Title: Sprinklings in Causal Set Theory and Local Structures to Discretize Field Propagators
Speakers: Christoph Minz
Collection: Quantum Gravity 2020
Date: July 17, 2020 - 9:20 AM
URL: http://pirsa.org/20070017
Sprinklings in Causal Set Theory and Local Structures to Discretize Field Propagators

Christoph Minz
(in joined work with Christopher J. Fewster, Eli Hawkins, and Kasia Rejzner)\(^1\)

\(^1\)Department of Mathematics
University of York

Quantum Gravity, July 17, 2020
• It is a framework for quantum gravity, see introductions by Henson [Hen09] or Sorkin [Sor11].

• Continuum spacetime manifold is replaced by a locally finite partial ordered set \((\mathcal{P}, \preceq)\), i.e. it fulfills the axioms

  \begin{align*}
  \text{transitivity:} & \quad x \preceq z \preceq y \Rightarrow x \preceq y, \\
  \text{acyclicity (anti-symmetry):} & \quad (x \preceq y \land y \preceq x) \iff x = y, \\
  \text{local finiteness:} & \quad |\{z \in \mathcal{P} | x \preceq z \preceq y\}| < \infty.
  \end{align*}

• The causal interval or Alexandrov subset of two causally related events is denoted by

  \[ I(x, y) := \{z \in \mathcal{P} | x \preceq z \preceq y\}. \]

• We write \( x \prec y \) if and only if \( x \preceq y \) and \( x \neq y \).

• An event \( y \) is linked to \( x \), \( x \prec^* y \) (or \( x \prec^* y \)) if and only if \( I(x, y) = \{x, y\} \) and \( x \neq y \).
Sprinkling is the Poisson process of obtaining a sprinkled causal set (causet) from a spacetime:

1. Let \((M, g)\) be a smooth spacetime manifold with metric \(g\).
2. Select a locally finite set called \(\text{sprinkle}\) of spacetime points \(S \subset_{\text{loc. finite}} M\) by a Poisson process.
3. Restrict the causal relation from the spacetime to the sprinkle

\[ x \preceq y \iff x \in J^-(y) \]

We will derive the corresponding probability measure in the following.
- Let $U \in L$ be a subset of $M$ in the set $L$ of open, pre-compact subsets.
- The configuration space of sprinkles on the manifold $M$ is

$$Q := \{ S \subset M \mid \forall U \in L : |S \cap U| < \infty \}.$$
- Let $U \in L$ be a subset of $M$ in the set $L$ of open, pre-compact subsets.
- The configuration space of sprinkles on the manifold $M$ is

$$Q := \{ S \subset M \mid \forall U \in L : |S \cap U| < \infty \}.$$
For the construction [AKR98], consider \( n \)-tuples of non-identical events in \( U \):

\[
\widetilde{Q}_{U,n} := \{(x_1, x_2, \ldots, x_n) \in U^n \mid \forall i, j \in [1, n]: x_i = x_j \iff i = j\}
\]
For the construction [AKR98], consider \( n \)-tuples of non-identical events in \( U \):

\[
\widetilde{Q}_{U,n} := \{(x_1, x_2, \ldots, x_n) \in U^n \mid \forall i, j \in [1, n]: x_i = x_j \iff i = j\}
\]
For the construction [AKR98], consider $n$-tuples of non-identical events in $U$:

$$\widehat{Q}_{U,n} := \{(x_1, x_2, \ldots, x_n) \in U^n \mid \forall i, j \in [1, n] : x_i = x_j \iff i = j\}$$
Every $n$-tuple with the same events $x_i$ corresponds to the same sprinkling:

$$\sum_{U,n} \widetilde{Q}_{U,n} \rightarrow Q_{U,n}; \quad (x_1, x_2, \ldots, x_n) \mapsto \{x_1, x_2, \ldots, x_n\}.$$

$$Q_{U,n} := \{S \subset U \mid |S| = n\}.$$
• Take the union over all cardinalities to get the config. space $Q_U$.
• The intensity measure for every open subset of $M$, derived from its volume and the sprinkling density parameter $\rho$

$$\lambda(U) = \rho \int_U \sqrt{|g|} \, d^d \! x$$
- Poisson (probability) measure on $U$ [AKR98] assigns a probability

$$\mu_U(B) := e^{-\lambda(U)} \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^\otimes_n \circ \Sigma_{U,n}^{-1}(B_n)$$

for any measurable subset $B = (B_n)_{n \in \mathbb{N}_0}$ such that $B_n \in \mathcal{B}(Q_{U,n})$.

