Title: Type D quiver representation varieties, double Grassmannians, and symmetric varieties

Speakers: Jenna Rajchgot

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Abstract: Since the 1980s, mathematicians have found connections between orbit closures in type A quiver representation varieties and Schubert varieties in type A flag varieties. For example, singularity types appearing in type A quiver orbit closures coincide with those appearing in Schubert varieties in type A flag varieties (Bobinski-Zwara); combinatorics of type A quiver orbit closure containment is governed by Bruhat order on the symmetric group (follows from work of Zelevinsky, Kinser-R); and multiple researchers have produced formulas for classes of type A quiver orbit closures in equivariant cohomology and K-theory in terms of Schubert polynomials, Grothendieck polynomials, and related objects.

After recalling some of this type A story, I will discuss joint work with Ryan Kinser on type D quiver representation varieties. I will describe explicit embeddings which completes a circle of links between orbit closures in type D quiver representation varieties, B-orbit closures (for a Borel subgroup B of GL_n) in certain symmetric varieties GL_n/K, and B-orbit closures in double Grassmannians Gr(a, n) x Gr(b, n). I will end with some geometric and combinatorial consequences, as well as a brief discussion of joint work in progress with Zachary Hamaker and Ryan Kinser on formulas for classes of type D quiver orbit closures in equivariant cohomology.
Type D quiver representation varieties, double Grassmannians, and symmetric varieties

Jenna Rajchgot

University of Saskatchewan
Saskatoon, SK, Canada

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Determinantal varieties

Consider the affine space of matrices

\[ \text{Mat}(d_0, d_1) = \{ V \mid \mathbb{C}^{d_0} \xrightarrow{V} \mathbb{C}^{d_1} \} \]

Let \( X = (x_{ij}) \) be a \( d_0 \times d_1 \) matrix of variables.

- There is a natural (right) action of \( \text{GL}(d_0) \times \text{GL}(d_1) \) on \( \text{Mat}(d_0, d_1) \) by \( V \cdot (g_0, g_1) = g_0^{-1} V g_1 \).
- Orbits are determined by matrix rank \( r \).
- Orbit closures are determinantal varieties, defined by the prime ideals
  \[ l_r = \langle (r + 1 \times r + 1) \text{ minors of } X \rangle \subseteq \mathbb{C}[x_{ij}] \]
- Orbit closures normal and Cohen-Macaulay, have nice Gröbner bases, orbit closure containment understood, multigraded Hilbert series understood...
Quiver representations

- A quiver $Q$ is a finite directed graph.

- A representation of $Q$ is an assignment of vector space to each vertex and linear map to each arrow.

- A dimension vector $\mathbf{d} = (d_0, d_1, \ldots, d_n)$ for $Q$ assigns vector space $\mathbb{C}^{d_i}$ to vertex of $Q$ with label $i$. 
- A **representation space** $\text{rep}_Q(d)$ is the space of all representations for $Q$ with fixed dimension vector.
- $\mathbf{GL}(d) := \mathbf{GL}(d_0) \times \mathbf{GL}(d_1) \times \cdots \times \mathbf{GL}(d_n)$ acts on $\text{rep}_Q(d)$ by conjugation.
- A **quiver locus** is a $\mathbf{GL}(d)$ orbit closure.

**Example**

Let $Q = \bullet \rightarrow \bullet$, and let $d = (2, 3)$.
- $\text{rep}_Q(d) = 2 \times 3$ matrices.
- There are three $\mathbf{GL}(d) = \mathbf{GL}(2) \times \mathbf{GL}(3)$ orbits (determined by matrix rank):

$$
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \mathbf{GL}(d), \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \mathbf{GL}(d), \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \mathbf{GL}(d)
$$
- Quiver loci are determinantal varieties.
Dynkin quiver loci

