Title: Fundamental local equivalences in quantum geometric Langlands

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Collection: Geometric Representation Theory

Date: June 22, 2020 - 3:15 PM

URL: http://pirsa.org/20060038

Abstract: In quantum geometric Langlands, the Satake equivalence plays a less prominent role than in the classical theory. Gaitsgory--Lurie proposed a conjectural substitute, later termed the fundamental local equivalence, relating categories of arc-integrable Kac--Moody representations and Whittaker D-modules on the affine Grassmannian. With a few exceptions, we verified this conjecture non-factorizably, as well as its extension to the affine flag variety. This is a report on joint work with Justin Campbell and Sam Raskin.
Fundamental local equivalences in quantum geometric Langlands

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Perimeter-MPI Geometric Representation Theory conference

June 22, 2020
Overview

- Gaitsgory and Lurie conjectured deformations of the geometric Satake equivalence termed *fundamental local equivalences*
- Under a mild restriction on the deformation parameter, we proved these conjectures non-factorizably, in joint work with J. Campbell and S. Raskin
- Has arithmetic and representation-theoretic applications which are work in progress.
Motivation

- Everything over $k$ alg. closed of characteristic zero
- $X$ smooth projective curve
- $G$ reductive group
- $\text{Bun}_G$ moduli stack of $G$-bundles on $X$
- Given $x \in X(k)$, have completed local ring and its fraction field
  \[ \mathcal{O} \quad \text{and} \quad K \]

and corresponding arc and loop groups

\[ G(\mathcal{O}) \quad \text{and} \quad G(K). \]
Motivation

- Hecke correspondences give action

\[ D(G(\mathcal{O}) \backslash G(\mathcal{K})/G(\mathcal{O})) \otimes D(\text{Bun}_G) \to D(\text{Bun}_G). \]

- Appearance of Langlands dual group \( \check{G} \):

Theorem

(Lusztig–Drinfeld–Ginzburg–Mirkovic–Vilonen) There is a canonical (up to signs symmetric) monoidal equivalence of abelian categories

\[ D(G(\mathcal{O}) \backslash G(\mathcal{K})/G(\mathcal{O}))^{\heartsuit} \simeq (\text{Rep} \check{G})^{\heartsuit}. \]

- Count parameters - double cosets on the LHS are indexed by dominant coweights for \( G \), i.e. dominant weights for \( \check{G} \).
Motivation

- Quantum Langlands concerns the deformation

\[ D\text{-modules on } Bun_G \sim \text{twisted } D\text{-modules on } Bun_G. \]

- Parameters given by

\[ \text{Sym}^2(g^*)^G \to \text{Pic}(Bun_G) \otimes k \]

\[ \kappa \sim D_{\kappa}(Bun_G) \]

- Basic question - find helpful deformation of Satake isomorphism

- Basic problem - twisted spherical Hecke category

\[ D_{\kappa}(G(\mathcal{O}) \backslash G(\mathcal{K})/G(\mathcal{O})) \]

is often 'too small', e.g. is supported on trivial coset for $\kappa$ generic.
Formulation of the conjectures

- Idea of Gaitsgory–Lurie: fewer Hecke operators, but same number of Whittaker coefficients
- Fix $N \subset B \subset G$ the unipotent radical of a Borel, and a nondegenerate character of conductor zero
  \[ \psi : N(\mathbb{K}) \to \mathbb{G}_a. \]

Theorem

(Frenkel–Gaitsgory–Vilonen) There is a canonical equivalence of abelian categories

\[ D(G(\mathcal{O}) \backslash G(\mathbb{K})/G(\mathcal{O}))^\heartsuit \simeq D(N(\mathbb{K}), \psi \backslash G(\mathbb{K})/G(\mathcal{O}))^\heartsuit \]

- Parameter count - relevant orbits both indexed by dominant coweights
- One side of quantum deformation:
  \[ D(G(\mathcal{O}) \backslash G(\mathbb{K})/G(\mathcal{O})) \rightsquigarrow D_\kappa(N(\mathbb{K}), \psi \backslash G(\mathbb{K})/G(\mathcal{O})). \]
Formulation of the conjectures

- Need matching deformation of $\text{Rep } \check{G}$
- Basic idea - pass to representations of quantum group, or equivalently Kazhdan–Lusztig category for $\check{G}$
- Duality for levels via dual inner products on Cartan subalgebras

$$\text{Sym}^2(g^*)^G \backslash 0 \sim \text{Sym}^2(\check{g}^*)^{\check{G}} \backslash 0.$$  

$$\kappa \leftrightarrow \check{\kappa}$$

- The level $\check{\kappa}$ yields a Kac–Moody extension

$$0 \rightarrow k \rightarrow \hat{g}_{\check{\kappa}} \rightarrow \check{g}(\mathcal{K}) \rightarrow 0.$$  

- Associated Kazhdan–Lusztig category of $\check{G}(\mathcal{O})$-integrable modules

$$\hat{g}_{\check{\kappa}} \text{-mod} \check{G}(\mathcal{O}).$$
Formulation of the conjectures

**Conjecture**

*(Gaitsgory–Lurie, 2006)* For any nonzero $\kappa$, there is an equivalence of triangulated categories

$$D_\kappa(N(\mathcal{K}), \psi \backslash G(\mathcal{K})/G(\emptyset)) \simeq \widehat{g}_\kappa \text{-mod} \mathcal{G}(\emptyset).$$

- Count parameters - both sides indexed by dominant coweights for $G$.

**Remark**

In fact, they conjectured more, namely an equivalence of factorization categories (informally, compatibilities with moving and colliding multiple points of the curve).
Formulation of the conjectures

- Gaitsgory also conjectured a tamely ramified variant, concerning the Iwahori subgroups $I$ and $\tilde{I}$.

