Title: K-Motives and Koszul Duality

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Collection: Geometric Representation Theory

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Abstract: Koszul duality, as conceived by Beilinson-Ginzburg-Soergel, describes a remarkable symmetry in the representation theory of Langlands dual reductive groups. Geometrically, Koszul duality can be stated as an equivalence of categories of mixed (motivic) sheaves on flag varieties. In this talk, I will argue that there should be an 'ungraded' version of Koszul duality between monodromic constructible sheaves and equivariant K-motives on flag varieties. For this, I will explain what K-motives are and present preliminary results.
\( pt = \mathrm{Spec} \mathbb{F}_p, \text{ coeff. } = \mathbb{Q} \)

\[ \text{K-motives and Koszul duality} \]

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split alg. torus

\[ T \xrightarrow{\text{duality}} T^\vee \]
dual torus

character lattice

\[ X(T) \cong Y(T^\vee) \]
cocharacter lattice

\[ R := \mathbb{Q} [C \times \mathbb{G}_m] \]
group algebra
of fundamental group

\[ K_0^T (pt) = R = \mathbb{Q} [\pi_1(T^\vee(C))] \]

\[ \text{D}K^T_{\text{const}} (pt) = \text{D}^b(R \text{-mod}) = \text{D}^b(\text{LC}(T^\vee(C))) \]
der. cat. of locally const. sheaves

\[ \text{Today: } K \text{-motive?} \]
\[ \times T \sim G \]
II $K$-motives

(constructible sheaves : singular cohomology)

$=\quad$

($K$-motives : alg. $K$-theory)

variety $X \mapsto DK(X)$

triangulated category of $K$-motives on $X$

* 6 functor formalism ($f_!, f_*, f^!, f^*, \otimes, \hom$)

* equivariant version $DK^T(X)$ for $T \to X$

for $X$ regular

$\text{Hom}_{DK^T(X)}(\varnothing, \varnothing[n]) = K^T_n(X)$ $T$-equivariant (higher) alg. $K$-theory
III. "Intersection K-theory complexes"

\[ \text{DK}_0^T (X) \overset{\text{def}}{=} \langle \Omega \rangle_\Delta = \text{DK}^T (X) \]

\[ \text{DK}_0^T (pt) = \mathcal{D}^b (\mathcal{R} \text{-mod}) \]
\[ \Omega \leftrightarrow \mathcal{R} \]

\[ \text{DK}_0^T (X) \overset{\text{def}}{=} \{ M \mid \text{M is constant} \} \]

\[ \text{Hom}_{\text{DK}_0^T (pt)} (\Omega, \Omega[\mathcal{R}]) = \text{K}_0^T (pt) = \langle R_{n=0} \rangle \]

\[ X = \{ X_s \mid X_s \cong A^2 \} \]
affinely stratified + assumptions

\[ \text{Thm. (E.)} \]
\[ \times \text{DK}_0^T (X) \text{ has "weight structure"} \]
\[ \times \text{DK}_0^T (X)_{w=0} \text{ pure K-motives} \]
\[ \times \text{DK}_0^T (X) \Rightarrow \mathcal{K}^b (\text{DK}_0^T (X)_{n=0}) \]

analogous to

\[ \times \text{intersection coh. complexes} \]
\[ \times \text{parity sheaves} \]

"Intersection K-th. complexes"
IV Koszul duality à la Beilinson–Ginzburg–Soergel

\[ T \subset G \xleftrightarrow{\text{Langlands dual}} T' \subset G' \]

* Take twist (1)
  mixed sheaves: * "graded version" of contr. sheaves

\[ D^b(\mathfrak{g}(\mathfrak{g})) \]
graded cat. of \( \text{Lie}(G_{\mathfrak{g}}) \)

\[ (1) \] (not true without \( \text{mix} \))
I Ungraded Koszul duality (E.)

\[ (1) \quad \text{real} \]

\[ \text{id} \quad \text{??} \quad \text{id} \]

\[ \text{constructible sheaves on } (G^v/B^v)(\mathbb{A})^{an} \]
Ungraded Koszul duality (E.)

Thm (E.)
(1)(2) ⊲⊳ (1) 
(1)(2) \( \mathcal{D}_{GB}^\text{mix}(G/B) \) ⊲⊳ \( \mathcal{D}_{GB}^\text{mix}(G^v/B^v) \) (1) 

\( H^\bullet(R) = \mathcal{K} \) 
\( K_{(y')} = \mathcal{K} \) 

\( \text{id} \) \( \mathcal{D}_{KB}(G/B) \) \( \mathcal{D}_{KB}(G^v/B^v) \) \( \text{id} \) 

\( \text{real} \) 
\( \text{real} \) 

Constructible sheaves on \( (G^v/B^v)(G)_{\text{an}} \)
VI Work in progress

THANK YOU

\[ \text{Thm/Conj. (C.1)} \]

\[ \mathcal{D}K^{T}_{(B)}(G/B) \xrightarrow{\text{weight complex functor}} \mathcal{D}_{\text{mon}}(G^\ast/U^\ast) \]

\[ K^0(\text{R-SBim}) \]

K-theory Soergel bimodules

\[ \text{etc...} \]
THANK YOU