Finite quantum geometry, octonions and the theory of fundamental particles

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Octonions and the Standard Model

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General framework

- External geometry: Lorentzian spacetime M
  $\mathcal{C}(M)$ with Poincaré group action and equivariant $\mathcal{C}(M)$-modules.

- Internal geometry : Finite quantum geometry
  $J=$ finite-dimensional algebra of quantum observables with some further structure $\Rightarrow G \subset \mathrm{Aut}(J)$ and equivariant J-modules.

- $\Rightarrow \mathcal{J} = \mathcal{C}(M, J)$, $\mathcal{J}$-modules and connections
  $\Rightarrow$ gauge interactions, etc.
  $\mathcal{J}$ defines an "almost classical quantum geometry".

The theory of universal unital multiplicative envelope $U_1(J)$ of J makes the bridge between the present approach and the noncommutative one which is summarized in [6] and [7].

Internal space for a quark [1]

$E \simeq \mathbb{C}^3$ with (color) $SU(3)$ action

$$\begin{cases} SU(3) \subset U(3) \Rightarrow E \text{ is Hilbert with scalar product } \langle \bullet, \bullet \rangle \\ \text{Unimodularity of } SU(3) \Rightarrow \text{ volume} = \text{3-linear form on } E, vol(\bullet, \bullet, \bullet) \end{cases}$$

$\Rightarrow$ antilinear antisymmetric product x on E

$$vol(Z_1, Z_2, Z_3) = \langle Z_1 \times Z_2, Z_3 \rangle$$

$SU(3)$-basis = Orthonormal basis (e_k) of E such that

$$v(e_1, e_2, e_3) = 1$$

By chosing an origin $SU(3)$-basis $\leftrightarrow SU(3)$
2 products $x : E \times E \to E$ and $\langle , \rangle : E \times E \to \mathbb{C}$

Unital $SU(3)$-algebra

$$SU(3) = \{U \in GL(E) | x \text{ and } \langle , \rangle \text{ are preserved}\}$$

$$\| Z_1 \times Z_2 \|^2 = \| Z_1 \|^2 \| Z_2 \|^2 - |\langle Z_1, Z_2 \rangle|^2$$

add a unit $\Rightarrow \mathbb{C} \oplus E \qquad \mathbb{1} = (1, 0)$
$(1,0)(0,Z) = (0,Z) = (0,Z)(1,0), (z_1,0)(z_2,0) = (z_1 z_2, 0)$

$(0,Z_1)(0,Z_2) = (\alpha\langle Z_1, Z_2\rangle, \beta Z_1 \times Z_2), |\alpha| = |\beta| = 1$
$\Rightarrow \| (0,Z_1) \|^2 \| (0,Z_2) \|^2 = \| (0,Z_1)(0,Z_2) \|^2$

natural to require $\| (z_1,Z_1)(z_2,Z_2) \| = \| (z_1,Z_1) \| \| (z_2,Z_2) \|$
solution :
$(z_1,Z_1)(z_2,Z_2) = (z_1 z_2 - \langle Z_1, Z_2\rangle, \bar{z}_1 Z_2 + z_2 Z_1 + i Z_1 \times Z_2)$
$\Rightarrow (\bar{z}, -Z)(z, Z) = (z, Z)(\bar{z}, -Z) = \| (z, Z) \|^2 \, \mathbb{1}$

An interpretation of the quark-lepton symmetry

$SU(3)$ is the group of complex-linear automorphisms of $\mathbb{C} \oplus E$ which preserves the above product and E carries the fundamental representation of $SU(3)$ while $\mathbb{C}$ corresponds to the trivial one.

$\Rightarrow$ E being the internal space of a quark, it is "natural" to consider $\mathbb{C}$ as the internal space of the corresponding lepton.

As a real algebra $\mathbb{C} \oplus E$ is 8-dimensional isomorphic to the octonion algebra $\mathbb{O}$.
$SU(3) \subset G_2 = Aut(\mathbb{O})$ is the subgroup preserving i, a given imaginary element of $\mathbb{O}$ with $i^2 = -1$.

The 3 generations

6 flavors of quark-lepton
$(u, \nu_e), (d, e), (c, \nu_\mu), (s, \mu), (t, \nu_\tau), (b, \tau)$
grouped in 3 generations, columns of

generations			
quarks $Q = 2/3$ leptons $Q = 0$	u ν_e	c ν_μ	t ν_τ
quarks $Q = -1/3$ leptons $Q = -1$	d e	s μ	b τ

This sort of "triality" combined with the above interpretation of the quark-lepton symmetry suggest to add over each space-time point the finite quantum system corresponding to the exceptional Jordan algebra.

Quantum geometry - I

J (real vector space) quantum analog of a space of real functions.
Squaring $x \mapsto x^2$ for $x \in J$ such that $x.y = \frac{1}{2}((x+y)^2 - x^2 - y^2)$ is bilinear.
J is *power associative* if by defining $x^{n+1} = x.x^n$
(i) $x^r.x^s = x^{r+s}$

J is *formally real* if one has
(ii) $\sum_{k \in I} (x_k)^2 = 0 \Rightarrow x_k = 0, \ \forall k \in I$

Theorem (1)

A finite-dimensional commutative real algebra J which is power associative and formally real is a Jordan algebra, that is one has

$$x^2.(y.x) = (x^2.y).x, \quad \forall x, y \in J.$$

Such a Jordan algebra is also called an *Euclidean* Jordan algebra.

