatrix}
\]

“Smooth boundary”

[Kitaev-Kong '12]
Boundary partition function of $\mathbb{Z}_2$ topological order

$$H_{\text{bdy}} = -\frac{U}{2} \sum_j \sigma_j^z - J \sum_j \sigma_j^x \sigma_{j+1}^x - \epsilon_0 L$$

$$Z_{\text{anyon } a} = \text{Tr}_{\mathcal{H}_a} e^{-\beta H_a}$$

**Gapped boundaries**

- $|J| < \frac{U}{2}$
- $J > \frac{U}{2}$

$$Z^{\text{m-condensed}} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad Z^{\text{e-condensed}} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

“Smooth edge” \quad “Rough edge”

[Kitaev-Kong '12]
Boundary partition function of $\mathbb{Z}_2$ topological order

$$Z_{\text{anyon } a} = \text{Tr}_{H_a} e^{-\beta H_a}$$

Gapless boundaries $J = \frac{U}{2}$

Rough answer: transverse Ising model at critical point = Ising CFT

What is $Z_a$?
Boundary partition function of $\mathbb{Z}_2$ topological order

Gapless boundaries

$$H_{\text{bdy}} = -\sum_j (\sigma_j^z + \sigma_j^x \sigma_{j+1}^x) - \epsilon_0 L$$

$$Z_{\text{anyon } a} = \text{Tr}_{\mathcal{H}_a} e^{-\beta H_a}$$

- **Gapless excitations** vacuum $\sigma$ $\psi \bar{\psi}$ $\mu$ $\psi$ $\bar{\psi}$

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<thead>
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<th>Boundary constraint</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>P.b.c. $\mathbb{Z}_2$ even</td>
<td>vacuum &quot;0&quot;, $\psi \bar{\psi}$</td>
</tr>
<tr>
<td>$e$</td>
<td>P.b.c. $\mathbb{Z}_2$ odd</td>
<td>$\sigma$</td>
</tr>
<tr>
<td>$m$</td>
<td>AP.b.c. $\mathbb{Z}_2$ even</td>
<td>$\mu$</td>
</tr>
<tr>
<td>$f$</td>
<td>AP.b.c. $\mathbb{Z}_2$ odd</td>
<td>$\psi$, $\bar{\psi}$</td>
</tr>
</tbody>
</table>

[Levin-Wen '03, unpublished, Chen-Jian-Kong-You-Zheng '19]
Vector of partition functions

**Bulk** Toric code

**Gapless boundaries** Ising CFT

\[
\begin{bmatrix}
Z_1 \\
Z_e \\
Z_m \\
Z_f
\end{bmatrix}
= 
\begin{bmatrix}
|\chi_1|^2 + |\chi_\psi|^2 \\
|\chi_\sigma|^2 \\
|\chi_\mu|^2 \\
\chi_\psi \bar{\chi}_1 + \chi_1 \bar{\chi}_\psi
\end{bmatrix}
\]

**Gapped boundaries**

\[
Z^{e-\text{cond}} = [1 1 0 0]^T \\
Z^{m-\text{cond}} = [1 0 1 0]^T
\]

A vector of partition functions describe various boundaries of anyon model. vector index = bulk anyon
Construct case by case?

**Boundary** \(\leftrightarrow\) **Bulk**

- More CFTs if add 4-spin interactions?
- \(U(1)\) CFT?
- Toric code
  \[
  K = \begin{bmatrix}
  0 & 2 \\
  2 & 0
  \end{bmatrix}
  \]

Look for

(i) A **schematic** way, given a TO/CFT pair, check if there is a solution of vector of partition function.

(ii) **Independent of particular microscopic construction**

(iii) Something **universal** about boundary/anyonic bulk correspondence?
Hint from Ising CFT ⇔ Toric code

Under modular transformation?

\[ Z_a(\tau, \bar{\tau}) = \sum_{ij} \chi_i(\tau) \, M^a_{ij} \, \bar{\chi}_j(\bar{\tau}) \]

\[
\chi_i(\tau + 1) = T^\text{CFT}_{ij} \, \chi_j(\tau) \\
T^\text{ls} = e^{-i \frac{2\pi}{24}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i2\pi \frac{1}{2}} & 0 \\ 0 & 0 & e^{i2\pi \frac{1}{16}} \end{bmatrix}
\]

\[ Z_a(\tau + 1, \bar{\tau} + 1) = \sum_{ij} \bar{\chi}_i(\bar{\tau}) \, \tilde{M}^a_{ij} \, \chi_j(\tau) \]

\[ \tilde{M}^a_{ij} = T^\text{CFT} \, M^a_{ij} \, T^\text{CFT} \]
Hint from Ising CFT $\leftrightarrow$ Toric code