- For $Q$ connected, $\text{rep}_Q(d)$ has finitely many orbits for all $d$ if and only if $Q$ has Dynkin type A, D, or E. (Gabriel’s Theorem)
- Orbit closures for Dynkin quivers $Q$ are determined set-theoretically by minors of certain somewhat complicated matrices. (Bongartz)

Example (Equioriented $A_4$)

$$\mathbb{C}^d \to \mathbb{C}^{d_1} \to \mathbb{C}^{d_2} \to \mathbb{C}^{d_3}$$

Orbits determined by ranks of: $V_1, V_2, V_3, V_1 V_2, V_2 V_3, V_1 V_2 V_3$

Example (Not Dynkin)

$$\mathbb{C}^d \to V \to \mathbb{C}^d$$

$\text{GL}(d)$ orbits characterized by Jordan canonical forms of $d \times d$ matrices (thus, infinitely many orbits).
A few motivations and connections

Motivations from commutative algebra and algebraic geometry

- up to radical, Dynkin quiver loci are generalized determinantal varieties, and include some classically studied varieties

- naturally arise in study of degeneracy loci of vector bundles

Another connection: representation theory of algebras

- The category \( \text{rep}(Q) \) of reps. of \( Q \) is equivalent to category of right modules over the path algebra \( kQ \).

- connections between geometry of quiver loci to rep. theory of \( Q \)

  Eg. Degeneration order

  - For \( V, W \in \text{rep}_Q(d) \), define \( V \leq_{\text{deg}} W \) iff \( \text{GL}(d) \cdot V \supseteq \text{GL}(d) \cdot W \).
  - \( V \leq_{\text{deg}} W \) iff \( \dim \text{Hom}_Q(V, X) \leq \dim \text{Hom}_Q(W, X) \) for all indecomposable reps. \( X \). (Bongartz)
Degeneracy loci:
Let $Y$ be a non-singular algebraic variety. Let $V$, $W$ be vector bundles over $Y$ of ranks $m, n$ and consider $\varphi : V \to W$. Let $r \leq \min(m, n)$.

- **degeneracy locus**: $Y_r = \{ y \in Y \mid \text{rank } \varphi_y \leq r \}$
- defined locally by vanishing of $(r + 1) \times (r + 1)$ minors of a matrix

To quiver loci:
- For general $\varphi$, the cohomology class $[Y_r]$ is expressible in terms of a Schur function. (Giambelli-Thom-Porteous formula)
- **Buch-Fulton**: Found an analogous formula for sequences of bundle maps $V_1 \to V_2 \to \cdots \to V_n$.
- **Knutson-Miller-Shimozono**: more formulas in the Buch-Fulton setting by computing *multidegrees* of equioriented type $A$ quiver loci (quiver polynomials)
- more formulas were subsequently found (in both cohomology and $K$-theory) in this and related settings (*Buch, Fehér, Kinser, Knutson, Kresch, Miller, R, Rimányi, Shimozono, Tamvakis, Yong*)
Part II: Type A quiver loci and Schubert varieties
Schubert varieties

Let $G = \text{GL}(n)$ and let $B_+ \leq G$ (resp. $B_- \leq G$) be subgroups of upper triangular (resp. lower triangular) matrices.

- **Flag variety:** $B_- \backslash G$
  This space is naturally identified with the space of complete flags: $0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq \mathbb{C}^n$, $\dim V_i = i$.

- **Schubert cell:** $B_+$-orbit in $B_- \backslash G$.

- **Schubert variety:** closure of a Schubert cell.

- Schubert cells and varieties are indexed by permutations in $S_n$.

**Woo-Yong, building on Kazhdan-Lusztig:** can study local properties (eg. singularities) via generalized determinantal varieties
Kazhdan-Lusztig varieties

The Kazhdan-Lusztig variety $X_{v,w}$ is a generalized determinantal variety associated to two permutations $v, w \in S_n$.

- $v$ determines a matrix of 0s, 1s, and variables.
- $w$ determines a collection of minors of the matrix assoc. to $v$.

