Title: Error correction with the color code

Speakers: Aleksander Kubica

Collection: Symmetry, Phases of Matter, and Resources in Quantum Computing

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Abstract: The color code is a topological quantum code with many valuable fault-tolerant logical gates. Its two-dimensional version may soon be realized with currently available superconducting hardware despite constrained qubit connectivity. In the talk, I will focus on how to perform error correction with the color code in $d \geq 2$ dimensions. I will describe an efficient color code decoder, the Restriction Decoder, which uses as a subroutine any toric code decoder. I will also present numerical estimates of the storage threshold of the Restriction Decoder for the triangular color code against circuit-level depolarizing noise.

Error correction with
the color code

Aleksander Kubica

work w/ N. Delfosse
work w/ C. Chamberland, T. Yoder, G. Zhu

arXiv:1905.07393
arXiv:1911.00355
Toward quantum computation

• We want to build a reliable, universal quantum computer.

• A path to fault-tolerant universal computation

  physical system & operations  →  quantum error correction  →  quantum algorithms
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[Images and diagrams: Kelly et al. '15; Corcoles et al. '15]
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  ![Physical system & operations](Kelsey et al.'15)
  ![Quantum error correction](Corcoles et al.'15)

  ![Quantum algorithms](diagram)
Toward quantum computation

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  physical system & operations  \rightarrow  quantum error correction  \rightarrow  quantum algorithms

- Desired properties of quantum error-correcting codes:
  - can be implemented in the lab,
  - easy fault-tolerant logical gates,
  - efficient decoders w/ high thresholds.
Decoding for stabilizer codes

- Stabilizer codes [G96]: commuting Pauli operators
code space = (+1)-eigenspace of stabilizers.
Decoding for stabilizer codes

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code space = \((+1)\)-eigenspace of stabilizers.

- Quantum error-correction game:

\[
|\psi\rangle \xrightarrow{\text{encode}} |\overline{\psi}\rangle \xrightarrow{\text{noise}} \mathcal{E}(\overline{\psi}) \xrightarrow{\text{recovery}} \mathcal{R} \circ \mathcal{E}(\overline{\psi}) \xrightarrow{\text{read off}} |\psi'\rangle
\]

move outside
the code space

Gottesman'96
Decoding for stabilizer codes

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\[ |\psi\rangle \xrightarrow{\text{encode}} |\overline{\psi}\rangle \xrightarrow{\text{noise}} |\overline{E(\overline{\psi})}\rangle \xrightarrow{\text{recovery}} R \circ E(|\overline{\psi}\rangle) \xrightarrow{\text{read off}} |\psi'\rangle \]

move outside
the code space

measure stabilizers to
discretize and diagnose errors

• Decoding = classical algorithm to find error correction from syndrome.

Gottesman'96
**Topological quantum codes**

- **Topological quantum codes:**
  - geometrically local generators,
  - logical info encoded non-locally.
Topological quantum codes

- Topological quantum codes:
  - geometrically local generators,
  - logical info encoded non-locally.

- Examples: toric code & (gauge) color code.

- Locality comes at a price — limitations and no-go theorems!
Why color code?

- Leading approach to scalable q. computing — **2D toric code**.

- Difficulty: **fault-tolerant non-Clifford** gate (needed for universality).

- **Color code** as an alternative to toric code
  - 😊 more qubit efficient,
  - 😊😊 transversal gates — $Z(\pi/2^d)$ rotation in $d$ dim \([B15,KB15]\),
  - 😊😊😊 avoiding magic state distillation \([B15,B18,BKS]\),
Why color code?

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- **Color code** as an alternative to toric code
  - more qubit efficient,
  - transversal gates — $Z(\pi/2^d)$ rotation in $d$ dim $[B15,KB15]$,
  - avoiding magic state distillation $[B15,B18,BKS]$,

- **Unfortunately**, color code
  - seems challenging to decode,
  - seems to perform worse than toric code.

Bombin’15; Kubica&Beverland’15; Beverland et al. (in prep.); Bombin’18
Main results & outline

**Results:** efficient decoders for color code in $d \geq 2$ dim w/ high thresholds.
Main results & outline

Results: efficient decoders for color code in $d \geq 2$ dim w/ high thresholds.