Under modular transformation?

$$Z_{\text{Ising}}^a(\tau + 1, \bar{\tau} + 1) = T^{\text{toric code}}_{ab} Z_b(\tau, \bar{\tau})$$
Hint from Ising CFT $\leftrightarrow$ Toric code

Under modular transformation?

\[
Z_{\text{Ising}}(\tau + 1, \bar{\tau} + 1) = T_{\text{toric code}}^{ab} Z_b(\tau, \bar{\tau}) \\
Z_{\text{Ising}}(-1/\tau, -1/\bar{\tau}) = S_{\text{toric code}}^{ab} Z_b(\tau, \bar{\tau})
\]

Boundary $\leftrightarrow$ Bulk

\[
Z_a(\tau) \quad \text{universal anyon data}
\] under modular transformation
How general? Gapped boundaries

Gapped boundaries
\[ Z^{e-\text{cond}} = [1 \ 1 \ 0 \ 0]^T \quad Z^{m-\text{cond}} = [1 \ 0 \ 1 \ 0]^T \]

\[ T^{\text{toric code}}_{ab} Z_b = Z_a \]
\[ S^{\text{toric code}}_{ab} Z_b = Z_a \]

Only two independent eigenvectors
How general? A Luttinger liquid boundary

**Bulk** Toric code/$\mathbb{Z}_2$ topological order

described by $U(1)$ Chern-Simons theory $K = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$

**Boundary** $U(1)$ CFT at level 4 $c = 1$ primaries $l = 0, 1, 2, 3$

$\begin{bmatrix} Z_1 \\ Z_e \\ Z_m \\ Z_f \end{bmatrix} = \begin{bmatrix} |\chi_0|^2 + |\chi_2|^2 \\ |\chi_1|^2 + |\chi_3|^2 \\ \chi_1 \bar{\chi}_3 + \chi_3 \bar{\chi}_1 \\ \chi_0 \bar{\chi}_2 + \chi_2 \bar{\chi}_0 \end{bmatrix}$

$Z^{(1)}_{\tau + 1, \bar{\tau} + 1} = T_{\text{toric code}}^{ab} Z_b(\tau, \bar{\tau})$

$Z^{(1)}_{-1/\tau, -1/\bar{\tau}} = S_{\text{toric code}}^{ab} Z_b(\tau, \bar{\tau})$
A 1 + 1d theory with non-invertible anomaly

1. defined on a space-time torus $\tau$, it has a multi-component partition function

\[ Z_\alpha(\tau, \bar{\tau}) \]

2. Under torus re-parametrization $T : \tau \rightarrow \tau + 1$, $S : \tau \rightarrow -\frac{1}{\tau}$, $Z_\alpha(\tau, \bar{\tau})$ transform covariantly according to 2+1d topological data 

\{anyon $a$\} anyon self-statistics $T^{\text{top}}$ mutual statistics $S^{\text{top}}$

\[ T^{\text{top}}_{ab} Z_b(\tau, \bar{\tau}) = Z_b(\tau + 1, \bar{\tau} + 1) \]

\[ S^{\text{top}}_{ab} Z_b(\tau, \bar{\tau}) = Z_b \left( -\frac{1}{\tau}, -\frac{1}{\bar{\tau}} \right) \]

Non-invertible anomaly = canceled by the 2 + 1d (non-invertible) topological order
An intuition about T transformation

Rotate $2\pi$ and glue back

Purely 1D ring nothing change
An intuition about T transformation

Rotate $2\pi$ and glue back

A boundary ring anony self-rotate and accumulate a phase
Construct case by case?

\begin{align*}
\text{Boundary} & \leftrightarrow \quad \text{Bulk} \\
\text{More CFTs if add 4-spin interactions?} & \quad \text{Toric code} \\
U(1) \text{ CFT?} & \quad K = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}
\end{align*}

Look for

(i) A \textbf{schematic} way, given a \text{TQFT/CFT} pair, check if there is a solution of vector of partition function.