Quantum geometry - II

Condition (i) and (ii) are necessary for spectral theory (with real spectra).

There are various infinite-dimensional extensions of the above theorem $\Rightarrow$ various formulations of "quantum geometry", etc.

In most cases the Jordan algebras which describe quantum geometries are hermitian (real) subspaces of complex $*$-algebras invariant by the anticommutator $x.y = \frac{1}{2}(xy + xy)$.

$\Rightarrow$ In these cases one can use the noncommutative geometric setting.

Properties of finite-dimensional Euclidean Jordan algebras

Let J be a finite-dimensional Euclidean Jordan algebra.
Then J has a unit $\mathbb{1} \in J$ and $\forall x \in J$

$$x = \sum_{r \in I_x} \lambda_r e_r, \;\; e_r e_s = \delta_{rs} e_r \in J, \;\; \lambda_r \in \mathbb{R}$$

with $\mathbb{1} = \sum_{r \in I_x} e_r$, $card(Ix) \leq n(J) \in \mathbb{N}$

$\Rightarrow$ functional calculus with $\mathbb{R}[X]$.

Furthermore J is a direct sum of a finite number of simple ideals.

Finite-dimensional simple Euclidean Jordan algebras

Theorem (2)

A finite-dimensional simple Euclidean Jordan algebra is isomorphic to one of

$c = 1$	$\mathbb{R}$
$c = 2$	$J_2^n = JSpin_{n+1} = \mathbb{R}\mathbb{1} + \mathbb{R}^{n+1}, \gamma^\mu.\gamma^\nu = \delta^{\mu\nu}\mathbb{1},\ n \geq 1$
$c = 3$	$J_3^1 = H_3(\mathbb{R}),\ J_3^2 = H_3(\mathbb{C}),\ J_3^4 = H_3(\mathbb{H}),\ J_3^8 = H_3(\mathbb{O})$
$c = n \geq 4$	$J_n^1 = H_n(\mathbb{R}),\ J_n^2 = H_n(\mathbb{C}),\ J_n^4 = H_n(\mathbb{H})$

These correspond to the "finite quantum spaces" (i.e. "real function's spaces" over the "quantum spaces").

The "octonionic factors" J_2^8 and J_3^8 [1], [4]

The above interpretation which connects the quark-lepton symmetry and the unimodularity of the color group points the attention to the factors

$$J_2^8 = H_2(\mathbb{O}) = JSpin_9$$

$$J_3^8 = H_3(\mathbb{O})$$

together with the subgroups of $\mathrm{Aut}(J_2^8) = O(9)$ and of $\mathrm{Aut}(J_3^8) = F_4$ which preserve the splitting $\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$ (and act $\mathbb{C}$-linearly on $\mathbb{C}^3$).

Remark : It is worth noticing here that there is another octonionic factor namely $J_2^7 = JSpin_8$ identified to the Jordan subalgebra of J_2^8 which consists of the 2×2 octonionic hermitian matrices with diagonals multiple of $\mathbb{1}$ (i.e. $\begin{pmatrix} \lambda & x \\ \bar{x} & \lambda \end{pmatrix}$ with $\lambda \in \mathbb{R}, x \in \mathbb{O}$).

Action of $G_{SM} = SU(3) \times SU(2) \times U(1)/\mathbb{Z}_6$ on J_2^8

$O(9) = \mathrm{Aut}(J_2^8)$, the subgroup which preserves the splitting $\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$ is the group $O(3) \otimes U(3)$. To express this action write

$$\begin{pmatrix} \zeta_1 & x \\ \bar{x} & \zeta_2 \end{pmatrix} \in J_2^8$$

as

$$\begin{pmatrix} \zeta_1 & x \\ \bar{x} & \zeta_2 \end{pmatrix} = \begin{pmatrix} \zeta_1 & z \\ \bar{z} & \zeta_2 \end{pmatrix} + Z \in J_2^2 \oplus \mathbb{C}^3$$

where $x = z + Z \in \mathbb{C} \ \oplus \mathbb{C}^3$ represents $x \in \mathbb{O}$. The action of $O(3) \otimes U(3)$ is then the action of $O(3) = \mathrm{Aut}(J_2^2)$ and the action of $U(3)$ on $\mathbb{C}^3$. The action of the connected part $SO(3) \times U(3)$ is in fact an action of $G_{SM}/\mathbb{Z}_2 = SO(3) \times U(3)$, i.e. of G_{SM} by forgetting the torsion part of the fundamental group.

Action of $SU(3) \times SU(3)/\mathbb{Z}_3$ on J_3^8

$F_4 = Aut(J_3^8)$, the subgroup which preserves the representations of the octonions occurring in the matrix elements of J_3^8 as elements of $\mathbb{C} \oplus \mathbb{C}^3$ is $SU(3) \times SU(3)/\mathbb{Z}_3$. To express this action write

$$\begin{pmatrix} \zeta_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \zeta_2 & x_1 \\ x_2 & \bar{x}_1 & \zeta_3 \end{pmatrix} \in J_3^8$$

as

$$\begin{pmatrix} \zeta_1 & z_3 & \bar{z}_2 \\ \bar{z}_3 & \zeta_2 & z_1 \\ z_2 & \bar{z}_1 & \zeta_3 \end{pmatrix} + (Z_1, Z_2, Z_3) \in J_3^2 \oplus M_3(\mathbb{C})$$

where $x_i = z_i + Z_i \in \mathbb{C} \oplus \mathbb{C}^3$ is the representation of $x_i \in \mathbb{O}$.