Each Kazhdan-Lusztig variety is isomorphic, up to an affine space factor, to an open patch of the Schubert variety $X_w$.

Example

$$
\begin{pmatrix}
a & b & c & 1 \\
d & e & 1 & 0 \\
f & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 2 \\
1 & 2 & 2 & 3 \\
1 & 2 & 3 & 4 \\
\end{pmatrix}
$$

matrix from $v = 4321$ \quad $w = 1423$ \quad NW ranks of $w$

$$X_{v,w} = \mathbb{V} \left( \det \begin{bmatrix} a & b & c \\ d & e & 1 \\ f & 1 & 0 \end{bmatrix}, (2 \times 2) - \text{minors of} \begin{bmatrix} a & b & c \\ d & e & 1 \end{bmatrix} \right)$$
Type A quiver loci and Schubert varieties

\[ \text{rep}_Q(d) = \{(V_1, V_2, V_3) \mid \mathbb{C}^{d_0} \xrightarrow{V_1} \mathbb{C}^{d_1} \xrightarrow{V_2} \mathbb{C}^{d_2} \xrightarrow{V_3} \mathbb{C}^{d_3}\} \]

Orbits determined by ranks of: \(V_1, V_2, V_3, V_1V_2, V_2V_3, V_1V_2V_3\).

The equioriented Zelevinsky map is defined to be:

\[
\text{rep}_Q(d) \rightarrow \begin{bmatrix}
* & * & * & l_{d_0} \\
* & * & l_{d_1} & 0 \\
* & l_{d_2} & 0 & 0 \\
l_{d_3} & 0 & 0 & 0 \\
\end{bmatrix}, \quad (V_1, V_2, V_3) \rightarrow \begin{bmatrix}
0 & 0 & V_1 & l_{d_0} \\
0 & V_2 & l_{d_1} & 0 \\
V_3 & l_{d_2} & 0 & 0 \\
l_{d_3} & 0 & 0 & 0 \\
\end{bmatrix}
\]

- Equioriented type A quiver loci are isom. to open subvarieties of Schubert varieties (Musili-Seshadri, Zelevinsky, Lakshmibai-Magyar)

- Consequences for these quiver loci: prime defining ideals, singularity results, F-splitting, orbit closure containment via Bruhat order

- In general: combinatorial and algebro-geometric aspects of arbitrary type A quiver loci are governed by corresponding aspects of Schubert varieties (Bobiński-Zwara, Kinser-R)
Multigraded Hilbert series

Set-up:
- Let $S = \mathbb{C}[x_1, \ldots, x_n]$ be positively $\mathbb{Z}^d$-graded.
- Let $M$ be a finitely generated $\mathbb{Z}^d$-graded module over $S$.

Definition
The multigraded Hilbert series of $M$ is

$$H(M; t) = \sum_{a \in \mathbb{Z}^d} \dim_{\mathbb{C}}(M_a)t^a = \frac{K(M; t)}{\prod_{i=1}^n (1 - t^{a_i})}, \quad \deg(x_i) = a_i$$

- K-polynomial of $M$: the numerator $K(M; t)$
- multidegree of $M$: sum of lowest deg. terms of $K(M; 1 - t)$. 
Multigraded Hilbert series for determinantal varieties

\[ \text{Mat}_{m \times n}(\mathbb{C}) = \{ M | \mathbb{C}^m \rightarrow \mathbb{C}^n \} \].

- Have a (right) action of \((\mathbb{C}^\times)^m \times (\mathbb{C}^\times)^n\) on \(\text{Mat}_{m \times n}(\mathbb{C})\):
  \[ M \cdot (g_0, g_1) = g_0^{-1} M g_1. \]

- It induces a \(\mathbb{Z}^{m+n}\)-grading on the coordinate ring \(\mathbb{C}[\text{Mat}_{m \times n}(\mathbb{C})]\).