- There exists a unique Poisson measure on the entire manifold $M$. 
Poisson (probability) measure on $U$ [AKR98] assigns a probability

$$
\mu_U(B) := e^{-\lambda(U)} \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^\otimes_n \circ \Sigma_{U,n}^{-1}(B_n)
$$

for any measurable subset $B = (B_n)_{n \in \mathbb{N}_0}$ such that $B_n \in \mathcal{B}(Q_{U,n})$.

- There exists a unique Poisson measure on the entire manifold $M$. 
Poisson (probability) measure on $U$ [AKR98] assigns a probability

$$\mu_U(B) := e^{-\lambda(U)} \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^\otimes n \circ \Sigma_{U,n}^{-1}(B_n)$$

for any measurable subset $B = (B_n)_{n \in \mathbb{N}_0}$ such that $B_n \in \mathcal{B}(Q_{U,n})$.

There exists a unique Poisson measure on the entire manifold $M$. 
- Classical and quantum fields on causal sets are described by discretized counterparts to the eom. on a continuum manifold.
- Most approaches to discretization are based on past $k$-layers [Sor09, ASS14],

\[ L_k^-(x) := \{ y \in \mathcal{C} \mid k = |I(y, x)| - 1 \}. \]
• Classical and quantum fields on causal sets are described by discretized counterparts to the eom. on a continuum manifold.
• Most approaches to discretization are based on past $k$-layers [Sor09, ASS14],

$$L_k^-(x) := \{ y \in \mathcal{C} \mid k = |I(y, x)| - 1 \}.$$
These discretizations need the spacetime dimension as an input, but in general the spacetime dimension of a given causal set is an emergent (local) property and not pre-defined.

A more recently proposed discretization method [DHFRW20] has the potential to be independent of the dimension, but needs a supplementary structure to a causal set called a *preferred past* as defined in the following.

We define a *path* from a causet event \( x \in \mathcal{C} \) to an event in its future \( y \succ x \) as the set of events \( \mathcal{P} \) that forms the linked chain

\[
x \prec x_1 \prec x_2 \prec \cdots \prec x_{n-2} \prec y.
\]

We denote the set of all *paths from* \( x \) *to* \( y \) by \( \text{paths}(x, y) \).

The length (number of links) of the shortest path is called the *rank*

\[
\text{rk}(y, x) := \begin{cases} 
\min_{\mathcal{P} \in \text{paths}(x, y)} |\mathcal{P}| - 1 & x \preceq y, \\
\infty & \text{otherwise}.
\end{cases}
\]
• The rank \( k \) past of an event \( x \in \mathcal{C} \) is the set
\[
R_k^-(x) := \{ y \in \mathcal{C} \mid r_k(x, y) = k \}.
\]

• A preferred past structure is a map \( \Lambda^- : \mathcal{C} \setminus C^2_2 \to \mathcal{C} \) such that \( \Lambda^-(x) \in R_2^-(x) \) for all \( x \in \mathcal{C} \setminus C^2_2 \).
• The rank $k$ past of an event $x \in \mathcal{C}$ is the set

$$R_k^-(x) := \{ y \in \mathcal{C} \mid \text{rk}(x, y) = k \}.$$  

• A preferred past structure is a map $\Lambda^- : \mathcal{C} \setminus C_2^- \to \mathcal{C}$ such that $\Lambda^-(x) \in R_2^-(x)$ for all $x \in \mathcal{C} \setminus C_2^-$. 
- For an event \( x \) and one of its rank 2 past events \( y \), we call the Alexandrov set \( I(y, x) \) a \( k \)-diamond if its size is
\[
k = |I(y, x) \setminus \{x, y\}|.
\]

- The number of internal events for a given \( k \)-diamond \( I(y, x) \) is
\[
itn(x, y) := k - |\{z \in I(y, x) \mid y \prec z \prec x\}|.
\]
For an event \( x \) and one of its rank 2 past events \( y \), we call the Alexandrov set \( I(y, x) \) a \( k \)-diamond if its size is

\[
k = |I(y, x) \setminus \{x, y\}|.
\]

The number of internal events for a given \( k \)-diamond \( I(y, x) \) is

\[
\text{itn}(x, y) := k - |\{z \in I(y, x) \mid y \prec z \prec x\}|.
\]
For an event $x$ and one of its rank 2 past events $y$, we call the Alexandrov set $I(y, x)$ a $k$-diamond if its size is

$$k = |I(y, x) \setminus \{x, y\}|.$$

The number of internal events for a given $k$-diamond $I(y, x)$ is

$$\text{itn}(x, y) := k - \left| \{ z \in I(y, x) \mid y \prec* z \prec* x \} \right|.$$
For an event $x$ and one of its rank 2 past events $y$, we call the Alexandrov set $I(y, x)$ a $k$-diamond if its size is

$$k = |I(y, x) \setminus \{x, y\}|.$$

The number of internal events for a given $k$-diamond $I(y, x)$ is

$$\text{itn}(x, y) := k - |\{z \in I(y, x) \mid y \prec z \prec \neg x\}|.$$
In order to find the rank 2 past events to prefer, we numerically investigated 6 criteria and compared their distributions of the number of rank 2 past events and their proper time separation.