The action of $(U, V) \in SU(3) \times SU(3)$ is then
$H \mapsto VHV^*$, $M \mapsto UMV^*$ on $H \oplus M \in J_3^2 \oplus M_3(\mathbb{C})$.

The action of U is the previous action of the color $SU(3)$.

The $\mathbb{Z}_3$-splitting principle

Yokota suggests a simpler formulation (Arxiv: 0909.0431), $i \in \mathbb{C}$ corresponds to $i \in \mathbb{O} \Rightarrow \mathbb{Z}_3 \subset SU(3) \subset G_2 = \mathrm{Aut}(\mathbb{O})$. The $\mathbb{Z}_3$ action on $\mathbb{O}$ is induced by $w \in \mathrm{Aut}(\mathbb{O})$

$$w(z + Z) = z + \omega_1 Z, \quad \omega_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$$

One has $w^3 = I$ and this also induces a $\mathbb{Z}_3$-action by automorphism, again denoted w, on J_2^8 (then $w \in SO(9)$) and on J_3^8 (then $w \in F_4$). The corresponding subgroups leaving w invariant are given by

$$(G_2)^w = SU(3)$$

$$(SO(9))^w = G_{SM}/\mathbb{Z}_2$$

$$(F_4)^w = SU(3) \times SU(3)/\mathbb{Z}_3$$

Exceptional quantum factor

$$J_3^8 = H_3(\mathbb{O}) = \{3 \times 3 \text{ hermitian octonionic matrices}\}$$

- Albert has shown that it cannot be realized as a part stable for the anticommutator of an associative algebra.
- It follows from the theory of Zelmanov that this is the only exceptional factor.

Center

A arbitrary $\mathbb{K}$-algebra; *the center* $Z(A)$ *of* A is the set of the $z \in A$ such that

$$[x, z] = 0, \ \forall x \in A$$

and

$$[x, y, z] = [x, z, y] = [z, x, y] = 0, \ \forall x, y \in A$$

where $[x, z] = xz - zx, [x, y, z] = (xy)z - x(yz), \ \forall x, y, z \in A$.
$Z(A)$ is a commutative associative subalgebra of A.

Lemma (1)

Assume that A is commutative. Then one has :
$z \in Z(A) \Leftrightarrow [x, y, z] = 0, \ \forall x, y \in A.$

Proof.

$[x, z] = 0$ is clear ; $[x, y, z] = -[z, y, x] = 0$ by commutativity and again by commutativity $[x, y, z] - [y, x, z] = 0$ implies $[x, z, y] = 0$.
$(\equiv [y, z, x] \equiv -[x, z, y])$. □

Derivations

A arbitrary $\mathbb{K}$-algebra ; a linear endomorphism δ of A is a *derivation of A* (into A) if it satisfies

$$\delta(xy) = \delta(x)y + x\delta(y), \ \ \forall x, y \in A$$

The space $\mathrm{Der}(A)$ of all derivations of A is a $Z(A)$-module

$$(z\delta)(x) = z\delta(x), \ \ \forall z \in Z(A), \forall x \in A$$

$\mathrm{Der}(A)$ is also a Lie algebra

$$[\delta_1, \delta_2](x) = \delta_1(\delta_2(x)) - \delta_2(\delta_1(x)), \forall \delta_1, \delta_2 \in \mathrm{Der}(A), \forall x \in A$$

One has

$$\delta(Z(A)) \subset Z(A), \ \ \forall \delta \in \mathrm{Der}(A)$$

and

$$[\delta_1, z\delta_2] = z[\delta_1, \delta_2] + \delta_1(z)\delta_2, \ \ \forall \delta_1, \delta_2 \in \mathrm{Der}(A), \ \ \forall z \in Z(A)$$

that is $(\mathrm{Der}(A), Z(A))$ is a *Lie Rinehart algebra*

Categories of algebras

$\mathbb{K}$ a fixed field ; all vector spaces, algebras are over $\mathbb{K}$
A category of algebras is a category $\mathcal{C}$ such that its objects are algebras and its morphisms are algebra-homomorphisms.
$\mathcal{C}_{\mathbf{Alg}}$ = category of all algebras and all algebra-homomorphisms
$\mathcal{C}_{\mathbf{Alg}_1}$ = category of unital algebras and unital algebra-homomorphisms
$\mathcal{C}_{\mathbf{Lie}}$ = category of Lie algebras
$\mathcal{C}_{\mathbf{Jord}}$ = category of Jordan algebras
$\mathcal{C}_{\mathbf{Jord}_1}$ = category of unital Jordan algebras
$\mathcal{C}_{A}$ = category of associative algebras
$\mathcal{C}_{A_1}$ = category of unital associative algebras
$\mathcal{C}_{A_Z}$ = category of all associative algebras but morphisms sending centers into centers.
$\mathcal{C}_{\mathbf{Com}}$ = category of commutative algebras, etc.