- The \(K\)-polynomial of the determinantal variety \(X_r \subseteq \text{Mat}_{m \times n}(\mathbb{C})\) is a double Grothendieck polynomial.

- The multidegree is a double Schubert polynomial.

This has been generalized, in multiple ways, to general type A quiver loci.

- **E.g:** the \(K\)-poly. of a type A quiver locus is a ratio of specialized double Grothendieck polynomials (Knutson-Miller-Shimozono, Kinser-Knutson-R)
Another generalization: type $A$ quiver component formulas

In 2005, Buch and Rimányi proved a positive multidegree formula for type $A$ quiver loci and conjectured an alternating $K$-polynomial formula.

**Theorem (Kinser-Knutson-R)**

The Buch-Rimányi conjecture is true.

**Proof sketch:** We reduce to the bipartite case where the goal is to show:

$$
K(\Omega; t, s) = \sum_{w \in KW(\Omega)} (-1)^{|w| - \text{codim}(\Omega)} G_w(t; s).
$$

We give a geometric proof of this combining:

- degenerations of bipartite quiver loci;
- a bipartite version of the Zelevinsky map (Kinser-R);
- properties of Kazhdan-Lusztig ideals;
- combinatorial arguments.

**Remark:** proofs in *equioriented* type $A$ came earlier
(Knutson-Miller-Shimozono, Buch, Miller, Yong)
5 minute break: summary slide

**General:**
- We are interested in quiver loci of Dynkin quivers a.k.a. closures of $\text{GL}(d)$-orbits of representations of Dynkin quivers.
- Our main motivation is to find explicit formulas for *quiver polynomials*.

**Type A:**
- Geometry and combinatorics of type A quiver loci is closely tied to that of Schubert varieties in type A flag varieties.
- Formulas for quiver polynomials are expressible in terms of well-known polynomials from Schubert calculus.
- Helpful idea: reduce all orientations of type A quivers to the bipartite (source-sink) orientation.

**After the break: type D**
Part III: Type D quiver representation varieties, double Grassmannians, and symmetric varieties
Type $D$ quiver loci

Let $G = GL(a+b)$ and $K = GL(a) \times GL(b)$ embedded as two blocks along the diagonal. Then $K \backslash G$ is a variety. It can be embedded in the double Grassmannian $Gr(a,n) \times Gr(b,n)$ as an open subvariety.

**Theorem (Kinser-R)**

*There is a series of links*

$\begin{align*}
\text{B-orbit closures in } K \backslash G & \iff \text{B-orbit closures in } Gr(a,n) \times Gr(b,n) \\
& \iff \text{GL(d)-orbit closures in } \text{rep}_Q(d), \text{ Q type D} \\
& \iff \text{B'-orbit closures in } K' \backslash G'
\end{align*}$

*each of which allows for transfer of various algebro-geometric and combinatorial properties from target to source.*

$\implies$ results on sings. (recovering work of Bobiński-Zwara), type $D$ poset as a subposet of clans poset (latter poset structure described by Wyser)

**Remark:** unlike in type $A$, don’t get prime defining ideals
Main idea

To a type $D$ quiver

$$Q = \begin{array}{c}
1 \\
3 \\
2 \\
4 \\
5 \\
6
\end{array}$$

with fixed dimension vector $d$, associate a (non-type $D$) bipartite quiver of the form

$$Q^* = \begin{array}{c}
x_0^1 \\
x_1^3 \\
y_0^3 \\
y_1^3 \\
x_2^4 \\
x_3^5 \\
y_2^4 \\
y_3^6 \\
\beta_0 & y_0^3 & \alpha_1 & x_0^1 & y_1^3 & \alpha_2 & x_1^3 & y_2^4 & \alpha_3 & x_2^4 & \beta_1 & y_3^6 & \alpha_1
\end{array}$$

and an associated dimension vector $d^*$ where $d^*(v^m) = d(m)$.

