Abstract: I'll discuss elliptic cohomology from a physical perspective, indicating the importance of the Segal-Stolz-Teichner conjecture and joint work with D. Berwick-Evans on rigorously proving some of these physical predictions.
The de Rham model for elliptic cohomology

Arnav Tripathy
based on joint work with Dan Berwick-Evans

Harvard University

Elliptic Cohomology and Physics
Perimeter, May 2020
Table of Contents

1. What is elliptic cohomology?

2. The Segal-Stolz-Teichner conjecture

3. Life goes on
What is elliptic cohomology?

Following Grojnowski, Hopkins, Segal, Stolz-Teichner, Witten

- First, the elliptic genus.
- Given a 2d QFT, we have the partition function

\[
\text{Tr}_\mathcal{H} e^{\beta H} = \int e^{iS} d(\text{fields on a torus}) = Z(\tau, \bar{\tau})
\]

satisfying the crucial property\(^1\)

\[
Z(\tau, \bar{\tau}) = Z(-1/\tau, -1/\bar{\tau}).
\]

- Simplify: suppose at least \(\mathcal{N} = (0,1)\) supersymmetry and insert \((-1)^F\) to obtain

\[
\text{Tr}_\mathcal{H} (-1)^F e^{\beta H} = Z_{EG}(\tau).
\]

- Witten index argument: \(Z_{EG}(\tau)\) holomorphic, deformation invariant, i.e. a deformation-invariant modular form on \(SL_2(\mathbb{Z}) \backslash \mathbb{H}\).

\(^1\)Assume all anomalies vanish at present.
What is elliptic cohomology?

- So, $Z_{EG}(\tau)$ a genus (à la Hirzebruch for $\sigma$-models + more) valued in modular forms.
- Behavior in families? What is a “family of modular forms” over some base space $B$? Supersymmetric QM: $H^*(B; MF_\mathbb{C}) =: \text{Ell}(B)_\mathbb{C}$.
- Ok, suppose we want a refinement. Natural idea: simply use deformation classes of families of the 2d $\mathcal{N} = (0,1)$ theories themselves. [Segal, Stolz-Teichner]
- Other obvious idea: quotient the complex cobordism spectrum $MU$ by the formal group law induced from (families of) elliptic curves. [Landweber-Ravenel-Stong, Hopkins-Mahowald-Miller]
- **Conjecture:** These constructions agree. **Applications:** Manifold.
What **is** elliptic cohomology?

- **Conjecture:** These constructions agree. **Applications:** Manifold.

**Topological Vafa-Witten [Gukov-Pei-Putrov-Vafa]**

Consider the 6d $\mathcal{N} = (0, 2)$ theory. Compactifying on an elliptic curve yields 4d $\mathcal{N} = 4$, and a further twisted compactification on a four-manifold $M$ yields the modular form $VW_M(\tau)$. Reversing the order of compactification, the twisted compactification of the 6d theory on $M$ yields a 2d theory whose elliptic genus would return $VW_M(\tau)$. Segal-Stolz-Teichner predicts a canonical lift of said modular form to a **topological** modular form. If one instead has a family of such manifolds parametrized by some base-space $B$ (for example, $G$-symmetry), one obtains a class in $\text{Ell}_G(B)$. 
What is equivariant elliptic cohomology?

- First, the equivariant elliptic genus.
- Given a 2d QFT with $G$ flavor symmetry, we have the partition function with background gauge fields

$$\text{Tr}_{\mathcal{H}_g} h e^{\beta H} = \int_{\text{twisted b.c.s}} e^{iS} d(\text{fields on a torus}) = Z(\tau, \bar{\tau}, g, h)$$

satisfying the crucial property

$$Z(\tau, \bar{\tau}, g, h) = Z(-1/\tau, -1/\bar{\tau}, h^{-1}, g).$$

- Simplify: suppose at least $\mathcal{N} = (0, 1)$ supersymmetry and insert $(-1)^F$ to obtain

$$\text{Tr}_{\mathcal{H}_g} (-1)^F h e^{\beta H} = Z_{EG}(\tau, g, h).$$

- Witten index argument: $Z_{EG}(\tau, g, h)$ holomorphic, deformation invariant, i.e. a deformation-invariant equivariant modular form on $\text{SL}_2(\mathbb{Z}) \times G \backslash \mathbb{H} \times \mathbb{C}^2(G) =: \text{Bun}_G(\mathcal{E})$. 
What is equivariant elliptic cohomology?