Bimodules

$\mathcal{C}$ a category of algebras
$A \in \mathcal{C}$ an object, M a vector space such that there are

$$A \otimes M \to M, a \otimes m \mapsto am \text{ and } M \otimes A \to M, m \otimes a \mapsto ma$$

define the product $(A \oplus M) \otimes (A \oplus M) \to A \oplus M$

$$(a \oplus m) \otimes (a' \oplus m') \mapsto aa' \oplus (am' + ma')$$

M is an *A-bimodule for $\mathcal{C}$* if

1. $A \oplus M \in \mathcal{C}$
2. $A \to A \oplus M$ is a morphism of $\mathcal{C}$
3. $A \oplus M \to A$ is a morphism of $\mathcal{C}$

Examples : Bimodules for the above categories (exercise !)

Jordan (bi)-modules I

J Jordan algebra, M vector space with

$$\begin{aligned} J \otimes M \to M, \quad & x \otimes \Phi \mapsto x\Phi \\ M \otimes J \to M, \quad & \Phi \otimes x \mapsto \Phi x \end{aligned}$$

such that the null-split extension $J \oplus M$

$$(x \oplus \Phi)(x' \oplus \Phi') = (xx' \oplus x\Phi' + \Phi x')$$

is again a Jordan algebra then M is a *Jordan bimodule*

$$\Leftrightarrow \begin{cases} (i) \;\; x\Phi = \Phi x \\ (ii) \; x(x^2\Phi) = x^2(x\Phi) \\ (iii) \, (x^2y)\Phi - x^2(y\Phi) = 2((xy)(x\Phi) - x(y(x\Phi))) \end{cases}$$

If J has a unit $\mathbb{1} \in J$, M is *unital* if
$(iiii)$ $\mathbb{1}\Phi = \Phi$

In view of (i), a Jordan bimodule is simply called a *Jordan module*.

Jordan (bi)-modules II

J, M being as before, set $L_x\Phi = x\Phi$ then (ii) reads

$$(ii)' \qquad [L_x, L_{x^2}] = 0$$

while (iii) reads

$$(iii)' \qquad L_{x^2y} - L_{x^2}L_y - 2L_{xy}L_x + 2L_xL_yL_x = 0$$

which is equivalent to

$$\begin{cases} (a) & L_{x^3} - 3L_{x^2}L_x + 2L_x^3 = 0 \\ (b) & [[L_x, L_y], L_z] + L_{[x,z,y]} = 0 \end{cases}$$

where $[x, z, y] = (xz)y - x(zy)$ is the associator. Condition $(iiii)$ reads

$$(iv)' \qquad L_{\mathbb{1}} = \mathbb{1}(= I_M)$$

Free J-modules and free $Z(J)$-modules I [3]

J a Jordan algebra is canonically a J-module which is unital whenever J has a unit.

Lemma (2)

Let J be a Jordan algebra, E and F be vector spaces and let $\varphi : J \otimes E \to J \otimes F$ be a J-module homomorphism. Then one has

$$\varphi(Z(J) \otimes E) \subset Z(J) \otimes F$$

Proof.

Choose basis (e_α) and (f_λ) for E and F. One has $\varphi(z \otimes e_\alpha) = m_\alpha^\lambda \otimes f_\lambda$ for $z \in Z(J)$ and some $m_\alpha^\lambda \in J$. On the other hand one has $(xy)z = x(yz)$ for any $x, y \in J$
$\Rightarrow \varphi((xy)z \otimes e_\alpha) = (xy)\varphi(z \otimes e_\alpha) = x\varphi(yz \otimes e_\alpha) = x(y\varphi(z \otimes e_\alpha))$
$\Leftrightarrow [x, y, m_\alpha^\lambda] = 0.$ □

Free J-modules and free $Z(J)$-modules II

Proposition (1)

Let J be a unital Jordan algebra. Then $J \otimes E \mapsto Z(J) \otimes E$ and $(\varphi : J \otimes E \to J \otimes F) \mapsto (\varphi \restriction Z(J) \otimes E : Z(J) \otimes E \to Z(J) \otimes F)$ is an isomorphism between the category of free unital J-modules and the category of free unital $Z(J)$-modules.

Indeed from the above lemma $\varphi \restriction (Z(J) \otimes E)$ is a $Z(J)$-module homomorphism of $Z(J) \otimes E$ into $Z(J) \otimes F$.
Conversely any $Z(J)$-module homomorphim $\varphi_0 : Z(J) \otimes E \to Z(J) \otimes F$ extends uniquely by setting $x\varphi_0(\mathbb{1} \otimes E) = \varphi(x \otimes E) \in J \otimes F$ as a J-module homomorphism.

Unital JSpin-modules I

$E, (\bullet, \bullet)$ pseudo euclidean $\to$ 3 unital $\mathbb{R}$-algebras generated by E

1. Jordan spin factor $JSpin(E) = \mathbb{R}\mathbb{1} + E \quad x \circ y = (x, y)\mathbb{1}$
2. Clifford algebra $C\ell(E) \quad xy + yx = 2(x, y)\mathbb{1}(= 2x \circ y)$
3. Meson algebra $B(E) \quad xyx = (x, y)x$ ($B(E)$ associative).