(ii) \textbf{Independent of particular microscopic construction}

(iii) Something \textbf{universal} about boundary/anyonic bulk correspondence?
CFT/TO pair

\[ T_{ab}^{\text{top}} (T^{CFT})^*_{ij} M_{bj} = M_{ai} \]
\[ S_{ab}^{\text{top}} (S^{CFT})^*_{ij} M_{bj} = M_{ai} \]

Boundary $\leftrightarrow$ Bulk
Pick a CFT Fix an anyon model

Solve an eigenvector problem for $M_{ai}$, $a$ labels anyon, $i$ labels primaries.
CFT/TO pair

Boundary $\leftrightarrow$ Bulk

Pick a CFT Fix an anyon model

Solve an eigenvector problem for $M_{ai}$, $a$ labels anyon, $i$ labels primaries.

\[
T^{\text{top}}_{\text{ab}} \ (T^{\text{CFT}})_{ij}^* \ M_{bj} = M_{ai}
\]

\[
S^{\text{top}}_{\text{ab}} \ (S^{\text{CFT}})_{ij}^* \ M_{bj} = M_{ai}
\]
More critical boundaries of toric code

Tricritical Ising $\mathcal{M}(4, 5)$ \hspace{1cm} $c_L = c_R = \frac{7}{10}$

\[
\begin{array}{cccccc}
 l & 1 & \sigma & \sigma' & \epsilon & \epsilon' & \epsilon'' \\
 h_f & 0 & \frac{3}{80} & \frac{7}{16} & \frac{1}{10} & \frac{3}{5} & \frac{3}{2}
\end{array}
\]

\[
\begin{pmatrix}
 Z_1 \\
 Z_e \\
 Z_m \\
 Z_f
\end{pmatrix} =
\begin{pmatrix}
 |\chi_0|^2 + |\chi_{\frac{1}{10}}|^2 + |\chi_{\frac{3}{5}}|^2 + |\chi_{\frac{3}{2}}|^2 \\
 |\chi_{\frac{7}{16}}|^2 + |\chi_{\frac{3}{80}}|^2 \\
 |\chi_{\frac{7}{16}}|^2 + |\chi_{\frac{3}{80}}|^2 \\
 \chi_0 \tilde{\chi}_{\frac{3}{2}} + \chi_{\frac{1}{10}} \tilde{\chi}_{\frac{3}{5}} + \chi_{\frac{3}{5}} \tilde{\chi}_{\frac{1}{10}} + \chi_{\frac{3}{2}} \tilde{\chi}_0
\end{pmatrix}
\]
All minimal models, except 4 cases
\( (\mathcal{M}(p, p + 1) \text{ with } p = 17, 18, 29, 30) \)
can be the gapless boundary of \( \mathbb{Z}_2 \) topological order.
Prediction

Stability: most relevant term in the vacuum sector $Z_1$.

\[
\begin{bmatrix}
Z_1 \\
Z_e \\
Z_m \\
Z_f
\end{bmatrix}
= 
\begin{bmatrix}
|\chi_1|^2 + |\chi_\psi|^2 \\
|\chi_\sigma|^2 \\
|\chi_\mu|^2 \\
\chi_\psi \bar{\chi}_1 + \chi_1 \bar{\chi}_\psi
\end{bmatrix}
\]

Compare with 1d Ising chain \( Z = |\chi_1|^2 + |\chi_\sigma|^2 + |\chi_\psi|^2 \)

In general more stable than CFT in purely 1D, since some excitations are projected out.
Example: Double semion topological order

**Bulk**

\[
\begin{array}{cccc}
  & i & 1 & s & s^* & b \\
\theta_i & 1 & e^{i2\pi\frac{1}{4}} & e^{-i2\pi\frac{1}{4}} & 1 \\
\end{array}
\]

\[
T^{\mathbb{Z}_2} = \begin{bmatrix}
  1 & 0 & 0 & 0 \\
  0 & i & 0 & 0 \\
  0 & 0 & -i & 0 \\
  0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
S^{\mathbb{Z}_2} = \frac{1}{2} \begin{bmatrix}
  1 & 1 & 1 & 1 \\
  1 & -1 & 1 & -1 \\
  1 & 1 & -1 & -1 \\
  1 & -1 & -1 & 1 \\
\end{bmatrix}
\]