- So, $Z_{EG}(\tau, g, h)$ a twisted, twined genus (à la Hirzebruch for $\sigma$-models + more) valued in equivariant modular forms.
- Behavior in families? What is a “family of modular forms” over some base space $B$? Supersymmetric QM: $H^*(B; MF_{G,\mathbb{C}}) =: \text{Ell}(B)_{G,\mathbb{C}}$. 
- Ok, suppose we want a refinement. Natural idea: simply use deformation classes of families of the 2d $\mathcal{N} = (0, 1)$ theories with $G$ flavor symmetry. [Segal, Stolz-Teichner]
- Other obvious idea: build a moduli space of derived algebro-geometric objects, oriented elliptic curves with equivariant structure. [Lurie, Gepner-Meier]
- Even more directly: build a family of algebras directly over $\text{Bun}_G(\mathcal{E})$, at least over $\mathbb{C}$. [Grojnowski]
- **Conjecture:** These constructions all agree.
Table of Contents

1. What is elliptic cohomology?
2. The Segal-Stolz-Teichner conjecture
3. Life goes on
The equivariant Segal-Stolz-Teichner conjecture

- **Conjecture:** Phases of 2d $\mathcal{N} = (0, 1)$ theories with $G$ flavor symmetry with some worldsheet torus $E$ and parametrized by a base-space $M$ yield a model for equivariant elliptic cohomology $\text{Ell}_G(M)$ (as defined in topology).
- Too hard to start with. Let’s try a 0-categorical, 0-chromatic height version first.
- **Conjecture:** The algebra of supersymmetric observables of the 2d $\mathcal{N} = (0, 1)$ $\sigma$-model to a $G$-manifold $M$, with background gauge fields turned on, yields $\hat{\text{Ell}}_G(M)_\mathbb{C}$ (as defined in Grojnowski, BE-T).
- Additional structure one could ask for:
  - Universal Euler classes in $\text{Ell}_{U(n)}(pt), \text{Ell}_{\text{Spin}(2n)}(pt)$ arising from $\mathcal{N} = (0, 1)$ free fermions in a (complex or real) representation. (Similar statement for Thom classes.) [Ando-Hopkins-Rezk]
  - Specializing the above to $U(1)$ intertwines the natural monoidal structures on both sides. [Ando-Hopkins-Strickland]
A theorem!

Theorem [Berwick-Evans–T]
For $G$ any compact Lie group and $M$ any compact $G$-manifold,

$$\mathcal{O}\left(\text{Maps}_0(\langle P \rangle E, [M/\Gamma G])\right) \sim \hat{\text{Ell}}_G(M).$$

Theorem [Berwick-Evans–T]
The function in the above model induced by $n$ gauged free fermions agrees with the universal elliptic Euler class of $\text{Ell}_{U(n)}(pt)$.

Theorem [Berwick-Evans–T]
The multiplicative structure on the universal elliptic Euler class $\sigma(\tau, z) \in \text{Ell}_{U(1)}(pt)$ induced from multiplying $U(1)$ gauge fields agrees with the (formal) elliptic group law defining elliptic cohomology.
Idea of proof

**Definition [Grojnowski]**
For $G$ a compact Lie group and $M$ a $G$-manifold, the fiber of $\text{Ell}_G(M)_\mathbb{C}$ at $(g_1, g_2) \in C^2(G)$ is

$$\left(\text{Ell}_G(M)_\mathbb{C}\right)_{(g_1, g_2)} := H^*(M(g_1, g_2); \mathbb{C})[\beta, \beta^{-1}].$$

**Example**
Consider $U(1)$ acting on $S^2$ by rotation, with fixed points the north and south poles. Then $\text{Ell}_{U(1)}(S^2)$ is a rank-two vector bundle over $\text{Bun}_{U(1)}(\mathcal{E})$.