$C\ell(E)$ and $B(E)$ are finite-dimensional unital $\mathbb{Z}_2$-graded associative real algebras and $x \mapsto \frac{1}{2}(x \otimes \mathbb{1} + \mathbb{1} \otimes x)$ defines an injective homomorphism $i : B(E) \to C\ell(E) \otimes C\ell(E)$

Theorem (3)

a - $C\ell(E)$ is the universal unital associative envelope of $JSpin(E)$
b - $B(E)$ is the universal unital multiplicative envelope of $JSpin(E)$
i.e. M unital left $B(E)$-module $\Leftrightarrow$ M unital $JSpin(E)$-module.

Proof.

Let M be a unital $JSpin(E)$-module. Then $L_{x \circ y} = (x, y)\mathbb{1}$ so (ii)' is satisfied in view of (iv)' and (iii)' while (iii)' reduces to $L_x L_y L_x = (x, y)Lx$ which means that M is a unital left $B(E)$-module. □

Unital JSpin-modules II

$O(E) = \mathrm{Aut}(JSpin(E)) = \mathrm{Aut}(C\ell(E)) = \mathrm{Aut}(B(E))$,
$\mathfrak{so}(E) = \mathrm{Der}(JSpin(E)) = \mathrm{Der}(C\ell(E)) = \mathrm{Der}(B(E))$
moreover the corresponding derivations are inner derivations in the above corresponding algebras $\Rightarrow$ they act on the modules for these algebras.
$JSpin(E)$ is an euclidean Jordan algebra iff. E is euclidean, in this case, one identifiess E with $\mathbb{R}^n$ ($n = \dim(E)$) endowed with the scalar product for which the canonical basis is orthonormal and use the notations $JSpin(E) = JSpin_n, C\ell(E) = C\ell_n, B(E) = B_n$, $O(E) = O_n$ and $\mathfrak{so}(E) = \mathfrak{so}_n$.

B_n **is the direct sum of a finite family of matrix algebras.**

Clifford algebras as JSpin-modules

$C\ell_{n+1}$ is a unital module over $J_2^n = JSpin_{n+1}$ via

$$L_\gamma(A) = \frac{1}{2}(\gamma A + A\gamma)$$

Canonical isomorphism of $\mathbb{Z}_2$-graded vector space (PBW)

$$\Gamma : \wedge\mathbb{R}^{n+1} \to C\ell_{n+1},\ \ \omega_{i_1}\ldots_{i_p} \mapsto \Gamma(\omega) = \omega_{i_1}\ldots_{i_p}\,\gamma^{i_1}\ldots\gamma^{i_p}$$

$$\Rightarrow C\ell_{n+1} = \oplus_{p=0}^{n+1}\Gamma^p \text{ with } \Gamma^p = \Gamma(\wedge^p\mathbb{R}^{n+1})$$

Proposition (2)

For any integer $p \leq \frac{1}{2}n$, $\Gamma^{2p} \oplus \Gamma^{2p+1}$ is an irreducible J_2^n-submodule of $C\ell_{n+1}$ and if $n+1 = 2m$ then $\Gamma^{2m} \simeq \mathbb{R}$ is also an irreducible submodule of $C\ell_{n+1} = C\ell_{2m}$.

The decomposition of $C\ell_{n+1}$ into irreducible J_2^n-modules follows.

The case of $J_2^{4k} = JSpin_{4k+1}$ for $k \geq 1$

$\hat{\varepsilon} = \gamma_0\gamma_1 \ldots \gamma_{4k} \in C\ell_{4k+1}$ is central with $(\hat{\varepsilon})^2 = \mathbb{1}$

$\Rightarrow C\ell_{4k+1} = C\ell_{4k}^{+} \oplus C\ell_{4k}^{-}$, $C\ell_{4k}^{\varepsilon} \simeq C\ell_{4k}$.

Setting $\gamma_0^{\varepsilon} = \varepsilon\gamma_1^{\varepsilon} \ldots \gamma_{4k}^{\varepsilon} \in C\ell_{4k}^{\varepsilon}$ and

$L_{\gamma_m}(\omega^{\varepsilon}) = \frac{1}{2}(\gamma_m^{\varepsilon}\omega^{\varepsilon} + \omega^{\varepsilon}\gamma_m^{\varepsilon})$, $\forall \omega^{\varepsilon} \in C\ell_{4k}^{\varepsilon}, m \in \{0, 1, \ldots, 4k\}$

$\Rightarrow C\ell_{4k}^{\varepsilon} \in \{J_2^{4k}\text{-modules}\} \Rightarrow$ a J_2^{4k}-module structure on $C\ell_{4k+1}$ *which is different* of the one induced by the J_2^{n}-module structure of $C\ell_{n+1}$ defined previously $\forall n$.

J_3^8-modules

Any Jordan algebra J is canonically a J-module which is unital whenever J has a unit.

The list of the unital irreducible Jordan modules over the finite-dimensional Euclidean Jordan algebras is given in [Jacobson]. In the case of the exceptional algebra one has the following proposition

Proposition (3)

Any unital irreducible J_3^8-module is isomorphic to J_3^8 (as module).

In particular, any finite unital module over J_3^8 is of the form $J_3^8 \otimes E$ for some finite-dimensional real vector space E. Thus the complexified $J_3^8 \otimes \mathbb{C}$ of J_3^8 is a free J_3^8-module.