1. Intro: toric & color codes in 2D.

2. Restriction Decoder: color code decoding by using toric code decoding.
2D toric code & decoding

- For CSS codes, we can correct correct X and Z errors separately.

- **2D toric code** [K97]:
  - Z-errors = 1D strings,
  - violated X-stabilizers = 0D points.
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- Global $\mathbb{Z}_2$ symmetry:
  excitations created in **pairs**.

- **Decoding** = finding position of errors
  from violated stabilizers = pairing up excitations!

Kitaev’97; Dennis et al.’02; Harrington’04; Duclos-Cianci&Poulin’10; Bravyi et al.’14; Torlai&Melko’16
**2D toric code & decoding**

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- **Successful decoding** iff error and correction differ by stabilizer.

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2D color code

- **Lattice**: triangles, 3-colorable vertices.
- **2D color code [BM06]**:
  - qubits = triangles,
  - stabilizers = X- & Z-vertices.
- Color and toric codes related [KYP15]…

Bombin & Martin-Delgado ’06; Kubica et al.’15; Wang et al.’10; Landahl et al.’11
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- Color and toric codes related [KYP15]…
- …but decoding seems to be challenging
  [WFHH10, LAR11] as excitations created
  in **pairs** & in **triples**!

Bombin&Martin-Delgado’06; Kubica et al.’15; Wang et al.’10; Landahl et al.’11
From toric to color code decoder

- Setup:
  - error
  - syndrome
  - 0D
  - 1D
  - TC decoder
  - [BDP12,D14]

- Two notions: **restricted lattice** $L_{RG}$ and **restricted syndrome** $s_{RG}$.

- Restriction Decoder:
From toric to color code decoder

- **Setup:**
  - error 2D
  - local lift
  - syndrome 0D
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- **Restriction Decoder:**
  1. Use toric code decoder for $\mathcal{L}_{RG}$ and $s_{RG}$ to find blue pairings. Repeat for $\mathcal{L}_{RB}$ and $s_{RB}$ to find green pairings.

Bombin et al.'12; Delfosse’14
From toric to color code decoder

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- **Restriction Decoder:**
  1. Use toric code decoder for $L_{RG}$ and $s_{RG}$ to find blue pairings. Repeat for $L_{RB}$ and $s_{RB}$ to find green pairings.
  2. For any $R$ vertex $v$ find neighboring faces $f(v)$, whose boundary locally matches blue/green pairings.

Bombin et al.'12; Delfosse'14
Comments on the Restriction Decoder

- Any toric code decoder can be used as a subroutine!

- Restriction Decoder threshold determined by toric code threshold!

- Improvements of Restriction Decoder over decoding by projection $[D14]$:
  - never aborts,
  - two (vs. three) restricted lattices,
  - local (vs. global) lift procedure,
  - can be generalized to $d \geq 2$ dim.

- Various modifications, e.g., no need for restricted lattices. Adaptation to fracton models $[BW19]$ or q. pin codes $[VB19]$?

Delfosse’14; Brown&Williamson’19; Vuillot&Breuckmann’19
Numerics

- Dual of 4.8.8. lattice on a torus, phase-flip noise, ideal measurements.
- Color code threshold $\sim 10.2\%$ on a par w/ toric code MWPM $\sim 10.3\%$.

Sarvepalli&Raussendorf'12; Bombin et al.'12; Delfosse'14; Delfosse&Nickerson'17
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- Color code threshold $\sim 10.2\%$ on a par w/ toric code MWPM $\sim 10.3\%$.
- Efficient high-threshold decoders: 7.8% $\sim 8.7\%$ [SR12,BDP12,D14].
- For almost-linear time decoder, instead of MWPM use UF [DN17].

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Going beyond 2D

- **Restriction Decoder** for the $d$-dim color code on the lattice $\mathcal{L}$: toric code decoding on restricted lattices $\mathcal{L}_C$ + local lifting procedure.

- **Theorem 1**: the $k^{\text{th}}$ homology groups of the color code lattice $\mathcal{L}$ and the restricted lattice $\mathcal{L}_C$ are isomorphic.
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- **Restriction Decoder** for the $d$-dim color code on the lattice $\mathcal{L}$: toric code decoding on restricted lattices $\mathcal{L}_C$ + local lifting procedure.