**Boundary**

**Gapped** \( Z^T = [1 
0 
0 
1] \) boson condensed
Bulk

Effective theory $U(1)$ Chern-Simons theory with a $K = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$ matrix

Boundary $u(1)_2$  $c_L = c_R = 1$

\[
\begin{array}{ccc}
I & 0 & 1 \\
\hline \\
h_1 & 0 & \frac{1}{4}
\end{array}
\]

\[
\begin{pmatrix}
Z_1 \\
Z_s \\
Z_s^* \\
Z_b
\end{pmatrix} = 
\begin{pmatrix}
|\chi_0|^2 \\
\chi_1 \bar{\chi}_0 \\
\chi_0 \bar{\chi}_1 \\
|\chi_1|^2
\end{pmatrix}
\]

**Prediction**  No relevant perturbation, from $Z_1 = |\chi_0|^2$.

exists marginal perturbation in $|\chi_0|^2$
Boundaries of double semion topological order

For a RCFT to be the boundary theory of a topological order, it must have primary fields $J$ with

$$e^{i2\pi(h'_L-h'_R)} = \theta_j$$

For double semion bulk, no minimal model solution for $p < 7$. Gapless boundaries of 2+1D double semion topological order must have central charge

$$c \geq \frac{25}{28}$$
A basis transformation

Example: $\mathbb{Z}_2$ topological order

\[
Z^{ls} = \begin{bmatrix}
|x_1|^2 + |x_\psi|^2 \\
|x_\sigma|^2 \\
|x_\mu|^2 \\
x_\psi \bar{x}_1 + \chi_1 \bar{x}_\psi
\end{bmatrix}
\]

\[
M = \begin{bmatrix}
1 & 1 & 0 & 0 \\
1 & -1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 1 & -1
\end{bmatrix}
\]

\[
\rightarrow Z^{ls'} = \begin{bmatrix}
|x_1|^2 + |x_\sigma|^2 + |x_\psi|^2 \\
|x_1|^2 - |x_\sigma|^2 + |x_\psi|^2 \\
x_\psi \bar{x}_1 + \chi_1 \bar{x}_\psi + |x_\mu|^2 \\
-x_\psi \bar{x}_1 - \chi_1 \bar{x}_\psi + |x_\mu|^2
\end{bmatrix}
\]

\[
S' = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
T' = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]
A CFT with an anomaly free $\mathbb{Z}_2$ symmetry
$S', T'$ orbit
An inverse basis transformation

\[
\begin{bmatrix}
Z_1 \\
Z_e \\
Z_m \\
Z_f
\end{bmatrix} = M^{-1}
\]

**Take-away**  Given any modular invariant CFT with an anomaly free \( \mathbb{Z}_2 \) symmetry, there exists a modular covariant CFT as the boundary of \( \mathbb{Z}_2 \) topological order.
Minimal model $\mathcal{M}(p, p + 1)$

- All minimal models, except 4 cases\(^1\) can be the gapless boundary of $\mathbb{Z}_2$ topological order. The most stable one has central charge $c = \frac{1}{2}$.
- The critical 3-Potts model and tricritical 3-Potts model can be the gapless boundary of $\mathbb{Z}_3$ topological order and $S_3$ topological order.
- The gapless boundary of all other (untwisted) topological order with discrete gauge group $G$ has central charge $c \geq 1$. 
Non-invertible anomaly and Anyon models

\[ Z_a(\tau) \]

(flux, charge)

(Bdy condition, charge)

\[ Z_a(\tau + 1) = T_{ab}^{\text{top}} Z_b \]

\[ Z_a(-1/\tau) = S_{ab}^{\text{top}} Z_b \]
Non-invertible anomaly

**Good for**

- **Schematic**: TO/CFT pair through eigenvector equation
- **Universal**: $Z_2$ anomaly free CFT $\Rightarrow$ Toric code boundary

Boundary of anyon models are more stable
Open remarks

- Examples are known yet that a faithful set of gapless boundary theories requires the mapping class group transformation on higher genus?
- Lattice construction for gapped and gapless boundaries?
- Relation between invertible and non-invertible anomaly (to appear)