- So, proof strategy: (i) understand local (super)geometry of $\text{Bun}_G(\mathcal{E})$, (ii) perform local calculation of the supersymmetric observables as de Rham cohomology, (iii) successfully glue together.
Table of Contents

1. What is elliptic cohomology?
2. The Segal-Stolz-Teichner conjecture
3. Life goes on
What is elliptic cohomology, physically?

- $\text{Ell}(M)$, a 2-category of boundary conditions of some 3d $\sigma$-model with target $M$. (*A priori*, needs 3d $\mathcal{N} = 4$ supersymmetry.)
- $\text{Ell}(M)$, phases of 2d $\mathcal{N} = (0, 1)$ theories parametrized by $M$.
- $\text{Ell}(M)_\mathbb{C}$, the BPS Hilbert space of a 3d $\mathcal{N} = 1$ $\sigma$-model to $M$ on a torus.
- $\text{Ell}(M)_\mathbb{C}$, the algebra of BPS observables for the 2d $\mathcal{N} = (0, 1)$ $\sigma$-model with torus worldsheet and target $M$. (And behavior of extended observables?)
- Compare to $K$-theory: boundary conditions for the $B$-model, phases of SQMs, BPS Hilbert space of 2d $\sigma$-model, algebra of BPS observables for SQMs. ($K_{\text{top}}$ from a category?)
- Why $K(M)_\mathbb{C}$ rather than $H^*(M; \mathbb{C})$ as the Hilbert space above? Discrete torsion. **Functoriality?**
So, what to attack next?

- Most obviously, return to the Segal-Stolz-Teichner conjecture but with increased chromatic height. We believe we have a model (cf. [Luecke]) for $KMF_G$ (cf. [Bunke-Naumann]), which fits in the square

\[
\begin{array}{ccc}
KMF_G & \rightarrow & K_{\text{Tate},G} \\
\downarrow & & \downarrow \\
TMF_{G,\mathbb{C}} & \rightarrow & K_{\text{Tate},G,\mathbb{C}}.
\end{array}
\]

- What about the $\sigma$-models above where the torus is replaced by a higher-genus surface? Enter $gll_G(M)$, which exists over $\mathbb{C}$ with necessarily poor integral properties but, for example, should contain the information of $Z_g(M)$ [Alvarez-Singer].

- M2-branes can end on M5-branes.
M-theory and TMF

- The D-brane charge lattice in type II string theories is most naturally $K$-theory (and type I, $KO$) as they represent boundary conditions for the fundamental string.
- Consider an F1-ending-on-D4 configuration in IIA and lift to M-theory to obtain an M2-ending-on-M5 configuration.
- Should the charge lattice of M5 branes most naturally be topological modular forms?
- Freed-Moore-Segal suggests TMF should then have some self-Pontryagin duality.
- Indeed, $Tmf_{\mathbb{C}}$ is self-dual with shift 21 by Serre duality and $\Delta(\tau)d\tau$ exhibiting $K_{\mathcal{M}1} \simeq \omega^{-10}$. [Stojanoska]
So, what to attack next?

- Most obviously, return to the Segal-Stolz-Teichner conjecture but with increased chromatic height. We believe we have a model (cf. [Luecke]) for $KMF_G$ (cf. [Bunke-Naumann]), which fits in the square

$$
\begin{align*}
KMF_G & \longrightarrow K_{\text{Tate},G} \\
\downarrow & \downarrow \\
TMF_{G,\mathbb{C}} & \longrightarrow K_{\text{Tate},G,\mathbb{C}}.
\end{align*}
$$

- What about the $\sigma$-models above where the torus is replaced by a higher-genus surface? Enter $gll_G(M)$, which exists over $\mathbb{C}$ with necessarily poor integral properties but, for example, should contain the information of $Z_g(M)$ [Alvarez-Singer].

- M2-branes can end on M5-branes.
Thank you!