**Conjecture**

*(Gaitsgory, 2006)* For any nonzero $\kappa$, there is an equivalence of triangulated categories

$$D_\kappa(N(\mathcal{K}), \psi \backslash G(\mathcal{K})/I) \simeq \tilde{\mathfrak{g}}_\kappa\text{-mod}^{\tilde{I}}.$$

- Analog for $\kappa = 0$:

**Theorem**

*(Arkhipov–Bezrukavnikov)* There is an equivalence of triangulated categories

$$D(N(\mathcal{K}), \psi \backslash G(\mathcal{K})/I) \simeq \text{QCoh}^{\tilde{G}}(T^*(\tilde{G}/\tilde{B})).$$
Results

Theorem

(Campbell-D.-Raskin) If $\kappa$ satisfies a mild technical hypothesis, then both conjectures are true (the former non-factorizably).

- Hypothesis - after restriction to each simple factor of $g$, $\kappa$ is either irrational or rational with denominator coprime to the bad primes of the root system.
- Hypothesis is vacuous in type $A$, in general should be removable by a variant of the argument.
- Gaitsgory and collaborators have a rich program which is expected to yield the conjectures with factorization
Methods

- In finite type, well known equivalence (Milicic–Soergel, Bezrukavnikov, etc.) between blocks of $\varnothing$ and partial Whittaker sheaves

$$g\text{-mod}_{\varnothing}^B \simeq D(B\backslash G/N, \chi_\lambda).$$

- (Aside) Does not arise from usual localization on $G/B$, but instead

Theorem

(Campbell-D.) Localization yields a fully faithful embedding

$$g\text{-mod}_\lambda \hookrightarrow D(G/N, \chi_\lambda).$$
Similarly, we relate

\[ D_\kappa(N(\mathcal{K}), \psi \backslash G(\mathcal{K})/I) \sim \text{category } \mathcal{O} \text{ for } \widehat{\mathfrak{g}}_\kappa \]

using categorical representation theory of loop groups.

- Match the combinatorial descriptions provided by Soergel–Fiebig for blocks of \( \mathcal{O} \) for \( \widehat{\mathfrak{g}} \) at level \( \kappa \) and \( \widehat{\mathfrak{g}} \) at level \( \tilde{\kappa} \).
Next steps

- Pass from \( k \) to \( \mathbb{F}_q \), so

\[ \mathcal{O} \cong \mathbb{F}_q[[t]] \quad \text{and} \quad \mathcal{K} \cong \mathbb{F}_q((t)). \]

- An unramified principal series representation \( \pi \) of \( G(\mathcal{K}) \) has one dimensional spaces of \( G(\mathcal{O}) \) invariants and \( (N(\mathcal{K}), \psi) \) coinvariants, hence yields a spherical Whittaker function

\[ f_\pi \in \text{Fun}(N(\mathcal{K}), \psi \backslash G(\mathcal{K})/G(\mathcal{O})). \]

- Frenkel–Gaitsgory–Vilonen equivalence

\[ \text{Perv}(N(\mathcal{K}), \psi \backslash G(\mathcal{K})/G(\mathcal{O})) \cong \text{Rep} \check{G} \]

recovers after trace of Frobenius the Casselman–Shalika formula for \( f_\pi \).
Next steps

- Fix a metaplectic cover \( \widehat{G(K)} \) of \( G(K) \)
- An unramified principal series representation \( \pi \) of \( \widehat{G(K)} \) has a line of \( G(\emptyset) \) invariants, but in general a greater than one dimensional space of \( (N(K), \psi) \) coinvariants, hence yields a vector space of spherical Whittaker functions

\[ V_\pi \subset \text{Fun}(N(K), \psi \backslash \widehat{G(K)}/G(\emptyset)) \]

(Kubota, Matsumoto, Kazhdan–Patterson, Brubaker, Bump, Chinta, Friedberg, Gunnells, McNamara, Lysenko, etc.)

- A variant of our proof of the tamely ramified conjecture should yield a description of

\[ \text{Perv}(N(K), \psi \backslash \widehat{G(K)}/G(\emptyset)) \]

in terms of the Kazhdan–Lusztig category, and after trace of Frobenius a metaplectic Casselman–Shalika formula for a canonical basis for \( V_\pi \).
Next steps

- Return to $k$ alg closed of characteristic 0
- To $\mathfrak{g}$ and $\kappa$ one may attach an (affine) $\mathcal{W}$-algebra
  \[ \mathcal{W}_\kappa. \]
- Problem (Frenkel–Kac–Wakimoto, Arakawa, etc.) - determine behavior of the obtained functor
  \[ C^{\infty}_{\frac{1}{2} + \ast}(\mathcal{K}, \mathcal{K}_\psi) : \widehat{\mathfrak{g}}_\kappa\text{-mod}^I \to \mathcal{W}_\kappa\text{-mod}. \]
Next steps

- This functor fits into a family of functors

\[ D_\kappa(N(\mathcal{K}), \psi \backslash G(\mathcal{K})/I) \otimes \widehat{\mathfrak{g}}_\kappa - \text{mod}^I \to \mathcal{W}_\kappa - \text{mod}. \]

- Similar arguments to those above yield a description of

\[ D_\kappa(N(\mathcal{K}), \psi \backslash G(\mathcal{K})/I) \otimes \widehat{\mathfrak{g}}_\kappa - \text{mod}^I \to \mathcal{W}_\kappa - \text{mod}. \]

in terms of ‘two sided antisphereflcial quotients’ of the affine Hecke category (at least for \( \kappa \) negative).

- Should imply a conjecture of Gaitgory on the compatibility of the above pairing with Feigin–Frenkel duality and the FLE
Thanks for listening!