- Define the open subvariety $X_Q \subseteq \text{rep}_{Q^*}(d^*)$ where red arrows are invertible.
- Realize each quiver locus in $\text{rep}_Q(d)$, up to smooth factor, as an orbit closure in $X_Q$. 
Embed $X_Q$ into a slice $S(d^*)$ of $K \backslash G$:

$$S(d^*) = \begin{bmatrix}
\begin{array}{cccccccccc}
  y_0 & y_1 & y_2 & x_0 & x_1 & x_2 & y_1 & y_0 + y_0' & y_1' & y_2' \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  y_0 & y_1 & y_2 & x_0 & x_1 & x_2 & y_1 & y_0 + y_0' & y_1' & y_2' \\
\end{array}
\end{bmatrix}$$

$$V \mapsto \begin{bmatrix}
\begin{array}{cccccccccc}
  0 & 0 & V_{\alpha_1} & V_{\beta_0} & 0 & 0 & \vdots & \vdots & \vdots & \vdots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  0 & 0 & V_{\alpha_1} & V_{\beta_0} & 0 & 0 & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\end{bmatrix}$$

Identify each orbit closure in $X_Q$ with an open subvariety of $\overline{O} \cap S(d^*)$, for some $P$-orbit closure $\overline{O}$ in $K \backslash G$. Then show that the latter is isomorphic, up to smooth factor, to an open subvariety of $\overline{O}$. 
A very small example

\[ Q = y_0 \xleftarrow{x_1} y_1, \quad d(y_0) = d(y_0') = 0, d(x_1) = d(y_1) = 1 \]

Then \( Q^* = Q, d^* = d, \) and

\[
S(d^*): \begin{bmatrix}
    x_1 & x_1^s & y_1 & y_1^s \\
    y_1 & \cdot & a & 1 & \cdot \\
    y_1 & \cdot & b & 1 & 1 \\
    x_1 & 1 & \cdot & \cdot & \cdot \\
    x_1 & 1 & 1 & \cdot & \cdot 
\end{bmatrix}
\]

- \( \text{rep}_Q(d) \) and the closed orbit \( \{0\} \subseteq \text{rep}_Q(d) \) embed in \( S(d^*) \) as the vanishing sets of \( \langle a - b \rangle \) and \( \langle a, b \rangle \) respectively.
- corresponding clans: \( 1 - +1 \) and \( 1122 \) respectively.
Example continued

\[ Q = \xymatrix{ y_0 \ar[rd]_{x_1} \ar[rr]^{y_1} & & y_0' } \]

\[ d(y_0) = d(y_0') = 0, \quad d(x_1) = d(y_1) = 1 \]

\( T = \mathbb{C}^* \times \mathbb{C}^* \) acts on \( \text{rep}_Q(d) \) by conjugation, which induces a grading on \( \mathbb{C}[a] := \mathbb{C}[\text{rep}_Q(d)] \), with \( \text{deg}(a) = t - s \).

We can compute this as a ratio of specialized Wyser-Yong polynomials, \( \Upsilon_{\gamma}(C; D) \):

\[ \Upsilon_{1-+1}(c_1, c_2, c_3, c_4; d_1, d_2, d_3, d_4) = c_1 - d_1 + c_2 - d_2, \]
\[ \Upsilon_{1122}(c_1, c_2, c_3, c_4; d_1, d_2, d_3, d_4) = (c_1 + c_2 - d_3 - d_4)(c_1 - d_1 + c_2 - d_2) \]

The quiver polynomial of the closed orbit is:

\[ \frac{\Upsilon_{1122}(-s, -s, -t, -t; -t, -s, -t, -s)}{\Upsilon_{1-+1}(-s, -s, -t, -t; -t, -s, -t, -s)} \]

Generalizing this is work in progress with Z. Hamaker and R. Kinser.
Thanks for listening!