J_3^8-modules for 2 families by generation [1]

$$J^u = \begin{pmatrix} \alpha_1 & \nu_\tau + t & \bar{\nu}_\mu - c \\ \bar{\nu}_\tau - t & \alpha_2 & \nu_e + u \\ \nu_\mu + c & \bar{\nu}_e - u & \alpha_3 \end{pmatrix}$$

$$J^d = \begin{pmatrix} \beta_1 & \tau + b & \bar{\mu} - s \\ \bar{\tau} - b & \beta_2 & e + d \\ \mu + s & \bar{e} - d & \beta_3 \end{pmatrix}$$

or with the previous representation

$$J^u = \begin{pmatrix} \alpha_1 & \nu_\tau & \bar{\nu}_\mu \\ \bar{\nu}_\tau & \alpha_2 & \nu_e \\ \nu_\mu & \bar{\nu}_e & \alpha_3 \end{pmatrix} + (u, c, t)$$

$$J^d = \begin{pmatrix} \beta_1 & \tau & \bar{\mu} \\ \bar{\tau} & \beta_2 & e \\ \mu & \bar{e} & \beta_3 \end{pmatrix} + (d, s, b)$$

α_i, β_j new Majorana particles $\Rightarrow$ OK for the cancellation of anomalies !

Quaternions and the $U(1) \times SU(2)$-symmetry

$$q = (z_1, z_2) = z_1 + z_2 j \in \mathbb{H}$$

The subgroup of $\mathrm{Aut}(\mathbb{H})$ which preserves i is $U(1)$

$$z_1 + z_2 j \mapsto z_1 + e^{i\theta} z_2 j$$

$$\begin{pmatrix} \xi_1 & q \\ \bar{q} & \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 & z_1 \\ \bar{z}_1 & \xi_2 \end{pmatrix} + z_2 \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix} \in J_2^4$$

Subgroup of $\mathrm{Aut}(J_2^4)$ which preserves $\cdots = U(1) \times SU(2)$

$$\begin{pmatrix} \xi_1 & z_1 \\ \bar{z}_1 & \xi_2 \end{pmatrix} + z_2 \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix} \mapsto U \begin{pmatrix} \xi_1 & z_1 \\ \bar{z}_1 & \xi_2 \end{pmatrix} U^* + e^{i\theta} z_2 \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix}$$

as for $U \in SU(2)$

$$U \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix} U^* = \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix}$$

Triality in J_3^8 and the 3 generations [4]

Two ways to describe the underlying triality of J_3^8:

W1 - this triality corresponds to the 3 octonions of the matrix of an element of J_3^8,

W2 - this triality corresponds to the 3 canonical subalgebras of hermitian 2×2 matrices of J_3^8 corresponding themselves to the 3 octonions of W1.

W1 and W2 are equivalent but lead naturally to 2 conceptually different interpretations. In fact $J_2^8 = JSpin_9$ corresponds to a complete generation.

$J_2^8 = JSpin_9$ for one generation

1. $Aut(J_2^8) = O(9)$
   $G_{SM}/\mathbb{Z}_2 = SO(3) \times U(3)$ is ($\simeq$) the subgroup of $SO(9)$ which preserves the splitting $\mathbb{C} \oplus \mathbb{C}^3$ of $\mathbb{O}$ and acts $\mathbb{C}$-linearly on $\mathbb{C}^3$.

2. The $*$-algebra $C\ell_9^c = M_{16}(\mathbb{C}) \oplus M_{16}(\mathbb{C})$ is generated by the relations

$$\begin{cases} \frac{1}{2}(xy + yx) = x \circ y, & \forall x, y \in J_2^8 \\ x^* = x, & \forall x \in J_2^8 \\ \mathbb{1} = \mathbb{1}_{J_2^8} & \end{cases}$$

3. J_2^8 is a unital Jordan subalgebra of the hermitian part $H(C\ell_9^c) = J_{16}^2 \oplus J_{16}^2$ of $C\ell_9^c$ which is therefore a J_2^8-module. Note that the diagonal $\Delta H(C\ell_9^c)$ of $H(C\ell_9^c)$ is **a maximal subspace of compatible observables in** $H(C\ell_9^c)$ is **of dimension** $32 = 2^5$. This property is common to $H(C\ell_9)$ $H(C\ell_{10})$ and $H(C\ell_{10}^c)$, i.e.
   $\dim(\Delta H(C\ell_9)) = \dim(\Delta H(C\ell_{10})) = \dim(\Delta H(C\ell_{10}^c)) = 2^5$.

The correspondence "triality-generation" in J_3^8 [4]

$P^2 = P$, primitive= pure state of J_3^8
$\leftrightarrow J_2^8(P) = (\mathbb{1} - P)J_3^8(\mathbb{1} - P) \simeq JSpin_9$
$Aut(J_2^8(P))$= subgroup of F_4 which preserves $P \simeq Spin_9$

P_i diagonal $\leftrightarrow J_2^8(P_i) \leftrightarrow$ generation $i \quad (i \in \{1, 2, 3\})$

$$Aut(J_2^8(P_i)) \cap \frac{SU(3)_c \times SU(3)}{\mathbb{Z}_3} = G_i \simeq \frac{SU(3)_c \times SU(2) \times U(1)}{\mathbb{Z}_6}$$

Each $J_2^8(P_i)$ with the identification $\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$ has automorphism group $G_i \subset F_4$ isomorphic to the standard model group for one generation