1: largest pure diamonds,
2: diamonds with the least internal events of the diamonds with the most rank 2 paths,
3: smallest diamonds,
4: diamonds with the most internal events of the diamonds with the most rank 2 paths,
5: consider all diamonds with the smallest number \( i \) of internal events of those diamonds with the greatest number \( p_{\text{max}} \) of rank 2 paths (as in criterion 2); furthermore, include diamonds that have \((p_{\text{max}} - j) \) rank 2 paths and up to \((i - j)\) internal events (or are pure), \( \forall j \in [1, p_{\text{max}} - 1] \). Split this set of diamonds and sort increasingly by the number of rank 2 paths, then count the diamonds in each subset. Select the first subset that has only one diamond so that it is a unique element - or choose the largest diamonds of the selection.
6: largest diamonds.
- For an event $x$ and one of its rank 2 past events $y$, we call the Alexandrov set $I(y, x)$ a $k$-diamond if its size is

$$k = |I(y, x) \setminus \{x, y\}|.$$

- The number of internal events for a given $k$-diamond $I(y, x)$ is

$$\text{itn}(x, y) := k - |\{z \in I(y, x) \mid y \prec z \prec x\}|.$$
In order to find the rank 2 past events to prefer, we numerically investigated 6 criteria and compared their distributions of the number of rank 2 past events and their proper time separation.

1: largest pure diamonds,
2: diamonds with the least internal events of the diamonds with the most rank 2 paths,
3: smallest diamonds,
4: diamonds with the most internal events of the diamonds with the most rank 2 paths,
5: consider all diamonds with the smallest number $i$ of internal events of those diamonds with the greatest number $p_{\text{max}}$ of rank 2 paths (as in criterion 2); furthermore, include diamonds that have $(p_{\text{max}} - j)$ rank 2 paths and up to $(i - j)$ internal events (or are pure), $\forall j \in [1, p_{\text{max}} - 1]$. Split this set of diamonds and sort increasingly by the number of rank 2 paths, then count the diamonds in each subset. Select the first subset that has only one diamond so that it is a unique element - or choose the largest diamonds of the selection.
6: largest diamonds.
- Probability that the criterion selects a unique event from the rank 2 past (for 10,000 causet ensembles in Alexandrov subsets of 1 + 1, 1 + 2 and 1 + 3 dimensional Minkowski spacetime)
Proper time separation between the past and future event of the diamond, for flat spacetime dimension $1+1$, $1+2$, $1+3$.

$\langle n \rangle = 6000$

criterion:
- 1
- 2
- 3
- 4
- 5
- 6
Proper time separation between the past and future event of the diamond, for flat spacetime dimension $1 + 1$, $1 + 2$, $1 + 3$

\[
\langle n \rangle = 6000
\]

criterion:
1
2
3
4
5
6
Proper time separation between the past and future event of the diamond, for flat spacetime dimension $1 + 1, 1 + 2, 1 + 3$.

![Graph showing probability distribution with critical points labeled crit. 1 to crit. 5. Each distribution curve is color-coded and labeled with the corresponding criterion number: 1, 2, 3, 4, 5, 6. The mean $\langle n \rangle = 6000$.](image-url)
Proper time separation between the past and future event of the diamond, for flat spacetime dimension $1 + 1$, $1 + 2$, $1 + 3$.
Sergio Albeverio, Yu G Kondratiev, and Michael Röckner.
Analysis and Geometry on Configuration Spaces.

Siavash Aslanbeigi, Mehdi Saravani, and Rafael D. Sorkin.
Generalized Causal Set d’Alembertians.

Edmund Dable-Heath, Christopher J Fewster, Kasia Rejzner, and Nick Woods.
Algebraic Classical and Quantum Field Theory on Causal Sets.

Joe Henson.
The Causal Set Approach to Quantum Gravity.
Approaches to Quantum Gravity: Towards a New Understanding of Space, Time and Matter, 393, 2009.

Rafael D. Sorkin.
Does Locality Fail at Intermediate Length-Scales.
Approaches to Quantum Gravity, pages 26–43, 2009.

Rafael D Sorkin.
Scalar Field Theory on a Causal Set in Histories Form.