- **Theorem 1**: the $k^{th}$ homology groups of the color code lattice $\mathcal{L}$ and the restricted lattice $\mathcal{L}_C$ are isomorphic.

- **Lemma**: morphism between color and toric code chain complexes

\[
\begin{align*}
C_{d-k-1}(\mathcal{L}) \xrightarrow{\partial_{d-k-1,d}} C_d(\mathcal{L}) & \xrightarrow{\partial_{d,k-1}} C_{k-1}(\mathcal{L}) \\
\downarrow \pi_C^{(2)} & \downarrow \pi_C^{(1)} & \downarrow \pi_C^{(0)} \\
C_{k+1}(\mathcal{L}_C) \xrightarrow{\partial_{k+1}^C} C_k(\mathcal{L}_C) & \xrightarrow{\partial_k^C} C_{k-1}(\mathcal{L}_C)
\end{align*}
\]

- $Z$-stabilizers $\rightarrow$ qubits $\rightarrow$ X-syndrome
More comments & numerics

• Efficient solution to the color code decoding problem:

\[ \text{d-dim color code} \rightarrow \text{d-dim toric code} \rightarrow \text{0D point-like excitations} \rightarrow \text{MWPM} \]

\[ \text{higher-dim excitations} \rightarrow \text{Sweep Decoder} \quad [KP19] \]

• Numerics for 3D bcc lattice, bit-/phase-flip noise, ideal measurements, two types of excitations:

0D point-like and 1D loop-like.

Kubica&Preskill’19; Kubica et al.’18
Realistic scenario

- A realistic setting:
  - a lattice w/ boundaries, e.g., triangular color code,
  - syndrome extracted via circuits w/ noisy components,
  - hardware-imposed limitations: 2D, connectivity,…

Tuckett et al.18; Maskara et al.’19
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- Restriction Decoder can adapted to circuit-level noise!

- Not so easy for high-threshold decoders based on tensor networks [TDCBBF18] or neural networks [MKJ19]!

Tuckett et al.18; Maskara et al.’19
Incorporating boundaries

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- Naive approach: two restricted lattices & lifting the boundary vertex.
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- Better way:
  — use three restricted lattices,
  — find connected components,
  — local lift not only for $R$ vertices.
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  - find connected components,
  - local lift not only for R vertices.

- For phenomenological noise:
  repeat stabilizer measurements and match excitations in (2+1)D.
Hardware implementation

- Superconducting qubit architecture:
  - 2D layout,
  - CNOTs between nearest-neighbor qubits.

- Data (white) and ancilla (colored) qubits on the hexagonal lattice w/ qubit connectivity = 3.

- For smaller connectivity — frequency collisions and cross-talk errors reduced!
Numerics

- Triangular color code & circuit-level depolarizing noise.

- Scaling of the logical error rate in the sub-threshold regime:
  \[ p_L \propto p^{(d-1)/2} \]

<table>
<thead>
<tr>
<th>code</th>
<th>connectivity</th>
<th>#qubits</th>
<th>threshold</th>
</tr>
</thead>
<tbody>
<tr>
<td>rotated surface</td>
<td>{4,4}</td>
<td>2d^2-1</td>
<td>0.7%</td>
</tr>
<tr>
<td>heavy hexagon</td>
<td>{12/5,3}</td>
<td>(5d^2-2d-1)/2</td>
<td>0.45% [CZYHC19]</td>
</tr>
<tr>
<td>heavy square</td>
<td>{8/3,4}</td>
<td>3d^2-2d</td>
<td>0.3% [CZYHC19]</td>
</tr>
<tr>
<td>triangular color</td>
<td>{3,3}</td>
<td>(3d^2-1)/2</td>
<td>0.2%</td>
</tr>
</tbody>
</table>

Chamberland et al.'19
Discussion

- Plug & play efficient **Restriction Decoder in \( d \geq 2 \) dim:
  color code decoder = toric code decoding + local lift.

- Restriction Decoder threshold \( \sim 10.2\% \)
  — better than efficient decoders for 2D color code,
  — on a par w/ 2D toric code MWPM \( \sim 10.3\% \).

- Adaptable to boundaries and circuit-level noise!