The extended electroweak symmetry $SU(3)_{ew}$

$$J_i = J_2^8(P_i), \;\; Aut(J_i) \simeq Spin_9$$

$$SU(3)_c \times SU(3)/\mathbb{Z}_3 \subset F_4 = \mathrm{Aut}(J_3^8)$$

$$Aut(J_i) \subset F_4$$

$$SU(3)_c \times SU(3)/\mathbb{Z}_3 \cap Aut(J_i) = G_i$$

$$G_i \simeq SU(3)_c \times SU(2) \times U(1)/\mathbb{Z}_6$$

$\Rightarrow$ The second $SU(3)$ project onto the electroweak symmetry for each generation .
This $SU(3)$ will be called extented electroweak symmetry and denoted by $SU(3)_{ew}$.
Internal symmetry $SU(3)_c \times SU(3)_{ew}/\mathbb{Z}_3 \subset F_4$

Differential graded Jordan algebras [1]

$\Omega = \oplus_{n\in\mathbb{N}}\Omega^n$ which is a Jordan superalgebra (for $\mathbb{N}/2\mathbb{N}$)

$$ab = (-1)^{|a||b|}ba \text{ for } a \in \Omega^{|a|}, b \in \Omega^{|b|}$$

and graded Jordan identity

$$(-1)^{|a||c|}[L_{ab}, L_c]\mathrm{gr} + (-1)^{|b||a|}[L_{bc}, L_a]\mathrm{gr} + (-1)^{|c||b|}[L_{ca}, L_b]\mathrm{gr} = 0$$

with a differential d

$$d^2 = 0$$

$$d\Omega^n \subset \Omega^{n+1}$$

$$d(ab) = d(a)b + (-1)^{|a|}ad(b)$$

Model for algebras of differential forms on quantum spaces.
Differential calculus over J = differential graded Jordan algebra Ω with $\Omega^0 = J$.

Derivation-based differential calculus

J unital Jordan algebra with center $Z(J)$

$$\Omega^n_{\mathrm{Der}}(J) = \mathrm{Hom}_{Z(J)}(\wedge^n_{Z(J)}\mathrm{Der}(J), J)$$

$\Omega_{\mathrm{Der}}(J) = \oplus_n \Omega^n_{\mathrm{Der}}(J)$ is canonically a differential graded Jordan algebra with

$$\begin{aligned} d\omega(X_0, \cdots, X_n) &= \textstyle\sum_{0\leq k\leq n}(-1)^k X_k\ \omega(X_0, \overset{k}{\overset{\vee}{\dots}}, X_n) \\ &+ \textstyle\sum_{0\leq r<s\leq n}(-1)^{r+s}\ \omega([X_r, X_s], X_0, \overset{r}{\overset{\vee}{\dots}}\cdot\cdot\overset{s}{\overset{\vee}{\dots}}, X_n) \end{aligned}$$

referred to as the *derivation-based differential calculus over J*.

Universal property for J_3^8 [1], [3]

Theorem (4)

Any homomorphism φ of unital Jordan algebra of J_3^8 into the Jordan subalgebra Ω^0 of a unital differential graded Jordan algebra $\Omega = \oplus \Omega^n$ has a unique extension $\tilde{\varphi} : \Omega_{Der}(J_3^8) \to \Omega$ as a homomorphism of differential graded Jordan algebras.

$\Omega_{\mathrm{Der}}(J_3^8) = J_3^8 \otimes \wedge \mathfrak{f}_4^*$ with the Chevalley-Eilenberg differential.

Derivation-based connections I

$J=$ unital Jordan algebra, center$=Z(J)$, $M=$ unital J-module.
A *derivation-based connection on M is a linear mapping* $X \mapsto \nabla_X$ of $\mathrm{Der}(J)$ into $\mathcal{L}(M)$ such that for $x \in J$ and $z \in Z(J)$

$$\begin{cases} \nabla_X(xm) = X(x)m + x\nabla_X(m) \\ \\ \nabla_{zX}(m) = z\nabla_X(m) \end{cases}$$

curvature of ∇

$$R_{X,Y} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$

$$\begin{cases} R_{X,Y}(xm) = xR_{X,Y}(m) \\ \\ R_{zX,Y}(m) = zR_{X,Y}(m) \end{cases}$$

$\mathfrak{g} \subset \mathrm{Der}(J)$, Lie subalgebra and $Z(J)$-submodule
$\Rightarrow$ *derivation-based $\mathfrak{g}$-connection on M* (by restriction).

Derivation-based connections II

$\Omega_{\mathrm{Der}}(M) = \mathrm{Hom}_{Z(J)}(\wedge \mathrm{Der}(J), M)$, ∇ linear endomorphism of $\Omega_{\mathrm{Der}}(M)$ such that

$$\begin{cases} \nabla(\Omega^n_{\mathrm{Der}}(M)) \subset \Omega^{n+1}_{\mathrm{Der}}(M) \\ \\ \nabla(\omega\Phi) = d(\omega)\Phi + (-1)^m \omega \nabla(\Phi) \end{cases}$$

for any $m, n \in \mathbb{N}$, $\omega \in \Omega^m_{\mathrm{Der}}(J)$ and $\Phi \in \Omega_{\mathrm{Der}}(M)$.
$\Rightarrow$ *curvature* ∇^2

$$\nabla^2(\omega\Phi) = \omega\nabla^2(\Phi)$$

Let ∇ be such a connection and define $\nabla_X(m)$ as in I by

$$\nabla_X(m) = \nabla(m)(X)$$

for $m \in M = \Omega^0_{\mathrm{Der}}(M)$, $X \in \mathrm{Der}(J)$
Conversely, ∇ as in I $\Rightarrow$ ∇ as here with

$$\begin{aligned} \nabla(\Phi)(X_0, \cdots, X_n) &= \textstyle\sum_{p=0}^{n} (-1)^p \nabla_{X_p}(\Phi(X_0, \overset{p}{\overset{\vee}{\cdots}}, X_n)) \\ &+ \textstyle\sum_{0\leq r<s\leq n} (-1)^{r+s}\, \Phi([X_r, X_s], X_0, \overset{r}{\overset{\vee}{\cdots}}\overset{s}{\overset{\vee}{\cdots}}, X_n) \end{aligned}$$

General connection

$\Omega = \oplus\Omega^n =$ differential graded Jordan algebra, $\Gamma = \oplus\Gamma^n$ graded module over Ω.

A connection on Γ, is a linear endomorphism of Γ satisfying

$$\begin{cases} \nabla(\Gamma^n) \subset \Gamma^{n+1} \\ \nabla(\omega\Phi) = d(\omega)\Phi + (-1)^m\omega\nabla(\Phi) \end{cases}$$

for $\omega \in \Omega^n$, $\Phi \in \Gamma \Rightarrow$

$$\nabla^2(\omega\Phi) = \omega\nabla^2(\Phi)$$

∇^2 homogeneous Ω-module homomorphism of degree 2 is *the curvature of* ∇.

$$\nabla\nabla^2 = \nabla^2\nabla$$

is *the Bianchi identity of* ∇.

Connections on free modules I [3]

J unital Jordan algebra, $M = J \otimes E$ free J-module, Ω differential calculus over J such that Ω is generated by $J = \Omega^0$ as differential graded Jordan algebra.
$\nabla : \Omega \otimes E \to \Omega \otimes E$ connection induced by $\nabla : J \otimes E \to \Omega^1 \otimes E$.

Proposition (4)

1. $\overset{0}{\nabla} = d \otimes I_E : J \otimes E \to \Omega^1 \otimes E$ *defines a flat connection on* M *which is gauge invariant whenever the center of* J *is trivial.*
2. *Any other* Ω*-connection* ∇ *on* M *is defined by* $\nabla = \overset{0}{\nabla} + A : J \otimes E \to \Omega^1 \otimes E$ *where* A *is a* J*-module homomorphism of* $J \otimes E$ *into* $\Omega^1 \otimes E$.
3. *If* $\Omega = \Omega_{Der}$ *(i.e. for derivation-based connections) one has* $(\nabla^2)(X, Y) = R_{X,Y} = XA_Y - YA_X + [A_X, A_Y] - A_{[X,Y]},\ \forall X, Y \in Der(J)$.

Connections on free modules II

Theorem (5)

Let J be a finite-dimensional euclidean Jordan algebra and M be a finite free module i.e. $M = J \otimes \mathbb{R}^n$ for $n < \infty$. Then the curvature of a derivation-based connection $\nabla_X + A_X$ on M is given by $R_{X,Y} = [A_X, A_Y] - A_{[X,Y]}$.

Proof.

It follows from Proposition 1 that A_X us a $n \times n$ matrix with coefficients in the center $Z(J)$ of J, but $Z(J)$ is a finite-dimensional associative euclidean Jordan algebra on which any derivation vanishes. So one has $YA_X = XA_Y = 0$. The result follows then from 3 in Proposition 4. □

Connections on JSpin-modules I

Any $X \in \mathrm{Der}(J_2^n) = \mathfrak{so}(n+1)$ has an extension as inner derivation of the meson algebra $B_{n+1} \Rightarrow$ an action $m \mapsto Xm$ on any J_2^n-module $M \Rightarrow$

$$\overset{0}{\nabla}_X m = Xm, m \in M$$

defines a derivation-based connection which is flat

$$\overset{0}{R}_{XY} = [\overset{0}{\nabla}_X, \overset{0}{\nabla}_Y] - \overset{0}{\nabla}_{[X,Y]} \equiv 0$$

Any other connection (for Ω_{Der}) is of the form

$$\nabla_X = \overset{0}{\nabla}_X + A_X$$

where A_X is a J_2^n-module endormorphism of M which depends linearly of $X \in \mathfrak{so}(n+1)$.

Connections on JSpin-modules II and further prospects

Since a unital $JSpin_{n+1}$-module is the same as a untial left B_{n+1}-module and that B_{n+1} is a finite matrix algebra, one can use the noncommutative approach to noncommutative gauge theory developed in the years 1987-1989 which is summarized in reference [6] (see also in [7]).
This is also true for any finite-dimensional euclidean Jordan algebra J since then the universal unital multiplicative envelope $U_1(J)$ of J is also a finite-dimensional matrix algebra (i.e. a finite sum of complete matrix algebras). $U_1(J)$ is an associative unital algebra characterized by the fact that a unital left $U_1(J)$-module is the same thing as a unital J-module, e.g. $B_{n+1} = U_1(JSpin_{n+1})$.

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